

# Robust Stability and Performance of Stochastic Uncertain Systems on an Infinite Time Interval <sup>1</sup>

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## Abstract

In this paper, we consider a robust stability problem for continuous time stochastic uncertain systems. The uncertainty in the system is characterized in terms of an uncertain probability distribution on the noise input. This uncertainty is assumed to satisfy a certain relative entropy constraint. The solution to a specially parametrized risk-sensitive performance analysis problem is used to estimate the level of guaranteed performance for the stochastic uncertain system under consideration. This solution is obtained by solving an algebraic Riccati equation. The corresponding performance bound holds for all admissible uncertainties and is nonconservative.

## 1 Introduction

Consider an uncertain system involving an LFT type interconnection between the nominal system model and the uncertainty in the system [5]; see Figure 1. Given a set of admissible uncertainties  $\Delta(s)$ , the problem of robust stability with guaranteed performance is to evaluate the stability and worst-case performance of the system in the presence of these uncertainties. This problem has received much attention in the recent control literature; e.g., see [3, 1, 8, 11, 14, 7, 20]. References [8, 11] investigate the quadratic stability approach to this problem. A solution to the guaranteed cost control problem which employs absolute stability ideas is presented in references [14, 17, 20]. A linear matrix inequalities (LMIs) approach to solving the guaranteed cost control problem is developed in [7]. Alternatively, the issue of robust performance is addressed using the theory of robust  $H^\infty$  control in [22, 15]; see also [2] where the robustness of the  $L_2$  gain of the system is analyzed using LMIs.

A common feature of the uncertain system models considered in the above references is that they do not take into account the fact that many physical systems are subject to additive noise disturbances such as sensor noises. There-

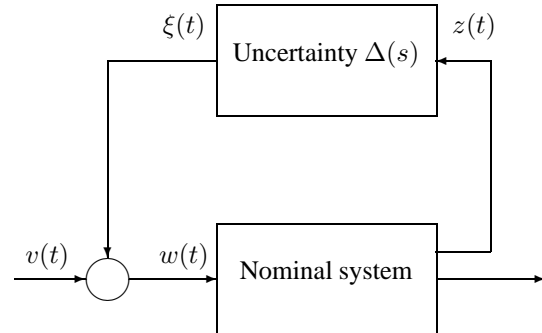


Figure 1: Uncertain system model block diagram

fore, the issue of robust stability and robust performance in the presence of noise disturbances was not adequately addressed. One possible approach to addressing this issue is to treat the disturbances as uncertain stochastic processes which satisfy a stochastic uncertainty constraint. This constraint restricts the *relative entropy* between an uncertain probability measure related to the distribution of the uncertain noise, and a reference probability measure corresponding to a nominal model of the noise input. This allows for the modeling of both additive disturbances and unmodeled dynamics. In many cases, the nominal noise is modeled using a Wiener process and the reference probability measure is assumed to be a Wiener measure. As shown in [18], the relative entropy constraint can be thought of as a stochastic counterpart to the integral quadratic constraint uncertainty description; see [14, 23, 17]. In this paper, we show that modeling the system uncertainty using uncertain stochastic processes which satisfy a relative entropy uncertainty constraint leads to a tractable solution to a problem of estimating the worst case of an infinite-horizon cost functional. Thus, a tractable solution to the robustness analysis problem is achieved. Furthermore, this solution is non-conservative in that it provides a precise characterization of the system worst-case performance in the face of uncertainty satisfying a relative entropy uncertainty constraint. It is shown that the worst-case performance can be found as the value of a certain minimization problem involving stabilizing solutions of a parameter-dependent Riccati equation.

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## 2 Definitions

Let  $(\Omega, \mathcal{F}, P)$  be a complete probability space on which a  $p$ -dimensional standard Wiener process  $W(\cdot)$  and an  $n$ -dimensional Gaussian random variable are defined. The space  $\Omega$  can be thought of as the noise space  $\mathbf{R}^n \times C([0, \infty), \mathbf{R}^p)$  [4]. The probability measure  $P$  can then be defined as the product of a given probability measure on  $\mathbf{R}^n$  and the standard Wiener measure on  $C([0, \infty), \mathbf{R}^p)$ . The space  $\Omega$  is endowed with a filtration  $\{\mathcal{F}_t, t \geq 0\}$  which has been completed by including all sets of  $P$ -probability zero. The random variable  $x_0$  and the Wiener process  $W(\cdot)$  are assumed to be stochastically independent in  $(\Omega, \mathcal{F}, P)$ . The mean and covariance of the Gaussian variable  $x_0$  are denoted by  $\bar{x}_0$  and  $Y_0$ , respectively, with  $Y_0 > 0$ .

### 2.1 Stochastic nominal system

On the probability space  $(\Omega, \mathcal{F}, P)$  defined above, we consider the system dynamics driven by the noise input  $W(\cdot)$ . These dynamics are described by the following stochastic differential equation

$$\begin{aligned} dx(t) &= Ax(t)dt + B_2 dW(t), & x(0) &= x_0, \\ z(t) &= C_1 x(t). \end{aligned} \quad (1)$$

In the the above equations,  $x(t) \in \mathbf{R}^n$  is the state,  $z(t) \in \mathbf{R}^q$  is the uncertainty output. The system (1) is referred to as the nominal system. All coefficients in equations (1) are assumed to be constant matrices of corresponding dimensions.

### 2.2 Stochastic uncertain system

In this paper, we use an uncertainty description for stochastic uncertain systems with additive noise which can be regarded as an extension of the uncertainty description considered in [10, 18] to the case of an infinite time horizon. The stochastic uncertain systems to be considered are described by the nominal system (1) considered over the probability space  $(\Omega, \mathcal{F}, P)$ , and also by a set of perturbations of the reference probability measure  $P$ . These perturbations are defined as follows. Consider the set  $\mathcal{M}$  of continuous positive martingales  $(\nu(t), \mathcal{F}_t, t \geq 0)$  such that for each  $T \geq 0$ ,  $\mathbf{E}\nu(T) = 1$ . Here,  $\mathbf{E}$  denotes the expectation with respect to the probability measure  $P$ . Note that the set  $\mathcal{M}$  is convex.

Every martingale  $\nu(\cdot) \in \mathcal{M}$  gives rise to a probability measure  $Q^T$  on the measurable space  $(\Omega, \mathcal{F}_T)$  defined by the equation

$$Q^T(d\omega) = \nu(T)P^T(d\omega). \quad (2)$$

Here,  $P^T$  denotes the restriction of the reference probability measure  $P$  to  $(\Omega, \mathcal{F}_T)$ . From this definition, for every  $T > 0$ , the probability measure  $Q^T$  is absolutely continuous with respect to the probability measure  $P^T$ ,  $Q^T \ll P^T$ . The uncertain system is described by equation (1) considered over the probability space  $(\Omega, \mathcal{F}_T, Q^T)$  for every  $T > 0$ . The expectation in this probability space is denoted  $\mathbf{E}^{Q^T}$ .

We now present an infinite-horizon uncertainty description for stochastic uncertain systems with additive noise. This uncertainty description was originally introduced in [16]. As mentioned in [16], this uncertainty description may be regarded as an extension of the uncertainty description considered in [18] to the infinite-horizon case. Also, this uncertainty description can be thought of as an extension of the deterministic integral quadratic constraint uncertainty description [14, 23] and stochastic integral quadratic constraint uncertainty description [17] to the case of stochastic uncertain systems with additive noise. Recall that one of the motivations behind introducing the relative entropy uncertainty description in [16] was that such an uncertainty description allows one to accommodate bounded power processes such as those described by the system (1). This is in contrast to integral quadratic constraints considered in [14, 23, 17] which deal with bounded energy processes.

The uncertainty description considered in this paper exploits a sequence of continuous positive martingales  $\{\nu_i(t), \mathcal{F}_t, t \geq 0\}_{i=1}^{\infty} \subset \mathcal{M}$  which converges to a limiting martingale  $\nu(\cdot)$  in the following sense: For any  $T > 0$ , the sequence  $\{\nu_i(T)\}_{i=1}^{\infty}$  converges weakly to  $\nu(T)$  in  $L_1(\Omega, \mathcal{F}_T, P^T)$ . Using the martingales  $\nu_i(t)$ , we define a sequence of probability measures  $\{Q_i^T\}_{i=1}^{\infty}$  as follows:

$$Q_i^T(d\omega) = \nu_i(T)P^T(d\omega). \quad (3)$$

From the definition of the martingales  $\nu_i(t)$ , it follows that for each  $T > 0$ , the sequence  $\{Q_i^T\}_{i=1}^{\infty}$  converges to the probability measure  $Q^T$  corresponding to a limiting martingale  $\nu(\cdot)$  in the following sense: For any  $\mathcal{F}_T$ -measurable random variable  $\eta \in L_\infty(\Omega, \mathcal{F}_T, P^T)$ ,

$$\lim_{i \rightarrow \infty} \int_{\Omega} \eta Q_i^T(d\omega) = \int_{\Omega} \eta Q^T(d\omega). \quad (4)$$

We denote this fact by  $Q_i^T \Rightarrow Q^T$  as  $i \rightarrow \infty$ .

**Remark 1** The property  $Q_i^T \Rightarrow Q^T$  implies that the sequence of probability measures  $Q_i^T$  converges weakly to the probability measure  $Q^T$ . Indeed, consider the Polish space of probability measures on the measurable space  $(\Omega, \mathcal{F}_T)$  endowed with the topology of weak convergence of probability measures. Note that  $\Omega$  is a metric space. Hence, such a topology can be defined; e.g., see [6]. For the sequence  $\{Q_i^T\}$  to converge weakly to  $Q^T$ , it is required that equation (4) holds for all bounded continuous random variables  $\eta$ . Obviously, this requirement is satisfied if  $Q_i^T \Rightarrow Q^T$ .

As in the finite-horizon case [10, 18], we describe the class of admissible uncertainties in terms of the relative entropy functional  $h(\cdot \|\cdot)$ ; for the definition and properties of the functional  $h(\cdot \|\cdot)$ , see [6].

**Definition 1** Let  $d$  be a given positive constant. A martingale  $\nu(\cdot) \in \mathcal{M}$  is said to define an admissible uncertainty if there exists a sequence of continuous positive martingales  $\{\nu_i(t), \mathcal{F}_t, t \geq 0\}_{i=1}^{\infty} \subset \mathcal{M}$  which satisfies the following conditions:

- (i) For each  $i$ ,  $h(Q_i^T \| P^T) < \infty$  for all  $T > 0$ ;
- (ii) For all  $T > 0$ ,  $Q_i^T \Rightarrow Q^T$  as  $i \rightarrow \infty$ ;
- (iii) The following stochastic uncertainty constraint is satisfied: For any sufficiently large  $T > 0$ , there exists a constant  $\delta(T)$  such that  $\lim_{T \rightarrow \infty} \delta(T) = 0$  and

$$\inf_{T' > T} \frac{1}{T'} \left[ \frac{1}{2} \mathbf{E}^{Q_i^{T'}} \int_0^{T'} \|z(t)\|^2 dt - h(Q_i^{T'} \| P^{T'}) \right] \geq -\frac{d}{2} + \delta(T) \quad (5)$$

for all  $i = 1, 2, \dots$ . In (5), the uncertainty output  $z(\cdot)$  is defined by equation (1) considered on the probability space  $(\Omega, \mathcal{F}_T, Q_i^T)$ .

In the above conditions,  $Q_i^T$  is the probability measure defined by (3) corresponding to the martingale  $\nu_i(t)$  and time  $T > 0$ . We let  $\Xi$  denote the set of martingales  $\nu(\cdot) \in \mathcal{M}$  corresponding to admissible uncertainties. Elements of  $\Xi$  are also called admissible martingales.

Observe that the reference probability measure  $P$  corresponds to the admissible martingale  $\nu(t) \equiv 1$ . Hence, the set  $\Xi$  is not empty. Indeed, choose  $\nu_i(t) = 1$  for all  $i$  and  $t$ . Then,  $Q_i^T = P^T$  for all  $i$ . It follows from the identity  $h(P^T \| P^T) = 0$  that

$$\begin{aligned} \inf_{T' > T} \frac{1}{T'} \left[ \frac{1}{2} \mathbf{E}^{Q_i^{T'}} \int_0^{T'} \|z(t)\|^2 dt - h(Q_i^{T'} \| P^{T'}) \right] \\ = \inf_{T' > T} \frac{1}{2T'} \mathbf{E} \int_0^{T'} \|z(t)\|^2 dt. \end{aligned}$$

Note that the expectations are well defined. Also, the infimum on the right hand side of the above equation is non-negative for any  $T > 0$ . Therefore, for any constant  $d > 0$ , one can find a sufficiently small  $\delta = \delta(T)$  such that  $\lim_{T \rightarrow \infty} \delta(T) = 0$  and the constraint (5) is satisfied strictly in this case.

**Remark 2** Note that condition (5) implies that

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \left[ \frac{1}{2} \mathbf{E}^{Q_i^T} \int_0^T \|z(t)\|^2 dt - h(Q_i^T \| P^T) \right] \geq -\frac{d}{2}$$

for all  $i = 1, 2, \dots$

In the sequel, we will use the following notation. Let  $\mathcal{P}_T$  be the set of probability measures  $Q^T$  on  $(\Omega, \mathcal{F}_T)$  such that  $h(Q^T \| P^T) < \infty$ . Also, the notation  $\mathcal{M}_\infty$  will denote the set of martingales  $\nu(\cdot) \in \mathcal{M}$  such that  $h(Q^T \| P^T) < \infty$  for all  $T > 0$ . It is readily verified that the set  $\mathcal{M}_\infty$  is convex. Note that the martingales  $\nu_i(\cdot)$  from Definition 1 belong to  $\mathcal{M}_\infty$ .

### 2.3 A connection between uncertainty input signals and martingale uncertainty

A connection between the disturbance signal uncertainty model and the perturbation martingale uncertainty model is discussed in the paper [19]. It is observed in [19] that an arbitrary uncertainty input  $\xi(\cdot)$  satisfying the conditions of Novikov's theorem [9]) on every finite interval  $[0, T]$  can be associated with an uncertainty martingale  $\nu(\cdot) \in \mathcal{M}$ . This result is summarized in the following lemma; see [19].

**Lemma 1** Suppose a random process  $(\xi(t), \mathcal{F}_t)$ ,  $0 \leq t \leq T$  satisfies the conditions:

$$\begin{aligned} P \left( \int_0^T \|\xi(s)\|^2 ds < \infty \right) &= 1, \\ \mathbf{E} \exp \left( \frac{1}{2} \int_0^T \|\xi(s)\|^2 ds \right) &< \infty \end{aligned} \quad (6)$$

for all  $T > 0$ . Then the equation

$$\nu(t) = 1 + \int_0^t \nu(s) \xi(s)' dW(s). \quad (7)$$

defines a continuous positive martingale  $\nu(t)$ . Furthermore, the stochastic process

$$\tilde{W}(t) = W(t) - \int_0^t \xi(t) dt, \quad (8)$$

is a Wiener process with respect to the system  $\{\mathcal{F}_t, 0 \leq t \leq T\}$  and the probability measure  $Q^T$  defined by equation (2) where  $\nu(\cdot)$  is defined by equation (7).

The above result follows from Novikov's Theorem and Girsanov's Theorem (e.g., see Theorem 6.1 and Theorem 6.3 of [9]). In particular, it follows from the above results that the martingale  $\nu(\cdot)$  is given by the equation

$$\nu(t) = \exp \left( \int_0^t \xi'(s) dW(s) - \frac{1}{2} \int_0^t \|\xi(s)\|^2 ds \right). \quad (9)$$

Also, on the probability space  $(\Omega, \mathcal{F}_T, Q^T)$ , the system (1) becomes a system of the following form:

$$\begin{aligned} dx &= (Ax + B_2 \xi) dt + B_2 d\tilde{W}(t), \quad x(0) = x_0, \\ z &= C_1 x, \end{aligned} \quad (10)$$

This is the standard form for a stochastic uncertain system driven by an uncertainty input  $\xi(t)$  and an additive noise input described by the Wiener process  $\tilde{W}(t)$ . The equation (10) can be viewed as an equivalent representation of the original system (1).

A class of uncertain system models which is often considered in the literature involves an LFT type interconnection between the nominal system model and the LTI uncertainty [5]; see Figure 1. One of the results of the paper [19] shows

that that if a stable rational LTI uncertainty  $\Delta(s)$  shown in Figure 1 satisfies the  $H^\infty$  norm bound

$$\|\Delta(s)\|_\infty \leq 1, \quad (11)$$

then the corresponding stochastic uncertain system satisfies the relative entropy constraint defined above. It is also observed in [19] that the description of stochastic uncertainty presented in Definition 1 encompasses some other important classes of uncertainty arising in control systems such as, for example, cone-bounded uncertainty.

### 3 Robust stability and performance of systems with relative entropy constraints on the uncertainty

#### 3.1 Absolute stability

An important issue in any system analysis problem on an infinite time interval concerns the stability properties of the system. In this paper, the systems under consideration are subject to additive noise. The solutions of such systems do not necessarily belong to  $L_2[0, \infty)$ . A definition of absolute stability which properly accounts for this feature of the systems under consideration is presented below. This definition is a special case of the corresponding definition of absolute stabilizability introduced in [16]; also, see [19].

**Definition 2** An uncertain system (1), (5), is said to be absolutely stable, if there exist constants  $c_1 > 0$ ,  $c_2 > 0$  such that for any admissible uncertainty  $\nu(\cdot) \in \Xi$ ,

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \left[ \mathbf{E}^{Q^T} \int_0^T \|x(t)\|^2 dt + h(Q^T \|P^T) \right] \leq c_1 + c_2 d. \quad (12)$$

In the sequel, the following property of mean square stable systems will be used; see [12].

**Lemma 2** Suppose the stochastic nominal system (1) is mean square stable; i.e.

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \mathbf{E} \int_0^T \|x(t)\|^2 dt < \infty. \quad (13)$$

Also, suppose the pair  $(A, B_2)$  is stabilizable. Then, the matrix  $A$  must be stable.

**Theorem 1** Suppose that the pair  $(A, B_2)$  is controllable and the pair  $(A, C_1)$  is observable. Then, the following statements are equivalent.

- (i) The uncertain system (1), (5) is absolutely stable.
- (ii) For any nonnegative-definite symmetric matrix  $R$ , there exists a positive constant  $\tau$  such that the Riccati equation

$$XA + A'X + R + \tau C_1' C_1 + \frac{1}{\tau} X B_2 B_2' X = 0, \quad (14)$$

has a positive definite-stabilizing solution.

The proof of this theorem makes use of the relationship between Riccati equations arising in stochastic control [21] and risk-sensitive performance [13]. Also, we use a result of the theory of large deviations known as the duality between free energy and relative entropy; e.g., see [4].

#### 3.2 Robust performance

Associated with the system (1), (5), consider a cost functional  $J(\nu)$  of the form

$$J(\nu) = \limsup_{T \rightarrow \infty} \frac{1}{T} \mathbf{E}^{Q^T} \int_0^T x(t)' R x(t) dt, \quad (15)$$

where  $R$  is a nonnegative definite symmetric matrix. In this section, we are concerned with a robust performance problem associated with the system (1), cost functional (15) and the constraint (5). In this problem, we wish to find the worst case value of the functional  $J(\nu)$  in the face of uncertainty  $\nu \in \Xi$  satisfying the constraint (5). That is, we consider the maximization problem:

$$\sup_{\nu \in \Xi} J(\nu). \quad (16)$$

**Theorem 2** Suppose that the pair  $(A, B_2)$  is controllable, the pair  $(A, C_1)$  is observable and condition (i) of Theorem 1 holds. Then the worst case performance of the uncertain system (1), (5) can be found from the following minimization problem:

$$\sup_{\nu \in \Xi} J(\nu) = \inf_{\tau} \frac{1}{2} (\text{tr } B_2 B_2' X_\tau + \tau d). \quad (17)$$

The infimum on the right hand side of equation (17) is taken over the set of constants  $\tau > 0$  such that the Riccati equation (14) has a positive-definite stabilizing solution  $X_\tau$ .

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