GENERALIZED QUADRATIC LYAPUNOV FUNCTIONS FOR NONLINEAR/UNCERTAIN SYSTEMS ANALYSIS

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Abstract— **We consider the class of discrete-time nonlinear/uncertain systems described by the feedback connection of a linear time-invariant system and a "troublesome component," i.e. either a static nonlinearity or a time-varying parametric uncertainty. We propose a generalized quadratic Lyapunov function for stability analysis of such systems. In particular, the Lyapunov function is given by a quadratic form of a vector that depends on the state in a specific nonlinear manner. Introducing a quadratic-form model of the troublesome component in the spirit of integral quadratic constraints, we obtain sufficient conditions for the existence of such Lyapunov functions that proves global/regional stability. The conditions are given in terms of linear matrix inequalities that can be numerically verified in polynomial time.**

I. Introduction[0]

While the linear control system theory has reached a mature state, there is still much work to be done for analysis and synthesis of nonlinear and/or uncertain control systems. Recent progresses in nonlinear control theories (see e.g. [12], [15]) are nontrivial, but it appears difficult to have a universal design procedure that is guaranteed to "work" for a general class of nonlinear systems. A seemingly more tractable, yet practically important problem is to consider a special class of nonlinear systems that can be described by the feedback connection of a nonlinear component ϕ and a linear time-invariant (LTI) system G (see Fig. 1). This restricted class of nonlinear systems have been extensively studied in the literature [18], [19], [26]. Moreover, such feedback systems have constituted a paradigm for linear robust control theory [6], [13], [21] where ϕ acts as an uncertainty in the system rather than as a nonlinearity. The representation of nonlinear/uncertain systems as in Fig. 1 has lead to a powerful tool for stability and performance analysis — input-output multipliers.

Another powerful tool for analysis and synthesis of nonlinear and/or uncertain systems is the Lyapunov function. The problem of quadratic stabilization was extensively studied in the 1980's (e.g. [16], [1]), where a single quadratic Lyapunov function is used to prove stability of a family of systems. It has been shown [2], [7], [8], [25] that the notion of quadratic stability leads to robust control design techniques that are conservative but numerically efficient, with the aid of interior point methods [20] for solving linear matrix inequalities (LMIs) [3], [9]. More recently, attention has been paid to the so-called parameter-dependent Lyapunov functions that give less conservative stability conditions than the quadratic stability approach.

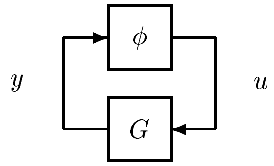

Fig. 1. Nonlinear/uncertain system

Some of the Lyapunov approaches can be interpreted within the framework of input-output multipliers, and vice versa. For example, satisfaction of the (multi-loop) circle criterion [24] implies the quadratic stability, while the Popov criterion for systems with sector-bounded nonlinearity is related to the existence of an affine parameter-dependent Lyapunov function for the corresponding uncertain systems [10], [3]. The Lyapunov interpretations of the frequency domain conditions given in terms of input-output multipliers have been found useful for developing *regional* (rather than global) stability conditions for nonlinear systems of the form in Fig. 1; For instance, the references [11], [22] consider saturating control systems and exploit the circle and the Popov criteria to give an estimate of the region of attraction, that is, the set of initial state vectors that generate trajectories converging to the origin.

In this paper, we propose a new class of Lyapunov functions for discrete-time systems described by the feedback connection in Fig. 1 where ϕ is either a static nonlinearity or an uncertainty. In particular, we generalize the class of parameter-dependent Lyapunov functions in [14] in order to allow for the analysis of both nonlinear and uncertain systems with reduced conservatism. The main idea can be explained as follows.

In [14], uncertain systems of the form in Fig. 1 is considered where G is a stable LTI system with a state space realization $C(sI-A)^{-1}B+D$ and ϕ is the function defined by $u=\Delta y$ with Δ being a matrix consisting of uncertain time-varying parameters. For this class of systems, the following parameter-dependent Lyapunov function is shown to provide a reasonable trade-off between the degree of conservatism and the computational burden to check ro-

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[0] We use the following notation. The sets of $n\times m$ real matrices and n-dimensional vectors are denoted by $\mathbf{R}^{n\times m}$ and $\mathbf{R}^n$, respectively. For a matrix (or a vector) A, A' means the transpose. For vectors v_i $(i=1,\ldots,k)$, we denote the composed vector $v := [\, v_1' \ \cdots \ v_k' \,]'$ by $v = [\, v_1; \cdots; v_k \,]$. For matrices A and B of the same dimensions, $A\cdot B$ denotes the entry-wise multiplication. The set of nonnegative integers is denoted by $\mathbf{Z}$ and ℓ_2^m is the set of square summable sequences of m dimensional real vectors.

bust stability: $V(x) := x'P_\Delta x$ where

$$P_\Delta := \begin{bmatrix} I \\ (I-\Delta D)^{-1}\Delta C \end{bmatrix}' P \begin{bmatrix} I \\ (I-\Delta D)^{-1}\Delta C \end{bmatrix}.$$

Note that this Lyapunov function can also be given by

$$V(x) := \begin{bmatrix} x \\ u \end{bmatrix}' P \begin{bmatrix} x \\ u \end{bmatrix} \qquad (1)$$

where u is uniquely determined from x as the solution[1] to $u = \Delta(Cx + Du)$. Note that the Lyapunov function in (1) does not depend explicitly on the uncertain parameter Δ and thus can also be defined for the system in Fig. 1 with *nonlinear* component ϕ. Here in this paper, we consider discrete-time feedback systems and propose the class of Lyapunov functions specified by

$$V(x_k) = \mathsf{x}_k' P \mathsf{x}_k, \qquad \mathsf{x}_k := \begin{bmatrix} x_k \ ; \ u_k \ ; \ \cdots \ ; \ u_{k+q-1} \end{bmatrix} \quad (2)$$

where x_k is the state at time k, u_k is the signal in Fig. 1 evaluated at time k, and q is a given positive integer. We call V in (2) a *generalized quadratic Lyapunov function* since V is quadratic in the auxiliary vector x but not in the original state x. Clearly, V is dependent upon either the uncertain parameters or the nonlinearity.

Based on the generalized quadratic Lyapunov function, we shall give sufficient conditions for global stability and for regional stability of the feedback system in Fig. 1. The quadratic-form model (QFM) of the "troublesome component" ϕ will play a crucial role in our analysis. The QFM of ϕ is the set of weighting matrices whose quadratic form with the vector v_k is nonnegative for all time k where v_k consists of the input-output graphs of ϕ at time instants k through $k+q$. Two specific QFMs are developed; One is for the class of nonlinear functions ϕ with a given first-order information[2]. The other is for the time-varying parametric uncertainty for which bounds on the magnitude and the rate of variation are known. We show that QFMs can be used as "multipliers" in our Lyapunov analysis, leading to stability conditions given in terms of LMIs.

II. FEEDBACK SYSTEM ANALYSIS

A. Problem formulation

Consider the discrete-time system in Fig. 1 described by

$$\begin{array}{rcl} x_{k+1} & = & Ax_k + Bu_k \\ y_k & = & Cx_k + Du_k \\ u_k & = & \phi(k, y_k) \end{array} \qquad (3)$$

where $x_k \in \mathbf{R}^n$ is the state vector, $u_k \in \mathbf{R}^p$ and $y_k \in \mathbf{R}^m$ are the signals in the feedback loop (see Fig. 1), and $\phi : \mathbf{Z} \times \mathbf{R}^m \to \mathbf{R}^p$ is a nonlinear and/or uncertain function. The following assumption is enforced throughout the paper.

Assumption 1:
(a) The function ϕ is continuous and satisfies $\phi(k, 0) = 0$ for all $k \in \mathbf{Z}$.
(b) For each $k \in \mathbf{Z}$ and $x \in \mathbf{R}^n$, there exists a unique $u \in \mathbf{R}^p$ such that $u = \phi(k, Cx + Du)$.

With this assumption, the existence and uniqueness of the solution to system equation (3) is guaranteed, and the origin is an equilibrium state. The condition (b) in Assumption 1 is often referred to as the well-posedness of the feedback connection.

To state our objective, we need the following:

Definition 1: Consider the system (3) and let A be a subset of $\mathbf{R}^n$ containing the origin. The system is said to be *globally stable* if the state trajectory x_k ($k \in \mathbf{Z}$) converges to the origin for any initial state $x_0 \in \mathbf{R}^n$. We say that the system is *regionally stable in* A if the state trajectory x_k ($k \in \mathbf{Z}$) converges to the origin whenever $x_0 \in \mathsf{A}$. Such A is called a *region of attraction.*

Given matrices A, B, C, D and certain information on the function ϕ to be specified later, our objective is to develop numerically tractable methods for analyzing global/regional stability of the system (3). In particular, we would like to give a (sufficient) condition for global stability of the system, and a characterization of the region of attraction.

The notion of global stability is sometimes too strong to ask for nonlinear systems. On the other hand, the notion of local stability is too weak to provide an engineer with a confidence since the set of initial states converging to the origin, the region of attraction, can be arbitrarily small. Hence, it is often the case in practice that the notion of regional stability, which lies in between the notions of global stability and local stability, is the right property to be questioned.

Before proceeding, let us introduce some notation. In view of Assumption 1, the vector u satisfying the relation $u = \phi(k, Cx + Du)$ is uniquely determined from k and x. We denote this mapping by $u = \varphi(k, x)$. Clearly, $\varphi(k, x) = \phi(k, Cx)$ when $D = 0$. Further introducing the nonlinear function

$$f(k, x) := Ax + B\varphi(k, x),$$

the system (3) can be described by

$$x_{k+1} = f(k, x_k), \qquad u_k = \varphi(k, x_k).$$

For integers k_0 and k_1 such that $0 \le k_0 \le k_1$, we see that the state at time k_1 is uniquely determined by the state at time k_0. Hence x_{k_1} can be expressed as $x_{k_1} = \mathsf{F}(k_0, k_1, x_{k_0})$, where the function $\mathsf{F} : \mathbf{Z} \times \mathbf{Z} \times \mathbf{R}^n \to \mathbf{R}^n$ is the "state transition mapping" and is defined by recursive applications of the function f. For instance, $\mathsf{F}(4, 7, x) = f(6, f(5, f(4, x)))$. Similarly, u_{k_1} is also a function of k_0, k_1 and x_{k_0}, and can be given by $u_{k_1} = \varphi(k_1, \mathsf{F}(k_0, k_1, x_{k_0})) =: \mathsf{G}(k_0, k_1, x_{k_0})$.

B. Generalized quadratic Lyapunov function

We use the Lyapunov function candidate described in Section I for our stability analysis. A precise definition of the class of such Lyapunov functions is as follows:

$$V(k, x) := \mathsf{x}' P \mathsf{x}, \qquad \mathsf{x} := T_k(x) \qquad (4)$$

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[1] The existence and the uniqueness of such u is guaranteed by the well-posedness $\det(I - \Delta D) \neq 0$ of the feedback system in Fig. 1.

[2] The slope-restricted nonlinearity is an example of such ϕ.

where P is a fixed symmetric matrix and $T_k(x)$ is a function defined by

$$T_k(x) := \left[\; x \; ; \; u_0 \; ; \; \cdots \; ; \; u_{q-1} \; \right],$$

$$u_i := \mathsf{G}(k, k+i, x), \qquad (i = 0, \ldots, q-1)$$

for a given integer $q \geq 1$. The function V is well defined if the system (3) is well posed. Clearly, $V(\cdot, 0) = 0$ holds due to the assumption $\phi(\cdot, 0) = 0$. Below, we shall develop a related auxiliary system, that has $\mathsf{x}_k := T_k(x_k)$ as its state.

Consider the system (3) and let x_k, u_k and y_k be any signals that satisfy the system equations. Then, it can be shown that the signals $\mathsf{x}_k \in \mathbf{R}^{n_q}$, $\mathsf{w}_k \in \mathbf{R}^p$, and $\mathsf{v}_k \in \mathbf{R}^{\ell_q}$, where $n_q := n + qp$, $\ell_q := (m+p)(q+1)$, defined by

$$\begin{bmatrix} \mathsf{x}_k \\ \mathsf{w}_k \end{bmatrix} := \begin{bmatrix} x_k \\ \mathsf{u}_k \end{bmatrix}, \quad \mathsf{v}_k := \begin{bmatrix} \mathsf{y}_k \\ \mathsf{u}_k \end{bmatrix},$$

$$\mathsf{u}_k := \left[\; u_k \; ; \; \cdots \; ; \; u_{k+q} \; \right], \quad \mathsf{y}_k := \left[\; y_k \; ; \; \cdots \; ; \; y_{k+q} \; \right]$$

satisfy

$$\begin{array}{lcl} \mathsf{x}_{k+1} & = & \mathcal{A}\mathsf{x}_k + \mathcal{B}\mathsf{w}_k \\ \mathsf{v}_k & = & \mathcal{C}\mathsf{x}_k + \mathcal{D}\mathsf{w}_k \end{array} \tag{5}$$

where

$$\mathcal{A} := \begin{bmatrix} A & B & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{B} := \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad \mathcal{C} := \begin{bmatrix} \mathcal{C}_y \\ \mathcal{C}_u \end{bmatrix} \tag{6}$$

$$\mathsf{C} := \left[\; C \quad D \quad 0 \; \right], \qquad \mathcal{D} := \begin{bmatrix} \mathcal{D}_y \\ \mathcal{D}_u \end{bmatrix} \tag{7}$$

$$\mathcal{C}_y := \begin{bmatrix} \mathsf{C} \\ \mathsf{C}\mathcal{A} \\ \vdots \\ \mathsf{C}\mathcal{A}^q \end{bmatrix}, \qquad \mathcal{D}_y := \begin{bmatrix} 0 \\ D \end{bmatrix}, \tag{8}$$

$$\mathcal{C}_u := \begin{bmatrix} 0 & I_{pq} \\ 0 & 0 \end{bmatrix}, \qquad \mathcal{D}_u := \begin{bmatrix} 0 \\ I_p \end{bmatrix}. \tag{9}$$

The above equations are obtained by recursive applications of the first two equations in (3) and noting the following identities:

$$\mathsf{C}\mathcal{A}^k\mathcal{B} = 0 \quad (k = 0, \ldots, q-2), \qquad \mathsf{C}\mathcal{A}^{q-1}\mathcal{B} = D.$$

The third equation $u_k = \phi(k, y_k)$ in (3) imposes a static constraint on the signal v_k as follows. Let $\boldsymbol{\phi} : \mathbf{Z} \times \mathbf{R}^{m(q+1)} \to \mathbf{R}^{p(q+1)}$ be the function consisting of ϕ's on the diagonal; more precisely, $\mathsf{u} = \boldsymbol{\phi}(k, \mathsf{y})$ means

$$u_i = \phi(k+i, y_i) \;\; (i = 0, \ldots, q),$$

$$\left[\; u_0 \; ; \; \cdots \; ; \; u_q \; \right] := \mathsf{u}, \quad \left[\; y_0 \; ; \; \cdots \; ; \; y_q \; \right] := \mathsf{y}.$$

Then it can be verified that the signal v_k is constrained to be a graph of the function $\boldsymbol{\phi}$ at time k;

$$\mathsf{v}_k \in \mathbf{G}_k(\phi) \tag{10}$$

where the set $\mathbf{G}_k(\phi)$ is defined by

$$\mathbf{G}_k(\phi) := \left\{ \begin{bmatrix} \mathsf{y} \\ \boldsymbol{\phi}(k, \mathsf{y}) \end{bmatrix} \in \mathbf{R}^{\ell_q} \; : \; \mathsf{y} \in \mathbf{R}^{m(q+1)} \right\}. \tag{11}$$

By construction, any solutions x_k, u_k and y_k of dynamical equations (3) satisfy equations (5) and constraint (10) for appropriately defined signals x_k, w_k and v_k. The converse statement also holds true as shown below.

Consider the auxiliary system described by (5) and (10). The system has an algebraic constraint (10) and hence there may be no solution for some initial state x_0. From the construction above, we see that the initial state is constrained to satisfy $\mathsf{x}_0 \in \mathbf{T}_0(\phi)$ where

$$\mathbf{T}_k(\phi) := \{ \; T_k(x) \in \mathbf{R}^{n_q} \; : \; x \in \mathbf{R}^n \; \}$$

if equations (5) and (10) are to generate a trajectory of the original system (3). The following lemma provides existence and uniqueness of the trajectory of the auxiliary system starting from $\mathsf{x}_0 \in \mathbf{T}_0(\phi)$.

Lemma 1: Consider the set $\mathbf{G}_k(\phi)$ in (11) and the augmented matrices in (6)–(9). Suppose that Assumption 1 holds. Then the following statements hold true:
(a) Given $\mathsf{x} \in \mathbf{R}^{n_q}$ and $k \in \mathbf{Z}$, there exists a unique vector $\mathsf{w} \in \mathbf{R}^p$ such that $\mathcal{C}\mathsf{x} + \mathcal{D}\mathsf{w} \in \mathbf{G}_k(\phi)$ if and only if $\mathsf{x} \in \mathbf{T}_k(\phi)$ holds.
(b) Given $\mathsf{x} \in \mathbf{R}^{n_q}$, $\mathsf{w} \in \mathbf{R}^p$ and $k \in \mathbf{Z}$, suppose $\mathcal{C}\mathsf{x} + \mathcal{D}\mathsf{w} \in \mathbf{G}_k(\phi)$. Then $\mathcal{A}\mathsf{x} + \mathcal{B}\mathsf{w} \in \mathbf{T}_{k+1}(\phi)$.
It follows from this lemma that, if $\mathsf{x}_0 \in \mathbf{T}_0$, then the resulting trajectory is unique and satisfies $\mathsf{x}_k \in \mathbf{T}_k$ for all $k \in \mathbf{Z}$. Therefore, for each x_k, there corresponds a vector x_k such that $\mathsf{x}_k = T_k(x_k)$. It can readily be verified that this x_k will be a solution to the original system (3).

Note from Lemma 1 that there exists a unique vector w such that $\mathcal{C}\mathsf{x} + \mathcal{D}\mathsf{w} \in \mathbf{G}_k(\phi)$ for each $k \in \mathbf{Z}$ and $\mathsf{x} \in \mathbf{T}_k(\phi)$. Let us denote such w by $\boldsymbol{\varphi}(k, \mathsf{x})$. That is,

$$\mathsf{w} = \boldsymbol{\varphi}(k, \mathsf{x}) \quad \Leftrightarrow \quad \mathcal{C}\mathsf{x} + \mathcal{D}\mathsf{w} \in \mathbf{G}_k(\phi).$$

Then the auxiliary system can be identified as a nonlinear time-varying system

$$\mathsf{x}_{k+1} = \mathsf{f}(k, \mathsf{x}_k), \qquad \mathsf{f}(k, \mathsf{x}) := \mathcal{A}\mathsf{x} + \mathcal{B}\boldsymbol{\varphi}(k, \mathsf{x}) \tag{12}$$

with algebraic constraints $\mathsf{x}_k \in \mathbf{T}_k(\phi)$ for all $k \in \mathbf{Z}$. As noted above, if the initial state satisfies the algebraic constraint, i.e. $\mathsf{x}_0 \in \mathbf{T}_0(\phi)$, then so does the state x_k for all $k \in \mathbf{Z}$.

In summary, the auxiliary system described by (5) and (10) is equivalent to the original system (3) in the sense that one generates a trajectory of the other. Thus the global/regional stability of the original system can be examined by studying the stability property of the auxiliary system. Moreover, the Lyapunov function in (4) is quadratic in the state x_k of the auxiliary system, although it is not quadratic in terms of the original state x_k. This fact enables us to consider the non-quadratic Lyapunov function using tools from the quadratic Lyapunov function analysis.

C. Global stability analysis

In this section, we shall give a sufficient condition for the existence of a Lyapunov function of the form (4) that proves global stability of the feedback system (3). As noted in the previous section, stability of the original system (3) is equivalent to stability of the auxiliary system described by (5) and (10), and a quadratic Lyapunov function for the auxiliary system will act as a (non-quadratic) Lyapunov function for the original system proving its stability.

To this end, let us define the following sets:

$$\mathbf{T}(\phi) := \bigcup_{k\in\mathbf{Z}} \mathbf{T}_k(\phi), \qquad \mathbf{G}(\phi) := \bigcup_{k\in\mathbf{Z}} \mathbf{G}_k(\phi),$$

$$\boldsymbol{\Theta}(\phi) := \left\{ \Theta = \Theta' \in \mathbf{R}^{\ell_q\times\ell_q} : \mathsf{v}'\Theta\mathsf{v} \geq 0, {}^\forall\, \mathsf{v} \in \mathbf{G}(\phi) \right\}. \quad (13)$$

Notice that, given any $\Theta \in \boldsymbol{\Theta}(\phi)$ and v_k in (10), the quadratic form $\mathsf{v}_k\Theta\mathsf{v}_k$ is nonnegative for all $k \in \mathbf{Z}$. Thus the set $\boldsymbol{\Theta}(\phi)$ may be considered as a "model" of the nonlinear/uncertain function ϕ in the spirit of IQC analysis [19]. We shall call the set $\boldsymbol{\Theta}(\phi)$ a quadratic-form model (QFM) of ϕ.

Theorem 1: Let an integer $q \geq 1$ be given. Consider the system (3) and define the set $\boldsymbol{\Theta}(\phi)$ as in (13) and augmented system matrices as in (6)–(9). Suppose that Assumption 1 holds and there exist symmetric matrices P and $\Psi, \Phi \in \boldsymbol{\Theta}(\phi)$ such that

$$\begin{bmatrix} \mathcal{C} & \mathcal{D} \\ I & 0 \end{bmatrix}' \begin{bmatrix} \Psi & \\ & -P \end{bmatrix} \begin{bmatrix} \mathcal{C} & \mathcal{D} \\ I & 0 \end{bmatrix} < 0 \quad (14)$$

$$\begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \\ I & 0 \end{bmatrix}' \begin{bmatrix} P & & \\ & \Phi & \\ & & -P \end{bmatrix} \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \\ I & 0 \end{bmatrix} < 0 \quad (15)$$

Then the system is globally stable. Moreover, $V(k, x)$ defined by (4) is a Lyapunov function that proves stability.

D. Regional stability analysis

In this section, we give a characterization of the region of attraction for the system (3). We consider for simplicity the case where the function ϕ in (3) is time-invariant. In order to perform regional stability analysis, we shall restrict our attention to the class of Lyapunov functions[3] $V(x)$ given in (4). As in the previous section, we consider the auxiliary system described by (5) and (10). Then the analysis problem reduces to the search for a quadratic Lyapunov function $\mathsf{V}(\mathsf{x}) = \mathsf{x}'P\mathsf{x}$ that is positive definite and monotonically decreasing along each state trajectory $\mathsf{x}_k \in \mathbf{T}_k(\phi)$ of the auxiliary system in some region $\mathbf{A}$ of the state space.

The following lemma provides a basis for our regional stability analysis. The essence of this approach has already appeared for example in [23], [11], [22].

Lemma 2: Consider a nonlinear system $\mathsf{x}_{k+1} = \mathsf{f}(\mathsf{x}_k)$ where $\mathsf{f} : \mathbf{R}^{\mathsf{n}} \to \mathbf{R}^{\mathsf{n}}$ is a continuous function satisfying $\mathsf{f}(0) = 0$. Let $\mathbf{X}$ be a subset of the state space $\mathbf{X} \subseteq \mathbf{R}^{\mathsf{n}}$. If there exists a function $\mathsf{V} : \mathbf{R}^{\mathsf{n}} \to \mathbf{R}$ such that

$$\mathsf{V}(\mathsf{x}) > 0, \quad {}^\forall \text{ nonzero } \mathsf{x} \in \mathbf{X}, \quad \text{and} \quad \mathsf{V}(0) = 0 \quad (16)$$

$$\mathsf{V}(\mathsf{f}(\mathsf{x})) - \mathsf{V}(\mathsf{x}) < 0, \quad {}^\forall \text{ nonzero } \mathsf{x} \in \mathbf{X} \quad (17)$$

$$\mathbf{A} \subseteq \mathbf{X} \quad (18)$$

where

$$\mathbf{A} := \{\, \mathsf{x} \in \mathbf{R}^{\mathsf{n}} : \ \mathsf{V}(\mathsf{x}) \leq 1 \,\}, \quad (19)$$

then $\mathbf{A}$ is a region of attraction, i.e., x_k approaches the origin as $k \to \infty$ whenever $\mathsf{x}_0 \in \mathbf{A}$.

The first step for the analysis is to choose the set $\mathbf{X}$. A possible criterion for the choice is that $\mathbf{X}$ allows for regional modeling of the nonlinearity $\boldsymbol{\phi}$. Suppose that the function ϕ (or $\boldsymbol{\phi}$) can be modeled, in a certain region $\mathbf{Y} \subseteq \mathbf{R}^{m(q+1)}$, as follows:

$$\mathbf{G}^{\mathsf{Y}}(\phi) := \left\{ \begin{bmatrix} \mathsf{y} \\ \boldsymbol{\phi}(\mathsf{y}) \end{bmatrix} \in \mathbf{R}^{\ell_q} : \ \mathsf{y} \in \mathbf{Y} \right\}$$

$$\boldsymbol{\Theta}^{\mathsf{Y}}(\phi) := \left\{ \Theta = \Theta' \in \mathbf{R}^{\ell_q\times\ell_q} : \mathsf{v}'\Theta\mathsf{v} \geq 0, {}^\forall\, \mathsf{v} \in \mathbf{G}^{\mathsf{Y}}(\phi) \right\}. \quad (20)$$

If we choose the set $\mathbf{X}$ as

$$\mathbf{X} := \{\, \mathsf{x} \in \mathbf{R}^{n_q} : \ {}^\exists \mathsf{w} \in \mathbf{R}^{p} \text{ s.t. } \mathcal{C}\mathsf{x} + \mathcal{D}\mathsf{w} \in \mathbf{G}^{\mathsf{Y}}(\phi) \,\},$$

then conditions (16) and (17) can be characterized as in Theorem 1 by replacing $\boldsymbol{\Theta}$ with $\boldsymbol{\Theta}^{\mathsf{Y}}$. In general, the set $\boldsymbol{\Theta}^{\mathsf{Y}}(\phi)$ is larger than $\boldsymbol{\Theta}(\phi)$ in the previous section due to the restriction of the "operating region" of the function ϕ. Hence, if we seek $\Psi, \Phi \in \boldsymbol{\Theta}^{\mathsf{Y}}(\phi)$ in (14) and (15), then the conditions become less restrictive.

Below, we consider the following set for $\mathbf{Y}$ among others:

$$\mathbf{Y} := \{\, \mathsf{y} \in \mathbf{R}^{m(q+1)} : \ \mathsf{v} := [\ \mathsf{y}\ ;\ \boldsymbol{\phi}(\mathsf{y})\ ],$$
$$\begin{bmatrix} \mathsf{v} \\ 1 \end{bmatrix}' L_i \begin{bmatrix} \mathsf{v} \\ 1 \end{bmatrix} \geq 0, \quad {}^\forall i = 1, \ldots, r \,\}$$

where $L_i \in \mathbf{R}^{(\ell_q+1)\times(\ell_q+1)}$ are given symmetric matrices. Partition L_i as

$$L_i = \begin{bmatrix} U_i & v_i \\ v_i' & w_i \end{bmatrix}$$

where w_i is a scalar. Considering the additional condition (18), we have the following regional stability result.

Theorem 2: Let an integer $q \geq 1$ be given. Consider the system (3) and define the set $\boldsymbol{\Theta}^{\mathsf{Y}}(\phi)$ as in (20) and augmented system matrices as in (6)–(9). Suppose that Assumption 1 holds and there exist positive scalars τ_i, σ_i symmetric matrices $\Psi, \Phi \in \boldsymbol{\Theta}^{\mathsf{Y}}(\phi)$, $\Omega_i \in \boldsymbol{\Theta}(\phi)$ and P satisfying (14), (15), and

$$\begin{bmatrix} \mathcal{C} & \mathcal{D} \\ I & 0 \end{bmatrix}' \begin{bmatrix} Z_i & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} \mathcal{C} & \mathcal{D} \\ I & 0 \end{bmatrix} > 0$$

$$\begin{bmatrix} \tau_i w_i - 1 & \tau_i \\ \tau_i & \sigma_i \end{bmatrix} > 0, \qquad Z_i := \tau_i U_i - \sigma_i v_i v_i' - \Omega_i$$

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[3] With a slight abuse of notation, we omit the first argument k in V since V is not an explicit function of the time k due to the time-invariance of ϕ. We use similar notation for $\boldsymbol{\phi}$, $\mathbf{T}_k(\phi)$, etc.

for all $i = 1, \ldots, r$. Then the system is regionally stable in A where

$$\mathsf{A} := \{\ x \in \mathbf{R}^n :\ V(x) \leq 1\ \}$$

with $V(x)$ given by (4).

Theorem 2 characterizes a region of attraction for system (3) in terms of LMIs. An estimate of the maximal region of attraction may be found by maximizing the "volume" of the set A subject to the conditions in Theorem 2. However, the set A is not an ellipsoid in $\mathbf{R}^n$ and it is difficult to measure its volume in a computationally tractable manner. One thing we could do is to maximize the volume of an ellipsoid that is contained in A. Let us explain this point in more detail.

Let a symmetric positive definite matrix $Q = Q' \in \mathbf{R}^n$ be given and consider the ellipsoid defined by $x'Qx \leq 1$. This ellipsoid is contained in A if and only if

$$\mathsf{x}'P\mathsf{x} \leq 1, \quad {}^{\forall}\mathsf{x} = T(x), \quad x'Qx \leq 1,$$

or equivalently,

$$\mathsf{x}'P\mathsf{x} \leq 1, \quad {}^{\forall}\mathsf{x},\ \mathsf{w} \text{ s.t. } \mathcal{C}\mathsf{x} + \mathcal{D}\mathsf{w} \in \mathbf{G}(\phi), \quad \mathsf{x}'J'QJ\mathsf{x} \leq 1$$

where $J := [\ I_n \quad 0\ ] \in \mathbf{R}^{n \times n_q}$. Then, using the S-procedure, it can be shown that a sufficient condition for this is given by the existence of $\mho \in \mathbf{\Theta}(\phi)$ such that

$$\begin{bmatrix} \mathcal{C} & \mathcal{D} \\ I & 0 \end{bmatrix}' \begin{bmatrix} \mho & 0 \\ 0 & P - J'QJ \end{bmatrix} \begin{bmatrix} \mathcal{C} & \mathcal{D} \\ I & 0 \end{bmatrix} < 0. \qquad (21)$$

Thus, approximating the volume of the ellipsoid $x'Qx \leq 1$ by $\mathbf{tr}(Q)$, an estimate of the maximal region of attraction can be found by minimizing $\mathbf{tr}(Q)$ subject to (14), (15), (21), and the conditions in Theorem 2 over the variables τ_i, σ_i, $\Psi, \Phi \in \mathbf{\Theta}^{\Upsilon}(\phi)$, $\Omega_i, \mho \in \mathbf{\Theta}(\phi)$, P, and $Q > 0$. This is an eigenvalue problem [3] and can be solved efficiently via interior point methods [20].

III. SPECIFIC QUADRATIC-FORM MODELS

In this section, we consider some specific functions ϕ and provide QFMs for them. It is difficult to have an exact characterization of the set $\mathbf{\Theta}(\phi)$ in a numerically tractable manner. Here, we try to develop, at the expense of conservatism, tractable QFMs that are suited for numerical computations involving LMIs. In particular, we show some inner approximations $\mathbf{\Phi}$ of the exact set $\mathbf{\Theta}(\phi)$. Replacing $\mathbf{\Theta}(\phi)$ by $\mathbf{\Phi}$ in the stability result (Theorem 1), we will have a computationally tractable sufficient condition for global stability. A similar comment applies to the regional stability result of Theorem 2.

Consider the function $\phi : \mathbf{Z} \times \mathbf{R}^m \to \mathbf{R}^p$ consisting of several functions ϕ_i $(i = 1, \ldots, \ell)$ as follows:

$$\phi = \mathbf{diag}\ (\phi_1, \ldots, \phi_\ell)$$

where $\phi_i : \mathbf{Z} \times \mathbf{R}^{m_i} \to \mathbf{R}^{p_i}$ and $\sum_{i=1}^{\ell} m_i = m$ and $\sum_{i=1}^{\ell} p_i = p$. A QFM of such function can be obtained by diagonal compositions once we have a QFM for each ϕ_i. Hence, we consider in the sequel the cases where ϕ is given by a single function.

A. Repeated static nonlinearity

Consider the class of nonlinear, time-invariant functions $\phi : \mathbf{R} \to \mathbf{R}$ satisfying

$$\begin{bmatrix} \xi \\ \zeta \\ \phi(\xi) \\ \phi(\zeta) \end{bmatrix}' \begin{bmatrix} x_{11} & x_{12} & y_{11} & y_{12} \\ x_{12} & x_{22} & y_{21} & y_{22} \\ y_{11} & y_{21} & z_{11} & z_{12} \\ y_{12} & y_{22} & z_{12} & z_{22} \end{bmatrix} \begin{bmatrix} \xi \\ \zeta \\ \phi(\xi) \\ \phi(\zeta) \end{bmatrix} \geq 0 \qquad (22)$$

for all $\xi, \zeta \in \mathbf{R}$ where x_{ij}, y_{ij} and z_{ij} are given scalars. This class of functions includes several important classes of nonlinearities that have been studied in the literature. Some examples are summarized in the following table.

Nonlinearity	Condition
Sector-bounded	$(\phi(\xi) - \alpha\xi)(\phi(\xi) - \beta\xi) \leq 0$
Gain-bounded	$\lvert\phi(\xi)\rvert \leq \gamma\lvert\xi\rvert$
Slope-restricted	$\sigma \leq \frac{\phi(\xi)-\phi(\zeta)}{\xi-\zeta} \leq \rho$
Lipschitz	$\lvert\phi(\xi) - \phi(\zeta)\rvert \leq \mu\lvert\xi - \zeta\rvert$

In the sequel, we consider the class of nonlinear time-invariant functions ϕ satisfying (22). It is also assumed that ϕ crosses the origin, i.e. $\phi(0) = 0$, and is possibly an odd function, i.e. $\phi(-\eta) = -\phi(\eta)$ for all $\eta \in \mathbf{R}$. Our objective here is to find a QFM for ϕI_m where m is a given positive integer. Below, we consider the special case where $x_{11} = x_{22}$, $y_{11} = y_{22}$, and $z_{11} = z_{22}$ for brevity. The case without this assumption can also be treated in a similar manner.

It turns out that applications of S–procedures lead to a QFM of ϕI_m characterized by following the set of diagonally dominant matrices:

$$\mathbf{S} := \{\ S \in \mathbf{R}^{r \times r} :\ s_{ii} \geq \sum_{j \neq i} \lvert s_{ij}\rvert, \qquad (23)$$

$$s_{ij} = s_{ji}, \quad (i, j = 1, \ldots, r)\}.$$

This class of matrices has been used [4], [5], [17] to define a class of multipliers for the analysis of systems with repeated nonlinearities. In this regard, our result can be considered as a generalization of these previous results.

To state the result, let us define

$$Y := \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{12} \\ y_{21} & y_{11} & \ddots & \vdots \\ \vdots & \ddots & \ddots & y_{12} \\ y_{21} & \cdots & y_{21} & y_{11} \end{bmatrix}, \qquad (24)$$

and X and Z similarly.

Theorem 3: Consider the static time-invariant nonlinearity ϕ that satisfies (22) with $x_{11} = x_{22}$, $y_{11} = y_{22}$, and $z_{11} = z_{22}$, and the set $\mathbf{\Theta}(\phi I_m)$ given by (13). Define Y by (24) and X and Z similarly. Let $\mathbf{S}$ be the set of diagonally dominant matrices given by (23) with $r := m(q + 1)$.

If ϕ is odd and crosses the origin, then

$$\mathbf{\Phi}_o := \{\ \begin{bmatrix} S \cdot X & S \cdot Y \\ S \cdot Y' & S \cdot Z \end{bmatrix} :\ S \in \mathbf{S}\ \}$$

is such that $\boldsymbol{\Phi}_o \subseteq \boldsymbol{\Theta}(\phi I_m)$.

If ϕ crosses the origin, then

$$\boldsymbol{\Phi} := \left\{ \begin{bmatrix} S{\cdot}X & S{\cdot}Y \\ S{\cdot}Y' & S{\cdot}Z \end{bmatrix} : \ S \in \mathbf{S} \cap \mathbf{P} \right\}$$

is such that $\boldsymbol{\Phi} \subseteq \boldsymbol{\Theta}(\phi I_m)$, where $\mathbf{P}$ is the set of symmetric matrices with positive entries.

B. *Time-varying parametric uncertainty*

Consider a static, linear, time-varying, uncertain function $\phi : \mathbf{R} \to \mathbf{R}$ given by $\phi(k,\xi) = \delta_k \xi$ for all $k \in \mathbf{Z}$ and $\xi \in \mathbf{R}$ where $\delta_k \in \mathbf{R}$ is an uncertain time-varying parameter satisfying

$$|\delta_k| \le \gamma, \qquad |\delta_{k+1} - \delta_k| \le \rho \tag{25}$$

for all $k \in \mathbf{Z}$. In this section, we develop a QFM for ϕI_m. We first consider the simple case where the parameter q in Section II-B is equal to one, and then generalize the result to the case $q > 1$.

Lemma 3: Consider the time-varying parametric uncertainty $\phi(k,\xi) = \delta_k \xi$ that satisfies (25) for all $k \in \mathbf{Z}$. Define the set $\boldsymbol{\Theta}(\phi I_m)$ by (13) for $q = 1$. Let

$$\boldsymbol{\Psi}_1 := \left\{ \begin{bmatrix} \gamma^2 D + \rho R & G \\ G' & -D \end{bmatrix} : \begin{array}{l} D \ge S \ge 0, \\ G + G' = 0, \end{array} \right.$$
$$\left. \begin{array}{l} R \ge \gamma(Q - S), \\ R_{12} = 0,\ S_{12} = 0, \end{array} \quad Q := \begin{bmatrix} D_{11} & G_{12} \\ G_{12}' & D_{22} \end{bmatrix} \right\}$$

where $D_{ij}, G_{ij}, R_{ij}, S_{ij} \in \mathbf{R}^{m \times m}$ $(i,j = 1,2)$ are partitioned block matrices of D, G, R and S, respectively. Then $\boldsymbol{\Psi}_1 \subseteq \boldsymbol{\Theta}(\phi I_m)$.

The above QFM reduces to the standard *D-G* scaling when the rate of parameter variation ρ is either zero or infinity. If $\rho = 0$, then $S = 0$ and $R = \mu I$ with sufficiently large $\mu > 0$ can be chosen without loss of generality, and all the constraints that define $\boldsymbol{\Psi}_1$ reduce to $D \ge 0$ and $G + G' = 0$. Thus we have the standard *D-G* scaling for a single uncertain parameter. On the other hand, when ρ approaches infinity, R must approach zero and we have $D \ge S \ge Q$ in the limit, implying that $G_{12} = 0$ and $D_{12} = 0$. Thus we have the standard *D-G* scaling for two independent uncertain parameters.

We now consider the case where $q > 1$. To state the result, we need to define, for each $i = 1, \ldots, q$, a mapping $E_i : \mathbf{R}^{2m \times 2m} \to \mathbf{R}^{m(q+1) \times m(q+1)}$ as follows: For a matrix U, $E_i(U)$ is a matrix given by $E_i(U) = \mathbf{diag}\ (0, U, 0)$ where 0 on the left of U is a square matrix of dimension $(i-1)m$ while 0 on the right is a square matrix of dimension $(q-i)m$.

Theorem 4: Consider the time-varying parametric uncertainty $\phi(k,\xi) = \delta_k \xi$ that satisfies (25) for all $k \in \mathbf{Z}$. Define the set $\boldsymbol{\Theta}(\phi)$ by (13). Let

$$\boldsymbol{\Psi} = \left\{ \sum_{i=1}^{q} \begin{bmatrix} E_i(U_i) & E_i(V_i) \\ E_i(V_i)' & E_i(W_i) \end{bmatrix} : \begin{bmatrix} U_i & V_i \\ V_i' & W_i \end{bmatrix} \in \boldsymbol{\Psi}_1 \right\}$$

where $\boldsymbol{\Psi}_1$ is defined in Lemma 3. Then $\boldsymbol{\Psi} \subseteq \boldsymbol{\Theta}(\phi)$.

This QFM has been obtained by "superposing" the previous result. For example, if $q = 3$, using

$$U_i := \begin{bmatrix} R_i & S_i \\ S_i' & Q_i \end{bmatrix}, \qquad (i = 1, 2, 3),$$

we have

$$\sum_{i=1}^{3} E_i(U_i) = \begin{bmatrix} R_1 & S_1 & 0 & 0 \\ S_1' & Q_1 + R_2 & S_2 & 0 \\ 0 & S_2' & Q_2 + R_3 & S_3 \\ 0 & 0 & S_3' & Q_3 \end{bmatrix}.$$

IV. CONCLUSION

We have considered the class of nonlinear/uncertain systems described by the feedback connection of a linear time-invariant system and a nonlinear/uncertain component. The generalized quadratic Lyapunov function is proposed for stability analysis of such systems. We have shown that a quadratic-form model of the nonlinear/uncertain component can be effectively utilized to obtain sufficient conditions for the existence of such Lyapunov functions that proves global/regional stability. The conditions are given in terms of linear matrix inequalities that can be numerically verified in polynomial time.

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