OPTIMAL CONTROL OF NONLINEAR DIFFERENTIAL ALGEBRAIC EQUATION SYSTEMS

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Abstract

A novel iterative procedure is described for solving non-linear optimal control problems subject to differential algebraic equations. The procedure iterates on an integrated modified linear quadratic model based problem with parameter updating in such a manner that the correct solution of the original non-linear problem is achieved. The resulting algorithm has a particular advantage in that the solution is achieved without the need to solve the differential algebraic equations . Convergence aspects are discussed and a simulation example is described which illustrates the performance of the technique.

1. Introduction

When modelling industrial processes often the resulting equations consist of coupled differential and algebraic equations (DAEs). In many situations these equations are nonlinear and cannot readily be directly reduced to ordinary differential equations. In chemical process modelling, for example, a common situation where such DAEs arise is when relatively fast transient differential equations are approximated by quasi-steady-state algebraic equations in order to avoid numerical integration problems in simulation (see Kumar and Daoutidis [1]). High differential index DAEs, where the index is defined as the minimum number of differentiations required to obtain an equivalent ordinary differential equation [1], are particularly difficult to solve. Even index-one DAEs require particular numerical methods, such as DASSL, for their solution as described in the text books by Brenan, Campbell and Petzold [2] and Asher and Petzold [3].

A foundation for the optimal control of nonlinear DAEs has been provided by Jonckheere [4] who provided first-order necessary conditions based on a Hamiltonian characterisation. This paper also uses a variational approach to determine necessary optimality conditions which are solved in an iterative fashion. The novelty of the technique is that the iterations are performed on the basis of an appropriately modified linear quadratic model based problem which contains time dependant modifiers and parameters calculated so that the final converged solution is that of the nonlinear DAE system optimal control. The formulation is based on dynamic integrated system optimisation and parameter estimation (DISOPE) which is a technique for solving complex nonlinear optimal control problems by iterating on simplified model based representations (see Roberts [5], Becerra [6]). An advantage of the resulting algorithm is that it avoids any requirement to numerically solve the DAEs during the iterative procedure.

2. Problem Formulation

Consider the following continuous optimal control problem (OCP), defined over the time horizon $t \in [t_o, t_f]$,

$$\min_{u(t)} \left\{ \Phi^*\left(x(t_f), z(t_f), t_f\right) + \int_{t_o}^{t_f} L^*\left(x(t), z(t), u(t), t\right) dt \right\} \quad (1)$$

subject to the semi-explicit type one differential-algebraic equations:

$$\left\{ \begin{array}{l} \dot{x}(t) = f^*\left(x(t), z(t), u(t), t\right) \;\; ; \quad x(t_o) = x_o \\ g^*\left(x(t), z(t), u(t), t\right) = 0 \end{array} \right\} \qquad (2)$$

where $u(t) \in \Re^m$, $z(t) \in \Re^{n_z}$ and $x(t) \in \Re^{n_x}$ are the control, algebraic variable and state vectors respectively, $\Phi^*: \Re^{n_x} \times \Re^{n_z} \times \Re \to \Re$ is the system terminal measure, $L^*: \Re^{n_x} \times \Re^{n_z} \times \Re^m \times \Re \to \Re$ is the system performance measure function, $f^*: \Re^{n_x} \times \Re^{n_z} \times \Re^m \times \Re \to \Re^{n_x}$, represents the system state equations, and $g^*: \Re^{n_x} \times \Re^{n_z} \times \Re^m \times \Re \to \Re^{n_z}$ represents the system algebraic equations.

Because, in complex situations, OCP is difficult to solve, the following simplified model based problem (MOP) is considered

$$\min_{u(t)} \left\{ \Phi\left(x(t_f), z(t_f), t_f, \gamma_1\right)\right.$$

$$+\int_{t_o}^{t_f} L\left(x(t),z(t),u(t),t,\gamma_2(t)\right)dt\Big\} \qquad (3)$$

subject to

$$\begin{Bmatrix} \dot{x}(t)= f\left(x(t),z(t),u(t),t,\alpha_1(t)\right) \;;\quad x(t_o)=x_o \\ g\left(x(t),z(t),u(t),t,\alpha_2(t)\right)=0 \end{Bmatrix} \qquad (4)$$

where $\gamma_1 \in \Re,\ \gamma_2(t) \in \Re,\ \gamma_3 \in \Re^{q},\ \alpha_1(t) \in \Re^{q_{n_x}}$ and $\alpha_2(t) \in \Re^{q_{n_z}}$ are identifiable parameters, $\Phi: \Re^{n_x} \times \Re^{n_z} \times \Re \times \Re \rightarrow \Re$ is the model terminal measure, $L: \Re^{n_x} \times \Re^{n_z} \times \Re^{m} \times \Re \times \Re \rightarrow \Re$ is the model performance measure function, $f: \Re^{n_x} \times \Re^{n_z} \times \Re^{m} \times \Re \times \Re^{n_x} \rightarrow \Re^{n_x}$, represents the model state equations, and $g: \Re^{n_x} \times \Re^{n_z} \times \Re^{m} \times \Re \times \Re^{n_z} \rightarrow \Re^{n_z}$ represents the model algebraic equations, where it is now assumed $g_z(.) = \partial g(.)/\partial z$ is non-singular. The mathematical structure of (3) and (4) is deliberately chosen so that MOP is readily solvable using standard techniques [7]. For instance, f(.).and g(.) may be linear and L(.) and Φ(.) may be quadratic. In particular it is assumed that the algebraic variables $z(t)$ can be eliminated from (3) and (4) using $g(.) = 0$. These assumptions and simplifications are not usually possible in the generally intractable original problem OCP defined by (1), (2). The objective, however, is to obtain the solution of OCP by appropriately iterating on MOP, matching the model to the original system through the parameters γ_1, $\gamma_2(t)$, $\alpha_1(t)$, and $\alpha_2(t)$. However, simply iterating between successive solutions of parameter estimation and solving MOP will not, in general, achieve the correct optimal solution of ROP. It is necessary to properly integrate the two problems taking account of the their mutual interaction. Dynamic Integrated System Optimisation and Parameter Estimation, DISOPE, provides the means for performing this integration and producing an appropriate iterative procedure [5].

3. DISOPE Approach

The key to the DISOPE approach is initially to define the following integrated problem, equivalent to OCP, which is known as the expanded optimal control problem (EOCP):

$$\begin{aligned} \min_{u(t)} \Big\{ & \Phi\left(x(t_f),z(t_f),t_f,\gamma_1\right) \\ & + \int_{t_o}^{t_f} \Big\{ L\left(x(t),z(t),u(t),t,\gamma_2(t)\right) \\ & + \tfrac{1}{2} r_0 \Big\| g\left(x(t),z(t),u(t),t,\alpha_2(t)\right) \\ & \qquad - g^*\left(w(t),y(t),v(t),t\right) \Big\|^2 \\ & + \tfrac{1}{2} r_1 \|u(t)-v(t)\|^2 + \tfrac{1}{2} r_2 \|x(t)-w(t)\|^2 \\ & \qquad + \tfrac{1}{2} r_3 \|z(t)-y(t)\|^2 \Big\} dt \Big\} \end{aligned} \qquad (5)$$

subject to (3) and (4) together with:

$$\begin{Bmatrix} f\left(w(t),y(t),v(t),t,\alpha_1(t)\right) = f^*\left(w(t),y(t),v(t),t\right) \\ g\left(w(t),y(t),v(t),t,\alpha_2(t)\right) = g^*\left(w(t),y(t),v(t),t\right) \\ L\left(w(t),y(t),v(t),t,\gamma_2(t)\right) = L^*\left(w(t),y(t),v(t),t\right) \\ \Phi\left(w(t_f),y(t_f),t_f,\gamma_1\right) = \Phi^*\left(w(t_f),y(t_f),t_f\right) \end{Bmatrix} \qquad (6)$$

$$\begin{Bmatrix} v(t)=u(t) \\ w(t)=x(t) \\ y(t)=z(t) \end{Bmatrix} \qquad (7)$$

where $v(t) \in \Re^{m}$ $w(t) \in \Re^{n_x}$ and $y(t) \in \Re^{n_z}$ are introduced to separate the controls, algebraic variables and states in the model based optimal control problem from the respective signals in the parameter estimation problem, defined by (6), and $r_0 \in \Re$, $r_1 \in \Re$, $r_2 \in \Re$ and $r_3 \in \Re$, denoted as convexification parameters, are positive weighting parameters introduced to improve convexity and aid convergence of the resulting iterative algorithm.

Define:

$$\begin{aligned} H(.) = {} & L\left(x(t),z(t),u(t),t,\gamma_2(t)\right) \\ & + p^T(t) f\left(x(t),z(t),u(t),t,\alpha_1(t)\right) \\ & + q^T(t) g\left(x(t),z(t),u(t),t,\alpha_2(t)\right) \\ & - \lambda^T(t)u(t) - \beta^T(t)x(t) - \theta^T(t)z(t) \end{aligned} \qquad (8)$$

where $p(t) \in \Re^{n_x}$ is the co-state vector, $q(t) \in \Re^{n_z}$ is a time dependent Lagrange multiplier attached to the algebraic equations; and $\lambda(t) \in \Re^{m}$, $\beta(t) \in \Re^{n_x}$ and $\theta(t) \in \Re^{n_z}$ are modifiers. Application of first-order variational calculus and algebraic manipulation then produces the following subsets of the necessary optimality conditions:

$$\begin{Bmatrix} \nabla_u H(.) + r_1\left(u(t)-v(t)\right) + r_0 g_u(.)^T\left(g(.)-g^*(.)\right) = 0 \\ \dot{p}(t) = -\nabla_x H(.) - r_2\left(x(t)-w(t)\right) \\ \qquad - r_0 g_x(.)^T\left(g(.)-g^*(.)\right) \\ \nabla_z H(.) + r_3\left(z(t)-y(t)\right) + r_0 g_z(.)^T\left(g(.)-g^*(.)\right) = 0 \end{Bmatrix} \qquad (9)$$

$$\begin{aligned} & \left[\nabla_x \Phi(.) + \Gamma_1 - p(t_f)\right]^T \delta x(t_f) \\ & \qquad + \left[\nabla_z \Phi(.) + \Gamma_3\right]^T \delta z(t_f) = 0 \end{aligned} \qquad (10)$$

$$\begin{Bmatrix} \lambda(t) = \left(f_v(.) - f_v^*(.)\right)p(t) \\ \quad + \left(g_v(.) - g_v^*(.)\right)^T q(t) + \left(\nabla_v L(.) - \nabla_v L^*(.)\right) \\ \beta(t) = \left(f_w(.) - f_w^*(.)\right)^T p(t) \\ \quad + \left(g_w(.) - g_w^*(.)\right)^T q(t) + \left(\nabla_w L(.) - \nabla_w L^*(.)\right) \\ \theta(t) = \left(f_y(.) - f_y^*(.)\right)^T p(t) \\ \quad + \left(g_y(.) - g_y^*(.)\right)^T q(t) + \left(\nabla_y L(.) - \nabla_y L^*(.)\right) \end{Bmatrix} \qquad (11)$$

$$\begin{cases}\Gamma_1 = -\left[\nabla_w \Phi(.) - \nabla_w \Phi^*(.)\right] \\ \Gamma_2 = -\left[\nabla_y \Phi(.) - \nabla_y \Phi^*(.)\right]\end{cases} \quad (12)$$

together with a repeat of (4), (6) and (7).

From (8), it is seen that conditions (9) and (10) can be satisfied by solving the modified model based optimal control problem (MMOCP):

$$\begin{aligned}&\min_{u(t)}\Big\{\Phi\big(x(t_f), z(t_f), t_f, \gamma_1\big) + \Gamma_1 x(t_f) + \Gamma_2 z(t_f) \\ &+ \int_{t_o}^{t_f}\Big\{L\big(x(t), z(t), u(t), t, \gamma_2(t)\big) \\ &+ \tfrac{1}{2} r_0 \Big\| g\big(x(t), z(t), u(t), t, \alpha_2(t)\big) \\ &\quad - g^*\big(w(t), y(t), v(t), t\big)\Big\|^2 \\ &+ \tfrac{1}{2} r_1 \|u(t) - v(t)\|^2 + \tfrac{1}{2} r_2 \|x(t) - w(t)\|^2 \\ &+ \tfrac{1}{2} r_3 \|z(t) - y(t)\|^2 \\ &- \lambda(t)^T u(t) - \beta(t)^T x(t) - \theta(t)^T z(t)\Big\} dt\Big\} \end{aligned} \quad (13)$$

subject to (4) under specified parameters $\gamma_1,\ \gamma_2(t),\ \alpha_1(t)$, and $\alpha_2(t)$; specified modifiers $\lambda(t),\ \beta(t)$ and $\theta(t)$ and where the co-state $p(t_f)$ has to satisfy (10) under specified modifiers Γ_1 and Γ_2. Note that the function $g(.)$ in (4) can be deliberately chosen to facilitate the solution of (13), for instance by expressing the algebraic variables $z(t)$ in terms of $x(t)$ and $u(t)$ and then applying elimination. The parameters $\gamma_1,\ \gamma_2(t),\ \alpha_1(t)$ and $\alpha_2(t)$ are determined from (6) while (11) and (12) define the calculations for computing the modifiers $\lambda(t),\ \beta(t),\ \theta(t),\ \Gamma_1$ and Γ_2.

3.1 Linear Quadratic MMOCP Situation

If we deliberately choose a linear quadratic model for MOCP in (4) such that

$$\begin{cases} f(.) = Ax(t) + Bu(t) + Cz(t) + \alpha_1(t) \\ g(.) = Dx(t) + Ez(t) + \alpha_2(t) \\ L(.) = \tfrac{1}{2}\big(x(t)^T Qx(t) + u(t)^T Ru(t) + z(t)^T S_0 z(t)\big) \\ \qquad + \gamma_2(t) \\ \Phi(.) = \tfrac{1}{2}\big(x(t_f)^T S_1 x(t_f) + z(t_f)^T S_2 z(t_f)\big) + \gamma_1 \end{cases} \quad (14)$$

noting that the dependence of $g(.)$ on u has been deliberately ignored in order to simplify the problem further, then the modified model based optimal control problem becomes

$$\begin{aligned}&\min_{u(t)}\Big\{\tfrac{1}{2}\big(x(t_f)^T S_1 x(t_f) + z(t_f)^T S_2 z(t_f)\big) + \gamma_1 \\ &+ \Gamma_1 x(t_f) + \Gamma_3 z(t_f) \\ &+ \int_{t_o}^{t_f}\Big\{\tfrac{1}{2}\big(x(t)^T Qx(t) + u(t)^T Ru(t) + z(t)^T S_0 z(t)\big) \\ &+ \gamma_2(t) + \tfrac{1}{2} r_0 \big\| Dx(t) + Ez(t) + \alpha_2(t) \\ &\quad - g^*\big(w(t), y(t), v(t), t\big)\big\|^2 + \tfrac{1}{2} r_1 \|u(t) - v(t)\|^2 \\ &+ \tfrac{1}{2} r_2 \|x(t) - w(t)\|^2 + \tfrac{1}{2} r_3 \|z(t) - y(t)\|^2 \\ &- \lambda(t)^T u(t) - \beta(t)^T x(t) - \theta(t)^T z(t)\Big\} dt\Big\} \end{aligned} \quad (15)$$

$$\text{s.t.: } \begin{cases}\dot{x}(t) = Ax(t) + Bu(t) + Cz(t) + \alpha_1(t) \\ Dx(t) + Ez(t) + \alpha_2(t) = 0\end{cases} \quad (16)$$

under the co-state terminal condition

$$\begin{aligned}&\left[S_1 x(t_f) + \Gamma_1 - p(t_f)\right]^T \delta x(t_f) \\ &+ \left[S_2 z(t_f) + \Gamma_2\right]^T \delta z(t_f) = 0\end{aligned} \quad (17)$$

Application of the Maximum Principle to (15), (16) and (17) produces the two-point boundary value problem (see Lewis and Syrmos [7])

$$\begin{cases}\dot{x}(t) = \overline{A}x(t) - B\overline{R}^{-1}B^T p(t) + \overline{\alpha}(t) \\ \dot{p}(t) = -\tilde{Q}x(t) - \overline{A}^T p(t) + \tilde{\beta}(t) \\ x(t_0) = x_o\ ,\ \ p(t_f) = \overline{S}_1 x(t_f) + \overline{\Gamma}_1\end{cases} \quad (18)$$

where

$$\overline{A} = A - CE^{-1}D \quad (19)$$

$$\overline{R} = R + r_1 I_m \quad (20)$$

$$\overline{\alpha}(t) = \alpha_1(t) - CE^{-1}\alpha_2(t) + B\overline{R}^{-1}\big(\lambda(t) + r_1 v(t)\big) \quad (21)$$

$$\tilde{Q} = Q + r_2 I_{n_x} + D^T E^{-T}\big(S_0 + r_3 I_{n_z}\big)E^{-1}D \quad (22)$$

$$\begin{aligned}&\tilde{\beta}(t) = \beta(t) + r_2 w(t) \\ &-D^T E^{-T}\Big(\theta(t) + r_3 y(t) + \big(S_0 + r_3 I_{n_z}\big)E^{-1}\alpha_2(t)\Big)\end{aligned} \quad (23)$$

$$\overline{S}_1 = \big(S_1 + D^T E^{-T} S_2 E^{-1} D\big) \quad (24)$$

$$\overline{\Gamma}_1 = \Gamma_1 - D^T E^{-T}\Gamma_3 + D^T E^{-T} S_2 E^{-1}\alpha_2(t_f) \quad (25)$$

with the optimal control law

$$u(t) = \overline{R}^{-1}\big(-B^T p(t) + \lambda(t) + r_1 v(t)\big) \quad (26)$$

Applying Riccati techniques, we assume

$$p(t) = K(t)x(t) + k(t),\ K(t_f) = \overline{S}_1,\ \ k(t_f) = \overline{\Gamma}_1 \quad (27)$$

which on substituting into (18) and (26) produces the Riccati equations

$$\begin{aligned}\dot{K}(t) &= K(t)B\overline{R}^{-1}B^T K(t) - K(t)\overline{A} \\ &\quad - \overline{A}^T K(t) - \tilde{Q},\ K(t_f) = \overline{S}_1\end{aligned} \quad (28)$$

$$\begin{aligned}\dot{k}(t) &= \big(K(t)B\overline{R}^{-1}B^T - \overline{A}^T\big)k(t) \\ &\quad - K(t)\overline{\alpha}(t) + \tilde{\beta}(t),\ k(t_f) = \overline{\Gamma}_1\end{aligned} \quad (29)$$

and a combined feedback - feedforward control law

$$u(t) = G(t)x(t) + h(t) \quad (30)$$

where

$$G(t) = -\overline{R}^{-1}B^T K(t) \quad (31)$$

$$h(t) = -\overline{R}^{-1}\left[B^T k(t) + \lambda(t) + r_1 v(t)\right] \quad (32)$$

From (16), (18) and (27) we can write

$$\dot{x}(t) = \left(\overline{A} - B\overline{R}^{-1}B^T K(t)\right)x(t)$$

$$+\overline{\alpha}(t) - B\overline{R}^{-1}B^T k(t),\ x(t_0) = x_0 \qquad (33)$$

$$z(t) = -E^{-1}\left(Dx(t) + +\alpha_2(t)\right) \qquad (34)$$

Solution of (27) to (34) produces an estimate of the optimal solution of $u(t)$, $x(t)$, $z(t)$ and $p(t)$ in terms of parameters, $\alpha_1(t)$ and $\alpha_2(t)$, and modifiers, $\lambda(t)$, $\beta(t)$, $\theta(t)$, Γ_1 and Γ_2. Expressions for the parameters and modifiers are derived using (14) with (6), (11) and (12) to give

$$\left\{\begin{aligned} \alpha_1(t) &= f^*\left(w(t), y(t), v(t), t\right) \\ &\quad -Aw(t) - Bv(t) - Cy(t) \\ \alpha_2(t) &= g^*\left(w(t), y(t), v(t), t\right) \\ &\quad -Dw(t) - Ey(t) \end{aligned}\right\} \qquad (35)$$

noting that it is not necessary to compute γ_1 or $\gamma_2(t)$, and

$$\left\{\begin{aligned} \lambda(t) &= \left(B - f_v^*(w(t), y(t), v(t), t)\right)^T p(t) \\ &\quad -g_v^*(w(t), y(t), v(t), t)^T q(t) \\ &\quad +\left(Rv(t) - \nabla_v L^*(w(t), y(t), v(t), t)\right) \\ \beta(t) &= \left(A - f_w^*(w(t), y(t), v(t), t)\right)^T p(t) \\ &\quad +\left(D - g_w^*(w(t), y(t), v(t), t)\right)^T q(t) \\ &\quad +\left(Qw(t) - \nabla_w L^*(w(t), y(t), v(t), t)\right) \\ \theta(t) &= \left(C - f_y^*(w(t), y(t), v(t), t)\right)^T p(t) \\ &\quad +\left(E - g_y^*(w(t), y(t), v(t), t)\right)^T q(t) \\ &\quad +\left(S_0 y(t) - \nabla_y L^*(w(t), y(t), v(t), t)\right) \end{aligned}\right\} \qquad (36)$$

$$\left\{\begin{aligned} \Gamma_1 &= -\left[S_1 w(t_f) - \nabla_w \Phi^*(w(t_f), y(t_f), v(t_f), t_f)\right] \\ \Gamma_2 &= -\left[S_2 y(t_f) - \nabla_y \Phi^*(w(t_f), y(t_f), v(t_f), t_f)\right] \end{aligned}\right\} \qquad (37)$$

An expression for $q(t)$ is obtained from the final equation in group (9) which, using (8), (14) and (34), produces

$$q(t) = E^{-T}\left(\overline{S}_0 E^{-1}\left(Dx(t) + \alpha_2(t)\right) - C^T p(t) + \overline{\theta}(t)\right)$$

$$+r_0 g^*\left(w(t), y(t), v(t), t\right) \qquad (38)$$

4. DAE-DISOPE Algorithm

Equations (27) to (38), together with (7), provide enough information to compute the solution of (1) subject to (2). The equations are tightly coupled and require an iterative procedure to achieve the solution. There are several possible alternative algorithms which may be employed. The advantage of the particular algorithm presented here, denoted by the acronym DAE-DISOPE, is that the final solution is obtained without any requirement to integrate the non-linear DAE equations (2). The algorithm is designed to solve the non-linear differential algebraic equation optimal control problem defined by (1) and (2) by repeated solution of the modified linear quadratic optimal control problem defined by (15), (16) and (17), and is stated as follows: (*note*: superscript (i) refers to iteration i.)

Data A, B, C, D, E, Q, R, S_0, S_1, S_2, k_u, k_x, k_z, k_p, k_q, r_0, r_1, r_2, r_3, and means for computing $f^*(.)$, $g^*(.)$, $L^*(.)$, $\Phi^*(.)$ together with their required derivatives.

Step 0 *Initialisation*: Compute the matrix expressions $\overline{A}$, $\overline{R}$, $\tilde{Q}$, and $\overline{S}_1$ from (19), (20), (22) and (24) respectively. Then compute $K(t)$ from (28) and $G(t)$ from (31). Then calculate or choose a nominal solution $v^{(0)}(t)$, $w^{(0)}(t)$, $y^{(0)}(t)$ $\hat{p}^{(0)}(t)$ and $\hat{q}^{(0)}(t)$; for instance, by solving the model based optimal control problem defined by (15) and (16) with all parameters and modifiers set to zero. Set $i = 0$.

Step 1 *Parameter Estimation*: Use (35) to compute the parameters $\alpha_1^{(i)}(t)$, and $\alpha_2^{(i)}(t)$.

Step 2 *Modifier Computation*: Use (36) and (37) to compute the modifiers $\lambda^{(i)}(t)$, $\beta^{(i)}(t)$, $\theta^{(i)}(t)$, $\Gamma_1^{(i)}$ and $\Gamma_2^{(i)}$. Then compute $\overline{\alpha}^{(i)}(t)$, $\tilde{\beta}^{(i)}(t)$ and $\overline{\Gamma}_1^{(i)}$ from (21), (23) and (25), respectively.

Step 3 *Modified Optimisation*

(a) Use (29) to compute $k^{(i)}(t)$ and (32) to compute $h^{(i)}(t)$.

(b) Use (33) to compute the predicted optimal state response $x^{(i+1)}(t)$. Then compute the predicted optimal control $u^{(i+1)}(t)$ from (30), costate $p^{(i+1)}(t)$ from (27), algebraic variable $z^{(i+1)}(t)$ from (34) and Lagrange multiplier $q^{(i+1)}(t)$ from (37).

Step 4 *Update* Use the following relaxation scheme to update $v^{(i)}(t)$, $w^{(i)}(t)$, $y^{(i)}(t)$ $\hat{p}^{(i)}(t)$ and $\hat{q}^{(i)}(t)$.

$$\left\{\begin{aligned} v^{(i+1)}(t) &= v^{(i)}(t) + k_u\left(u^{(i+1)}(t) - v^{(i)}(t)\right) \\ w^{(i+1)1}(t) &= w^{(i)}(t) + k_x\left(x^{(i+1)}(t) - w^{(i)}(t)\right) \\ y^{(i+1)}(t) &= y^{(i)}(t) + k_z\left(z^{(i+1)}(t) - y^{(i)}(t)\right) \\ \hat{p}^{(i+1)}(t) &= \hat{p}^{(i)}(t) + k_p\left(p^{(i+1)}(t) - \hat{p}^{(i)}(t)\right) \\ \hat{q}^{(i+1)}(t) &= \hat{q}^{(i)}(t) + k_q\left(q^{(i+1)}(t) - \hat{q}^{(i)}(t)\right) \end{aligned}\right\} \qquad (39)$$

Then repeat from Step 1 until convergence is achieved.

5. Convergence Considerations

A full analytical analysis of the convergence properties of the DAE-DISOPE algorithm has not yet been performed. However, by comparison with previous work, it is expected that convergence performance will be enhanced if the overall problem is at least locally convex [8] [9] [10]. Furthermore, the existence of the optimal solution and convergence to it will probably require that the non-linear system descriptions (1) and (2) are Lipschitz continuous. In the practical implementation of the algorithm relaxation gains, k_u, k_x, k_z, k_p, k_q and k_p and convexification term coefficients r_0, r_1, r_2 ,and r_3 are provided in order to regulate stability and convergence. In the initial application of the algorithm to a given problem, the relaxation gains would, in general, be chosen as unity, and the convexification term coefficients set to zero; and adjusted only if convergence difficulties arise.

6. Illustrative Example

The DAE-DISOPE algorithm as described in the previous section has been implemented in MATLAB. In order to illustrate the behaviour of the algorithm a seventh - order non-linear system with a quadratic performance index and two control inputs has been simulated. There are four algebraic variables together with four algebraic equations.

The non-linear optimal control problem (OCP) is defined as:

$$\min_{u_1(t),u_2(t)} \frac{1}{2}\left\{\sum_{1}^{7} x_i^2(1) + \int_0^1\left[\sum_{i=1}^{7} x_i^2(t) + 0.1\sum_{i=1}^{2} u_i^2(t)\right]\right\} \quad (40)$$

subject to:

$$\left.\begin{cases} \dot{x}_1(t) = -5x_1(t) + 0.2x_2(t) + 0.5x_3(t) + 0.1x_5(t) \\ \qquad +0.5x_6(t) + z_1(t) \\ \dot{x}_2(t) = -2x_2(t) + 0.5x_4(t) - 0.5x_5(t) + 0.2x_6(t) \\ \qquad -0.1x_7(t) + z_2(t) \\ \dot{x}_3(t) = 0.1x_2(t) - 1.5x_3(t) + x_4(t)^2 + 0.5x_5(t) \\ \qquad +0.1x_6(t) + z_3(t) \\ \dot{x}_4(t) = 0.2x_2(t) - 0.5x_3(t) - x_4(t) + 0.2x_5(t) \\ \qquad +0.2u_1(t) \\ \dot{x}_5(t) = 0.2x_1(t) + 0.17x_3(t) - x_5(t) + x_7(t) \\ \qquad +x_5(t)x_6(t) \\ \dot{x}_6(t) = 0.1x_1(t) - 0.2x_2(t) - x_4(t) - 0.5x_6(t) \\ \dot{x}_7(t) = 0.4x_1(t) + 0.1x_2(t) - x_3(t) - 0.5x_5(t) \\ \qquad -x_7(t) + 0.1u_2(t) \end{cases}\right\} \quad (41)$$

$$\left.\begin{cases} z_1^3(t) - 2z_1^2(t)x_1(t)x_2(t) + 1.5z_1(t)x_1^2(t)x_2^2(t) \\ \qquad - 0.5x_1^3(t)x_2^3(t) = 0 \\ z_2^3(t) - 2z_2^2(t)x_1^3(t) + 1.5z_2(t)x_1^6(t) - 0.5x_1^9(t) = 0 \\ z_3(t) - 0.2x_3^2(t) - 0.2z_4(t) = 0 \\ z_4(t) - u_1(t) = 0 \end{cases}\right\} \quad (42)$$

with initial conditions $x_1(0) = 1.0$, $x_2(0) = 0.8$, $x_3(0) = 0.5$, $x_4(0) = 0.6$, $x_5(0) = 1.5$, $x_6(0) = 1.1$ and $x_7(0) = 1.2$.

Note the algebraic equations (42) have the real solution

$$\left.\begin{cases} z_1(t) = x_1(t)x_2(t) \\ z_2(t) = x_1^3(t) \\ z_3(t) = 0.2x_3^2(t) + 0.2u_1(t) \\ z_4(t) = u_1(t) \end{cases}\right\} \quad (43)$$

However, this fact is not employed in the implementation of the DAE-DISOPE algorithm. It is employed to test the accuracy of the final results by comparing them with the results obtained from a standard DISOPE implementation [5] applied to (41) and (42) where the algebraic variables are substituted using (43). In the implementation of the algorithm the matrices *A*, *B*, *C*, *D* and *E* are set as Jacobian matrices derived by linearisation of (41) and (42) about $x(t) = z(t) = u(t) = 0$. The matrices *Q*, *R* and S_1 are chosen to match (41) with S_0 and S_2 both null.

Figures 1 and 2 compare the resulting optimal control and state signals while Figure 3 shows the corresponding algebraic signals. For this example it is observed that the DAE-DISOPE algorithm produces results that are in good agreement with the DISOPE results. The DAE-DISOPE results were obtained using relaxation gain and convexification parameters k_u=0.5, k_x=1, k_z=1, k_p=1, k_q=1, r_0=0.4, r_1=0.7, r_2=0 and r_3=0. The corresponding satisfactory convergence behaviour, plotted using absolute and semi-logarithmic scales, is demonstrated in Figures 4 and 5 which show how the norms $\left\|u^{(i)}(t) - u^{(i-1)}(t)\right\|$ and $\left\|g^{(i)}(t)\right\|$ converge as the iterations proceed.

7. Conclusions

An algorithm has been described which achieves the solution of nonlinear optimal control problems subject to differential algebraic equations. A significant advantage of the technique is that there is no requirement to solve the differential algebraic equations during the iterations. The utility of the algorithm has been demonstrated successfully. The technique as described is limited to type-one differential algebraic systems and research is continuing to remove this restriction and to perform local and global convergence analysis.

References

[1] Kumar, A. and P Daoutidis, *Control of Nonlinear Differential Algebraic Equation Systems with Applications to Chemical Process*, 1999, (Chapman & Hall/CRC, London, UK)

[2] Brenan K.E., Campbell S.E. and L.R. Petzold, *Numerical Solution of Initial-Value Problems in Differential-Algebraic Equations*, 1996, (SIAM, USA)

[3] Ascher U.M. and L.R. Petzold, *Computer Methods for Ordinary Differential Equations and Differential-Algebraic Equations*, 1998, (SIAM, USA)

[4] Jonckheere E., Variational calculus for descriptor problems, *IEEE Transactions on Automatic Control*, 1988, Vol. 33, No. 5, pp. 491-495.

[5] Roberts P.D., An algorithm for optimal control of nonlinear systems with model-reality differences, *Proceedings of 12th IFAC World Congress on Automatic Control*, Sydney, Australia, 18-23 July, 1993, Vol. 8, pp 407-412.

[6] Becerra V.M., *Development and Applications of Novel Optimal Control Algorithms*, Ph.D. Thesis, 1994, City University, London.

[7] Lewis F.L. and V.L. Syrmos, *Optimal Control*,. 1995, (Wiley, New York USA)

[8] Brdys M. and P.D. Roberts, Convergence and optimality of modified two-step algorithm for integrated system optimisation and parameter estimation, *Int. J. Systems Sci*. 1987, Vol. 18, No. 7, pp 1305-1322.

[9] Brdys M., Ellis J.E. and P.D. Roberts, Augmented integrated system optimisation and parameter estimation techniques: derivation, optimality and convergence, *IEE Proceedings Pt. D*, 1987, Vol. 134, No. 3, pp 201-209.

[10] Becerra V.M. and P.D. Roberts, Dynamic integrated system optimization and parameter estimation for discrete time optimal control of nonlinear systems, *Int.J.Control*, 1996, Vol. 63, No. 2, pp257-281.

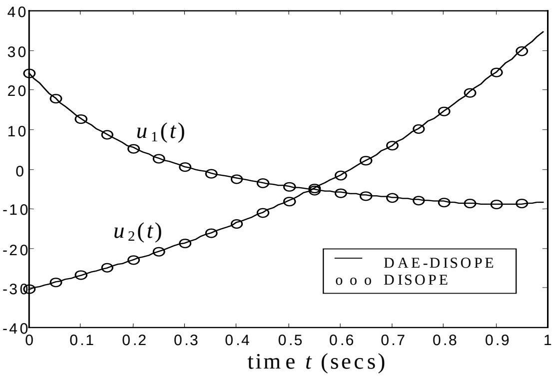

Fig.1 Optimal control signals

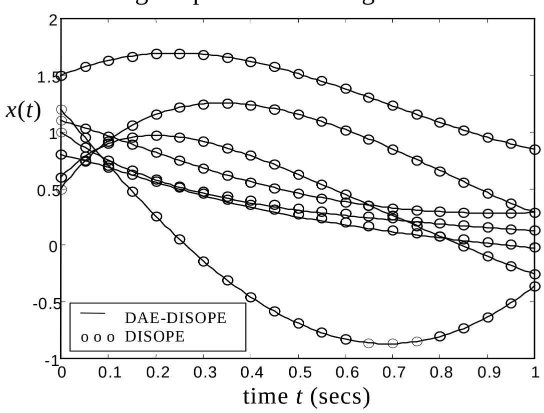

Fig.2 Optimal state signals

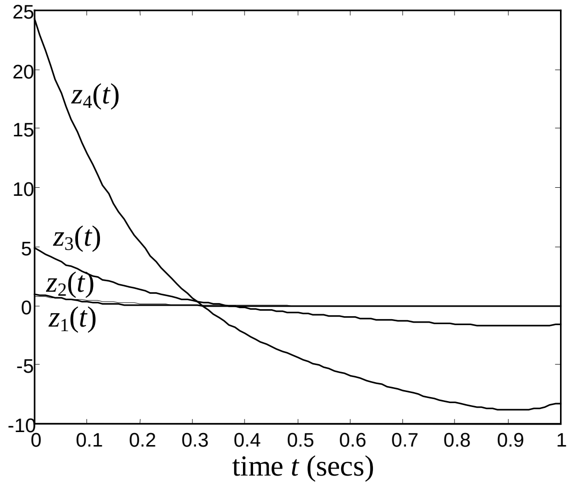

Fig.3 Optimal algebraic signals

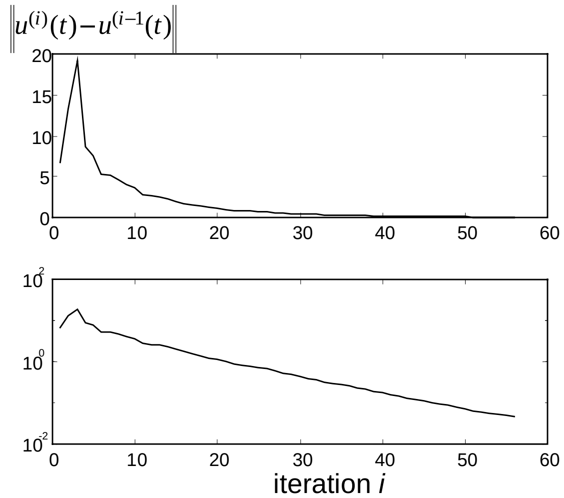

Fig.4 Control signal convergence

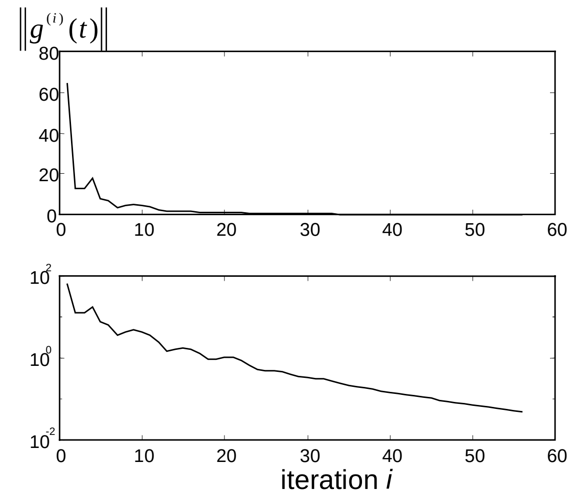

Fig.5 Algebraic constraints convergence