

# Uniqueness of solutions for control systems with degenerate Lagrangians

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## Abstract

We outline a method for dealing with controlled mechanical systems with degenerate Lagrangians, or singular optimal control problems with degenerate cost functions. We show how consistent equations of motion can be obtained despite the – implicit – constraint of degeneracy, and state conditions on which the system exhibits unique solutions (on a reduced phase space). This method is adopted from results in classical mechanics, viz. the Gotay-Nester-Hinds algorithm for the resolution of constraints in presymplectic systems. We illustrate with an example from singular linear optimal control theory.

## 1 Introduction

In most of the work on Hamiltonian systems with constraints, the assumption is made that the Lagrangian of the system is nondegenerate in the velocities or, for control systems, in the input parameters (cfr. [1]). This assumption renders the Legendre transformation between the Lagrangian and the Hamiltonian framework nondegenerate respectively enables one to select a unique input  $u(\cdot)$ .

When the Lagrangian is degenerate, it is still possible to set up a Hamiltonian description of the dynamical system; the momenta  $p$  will now no longer be independent functions of the velocities, giving rise to constraints. In the late fifties, Dirac and Bergmann [2] devised a method to obtain Hamiltonian equations of motion – on a reduced phase space defined by the constraints – that are consistent, i.e. that do not evolve off the reduced phase space.

Control theory addressed this question with the aid of the celebrated Maximum Principle [3], [4], again in a Hamiltonian setting. The question was raised [5] how consistent equations of motion could be obtained in the

Lagrangian setting.

In an effort to understand the geometry behind the Dirac-Bergmann method, Gotay, Nester and Hinds (cfr. [6, 7]) developed a new constraint resolution algorithm. Theirs is a global version of the Dirac-Bergmann algorithm and is furthermore applicable in the Lagrangian case as well. For a very general geometric setting, the reader should consult the literature on *Dirac structures* [8, 9].

In this paper we would like to outline the ideas behind the new algorithm and illustrate its use in a singular linear optimal control problem.

The paper is organised as follows: in section 2, preliminary notions on control systems and symplectic geometry are introduced; section 3 outlines the Gotay-Nester algorithm and studies uniqueness of solutions. And finally in the fourth section, the algorithm is applied to a simple optimal control problem.

## 2 Preliminary notions

### 2.1 A control system

We consider a control system to be a collection of vector fields. More precisely, let  $Q$  denote the configuration space of the system, and  $q$  a generic element of  $Q$ . We will, in this paper, assume that  $Q \cong \mathbb{R}^n$ . In each  $q$  we consider the tangent space  $T_qQ$ , and the union of these vector spaces constitutes the tangent bundle  $TQ$ . Let  $(x^i)$  denote the coordinates of  $q$  in  $\mathbb{R}^n$ , and similarly  $(\dot{x}^i)$  for coordinates in  $T_qQ$ . Further on we will consider curves  $q(t)$ , with (running) coordinates  $(x^i(t), \dot{x}^i(t))$ ,  $t \in \mathbb{R}$ ; it is understood that at time  $t$  the trajectory passes through  $q$  with tangent vector at  $q$  given – in coordinates – by  $\dot{x}^i$  at time  $t$ .

A control system then merely states which dynamics are ‘allowed’: it is a rule that restricts the system to a subset of the tangent bundle. For instance, a linear control system is defined by the rule – in local coordi-

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notes:

$$\dot{x}(t) = Ax(t) + Bu'(t), \quad A \in \mathbb{R}^{n \times n}, \quad B \in \mathbb{R}^{n \times m}. \quad (1)$$

Thus the only curves  $q(t)$  that are allowed, are those where, at each time instant  $t$  the particular tangent vector in  $T_{q(t)}Q$  satisfies (1).  $u'$  denotes the control parameter, with  $u'(\cdot) \in \mathcal{U} \subset \mathbb{R}^m$ ,  $m \leq n$ . The choice of  $u'$  enables one to tune  $m$  parameters in the expression of the components of the tangent vector to  $q(t)$  in  $T_{q(t)}Q$ .

In this paper, we will make some simplifying ‘controllability-like’ assumptions. Let  $u \in \mathbb{R}^n$  denote  $Bu'$  with zero components added where control is absent in the original system, and assume (cfr. [1]) that  $\exists u$  s.t.  $\dot{x} = Ax + u$ , and secondly that the optimal control cost function  $\hat{L}(x, u)$  can be written as a Lagrangian  $L(x, Ax + u)$ .

## 2.2 $TQ$ as a (pre)symplectic manifold

We assume the reader is familiar with the concept of a symplectic manifold, i.e. a necessarily even-dimensional manifold with a nondegenerate closed two-form  $\Omega$ . In a *presymplectic* manifold, the closed two-form can be degenerate.

A co-tangent bundle  $T^*Q$  has an a priori symplectic structure, given in so-called canonical coordinates by  $\Omega = dx^i \wedge dp_i$ ; a tangent bundle has not. However, if  $TQ$  comes with a Riemann-metric, the tangent bundle can inherit a symplectic structure from  $T^*Q$  via dualisation<sup>1</sup>. But if a manifold admits a Riemann-metric, it admits many and the question arises which one is most suited for our purposes. It is possible to single out a preferred metric by means of a Lagrangian function  $L : TQ \rightarrow \mathbb{R}$ .  $L$  determines a fibre derivative  $\mathbb{F}L : TQ \rightarrow T^*Q$  defined by

$$\langle v, \mathbb{F}L(w) \rangle = \frac{d}{dt}L(q, w + tv)|_{t=0} = \frac{\partial L}{\partial \dot{x}^i}(w)v^i \quad (2)$$

with both  $v$  and  $w$  elements of  $T_qQ$ . We recover the well-known expressions

$$x^i \circ \mathbb{F}L = x^i, \quad p_i \circ \mathbb{F}L = \frac{\partial L}{\partial \dot{x}^i}. \quad (3)$$

In this paper we will consider a linear system (1) together with a quadratic Lagrangian function

$$L(x, u) = \frac{1}{2}u^T Pu + u^T Qx + \frac{1}{2}x^T Rx. \quad (4)$$

One should consider  $L$  as a function on  $TQ$  parametrised by the control  $u$ . The matrices  $P$  and

<sup>1</sup>It is important to note that at this stage,  $T^*Q$  has no physical meaning, in particular it is not looked upon as the Hamiltonian phase space of the system; it is merely an abstract geometric concept.

$R$  are symmetric, but the usual condition  $P > 0$  ( $P$  positive definite) is weakened; instead  $P$  can be positive *semi*-definite,  $P \geq 0$ .

We define a two-form on  $TQ$  as

$$\Omega_L = \mathbb{F}L^* \Omega. \quad (5)$$

In coordinates

$$\Omega_L = dx^i \wedge d\left(\frac{\partial L}{\partial \dot{x}^i}\right) \quad (6)$$

$$= \frac{\partial^2 L}{\partial \dot{x}^i \partial x^j} dx^i \wedge dx^j + \frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j} dx^i \wedge d\dot{x}^j. \quad (7)$$

In the linear-quadratic case

$$\frac{\partial^2 L}{\partial \dot{x}^i \partial x^j} = [-PA + Q]_{ij} \quad (8)$$

$$\frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j} = P_{ij}. \quad (9)$$

It is clear that  $\Omega_L$  will be nondegenerate if and only if  $P$  is nonsingular.

The equations of motion on the tangent bundle are written in terms of the *action*  $S : TQ \rightarrow \mathbb{R}$ , where

$$S(v) = \langle v, \mathbb{F}L(v) \rangle. \quad (10)$$

With  $E = S - L$ , the vector field  $X \equiv (X_{x^i}, X_{\dot{x}^i})$  that governs the motion satisfies

$$i_X \Omega_L = dE, \quad (11)$$

or

$$\frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j} [X_{x^i} - \dot{x}^i] = 0, \quad (12)$$

$$\frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j} X_{\dot{x}^i} = \frac{\partial^2 L}{\partial \dot{x}^i \partial x^j} X_{x^i} - \frac{\partial^2 L}{\partial \dot{x}^i \partial x^j} \dot{x}^i + \frac{\partial L}{\partial x^j}.$$

In the nonsingular case, we have

$$\frac{dx^i}{dt} = \dot{x}^i, \quad (13)$$

$$\frac{d\dot{x}^i}{dt} = \left[\frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j}\right]^{-1} \left[\frac{\partial L}{\partial x^j} - \frac{\partial^2 L}{\partial \dot{x}^j \partial x^k} \dot{x}^k\right].$$

These equations are entirely equivalent to the Hamiltonian equations; that the latter look much more simple, comes from the fact that the co-tangent bundle’s natural coordinates are canonical.

In the singular linear-quadratic case, one immediately sees from (12), that the vector field  $X$  is only determined up to terms in  $\ker P$ : the system exhibits *gauge freedom*. This gauge freedom is however not completely arbitrary because the equations of motion need be ‘consistent’, i.e. they must define a vector field that satisfies the constraints (12) for all future  $t$ .

### 3 The Gotay-Nester algorithm

#### 3.1 Resolving the gauge freedom

The Gotay-Nester algorithm uses/reduces the arbitrariness in  $X$  to solve this problem of consistent equations of motion. The aim of any reduction algorithm is defining a submanifold  $N$  of the phase space (here  $TQ$ ) such that  $i_X\Omega = dE$  hold when restricted to  $N$ .

The main difficulty in singular problems derives from the fact that both the algebraic and the differential content of the equations of motion must be handled at the same time. Thus  $i_X\Omega = dE$  should not only hold as an algebraic equation on  $N$ , its solution  $X$  must be tangent to  $N$  as well in order not to evolve off the relevant manifold. The Gotay-Nester algorithm, as does the classic Dirac-Bergmann algorithm, imposes these conditions and iteratively generates further constraints (and corresponding submanifolds) which the final vector field  $X$  must satisfy. Once an  $X$  is found that obeys all constraints – both algebraic and differential – the algorithm terminates; of course  $N$  is then locally defined by all constraints generated along the way. However, it can occur that not all of the freedom we had originally in defining  $X$ , is removed by the algorithm. In this case, the final constraint manifold  $N$  will not be symplectic and give rise to a non-unique vector field  $X$ .

#### 3.2 The reduction algorithm

We start from the presymplectic dynamical system  $(M_1 \equiv TQ, \Omega, dE)$ .

**Step 1.** Look for all points in  $M_1$  such that  $dE$  is in the range of  $\flat : TQ \rightarrow T^*Q : X \mapsto i_X\Omega$ . Assume these points form a (sub)manifold  $M_2$ . This is an algebraic issue.

**Step 2.** The corresponding differential problem is then solved by further restricting  $M_2$  to  $M_3$ , consisting of all points  $m$  at which  $dE(m) \in \flat(TM_2)$ .

**Step 3.**  $X$  should now be tangent to  $M_3$  as well, and the algorithm iterates.

In general,  $M_{l+1} = \{m \in M_l \text{ such that } dE(m) \in \flat(TM_l)\}$ . If there exists a  $k$  such that  $M_{k+1} = M_k$  and  $\dim M_k \neq 0$ , we end up with consistent equations of motion; the – (pre)symplectic – nature of the final constraint manifold  $N \equiv M_k$  determines whether  $X$  is unique. In the finite-dimensional case, the algorithm is guaranteed to terminate.

**Remark 1** [10] *Let  $N$  be a submanifold of  $M$ , we define the symplectic complement  $TN^\perp$  of  $TN$  in  $TM$  as follows:*

$$TN^\perp = \{Z \in TM|_N \text{ s.t. } \Omega|_N(X, Z) = 0, \forall X \in TN\}. \quad (14)$$

*The sequence of manifolds in the algorithm can then alternatively be characterised as*

$$M_{l+1} = \{m \in M_l \text{ such that } \langle TM_l^\perp, dE \rangle(m) = 0\}, \quad (15)$$

*with*

$$TM_l^\perp = \{Z \in TM_l \text{ such that } k_L^*[i_Z\Omega] = 0\}, \quad (16)$$

*$k_l = j_2 \circ j_3 \circ \dots \circ j_l$ , and  $j_l$  the respective inclusion.*

*It can be shown that the Dirac-Bergmann algorithm – when applicable – is a local version of the Gotay-Nester algorithm in this simplified form: note that the vector field  $Z^\alpha$  on the neighbourhood  $U_{l+1}$  (in  $M_{l+1}$ ) defined by*

$$i_{Z^\alpha}\Omega = c^\alpha_\alpha d\zeta^\alpha \quad (17)$$

*is tangent to  $U_l$ , so that every Dirac-Bergmann  $(l+1)$ -constraint  $g^\alpha_\alpha \{\zeta^\alpha, H\}|_{U_l} = 0$  is equivalent to  $\langle Z^\alpha, dH \rangle = 0$  with  $Z^\alpha \in TU_l^\perp$ .*

#### 3.3 Uniqueness of solutions

The main question we would like to answer is to determine the conditions on which the final constraint manifold will be symplectic, in terms of the Lagrangian cost function (cfr. [5]).

It is clear that answering the above question amounts to examining the nature of the final constraint manifold.

**3.3.1 Classification of submanifolds:** Let  $(M, \Omega, N)$  be the system we consider, and let  $j : N \rightarrow M$  be the inclusion map. The final constraint submanifold  $N$  is called

- (1) *isotropic* if  $TN \subset TN^\perp$ ,
- (2) *co-isotropic (first-class)* if  $TN^\perp \subset TN$ ,
- (3) *symplectic (second-class)* if  $TN \cap TN^\perp = \{0\}$ ,
- (4) *Lagrangian* if  $TN = TN^\perp$ .

In any other case,  $N$  is called a *mixed* manifold.

Note that the induced two-form  $\Omega_N = j^*\Omega$  has as its kernel  $\ker \Omega_N = TN \cap TN^\perp$ . This leads immediately to the following result:

**Proposition 1** *The system will have unique solutions iff  $TN \cap TN^\perp = \{0\}$ .*

Obviously, this can only be examined once the – constructive – algorithm is completed.

### 3.3.2 Gauge vector field algorithm :

In local coordinates, the constraint manifold will be defined in terms of constraint functions. We distinguish two types of constraint functions: constraints in  $A(M, N) = \{\zeta \in C^\infty(M) \text{ such that } TN[\zeta] = 0 \text{ and } TN^\perp[\zeta] = 0\}$  are called first-class constraints; constraints which are not first-class are called second-class. The reader can verify that co-isotropic resp. symplectic constraint manifolds are locally described by first-class resp. second-class constraints; the other classes require both types.

*Example [10].* Consider  $S \subset Q$ , with respective dimensions  $m$  and  $n$ . Then  $T^*S$  is a second-class submanifold, while  $\pi_Q^{-1}(S)$  is first-class. In local coordinates  $(q, p)$  on the co-tangent bundle,  $T(T^*S)^\perp = \text{span} \{\partial/\partial q^{m+1}, \dots, \partial/\partial q^n, \partial/\partial p_{m+1}, \dots, \partial/\partial p_n\}$ , and thus  $T(T^*S)^\perp \cap T(T^*S) = \{0\}$ ; the constraint functions are  $q^{m+1} = 0, \dots, p_n = 0$ . Similarly,  $T(\pi_Q^{-1}(S))^\perp = \text{span} \{\partial/\partial p_{m+1}, \dots, \partial/\partial p_n\}$  and the constraints  $q^{m+1} = 0, \dots, q^n = 0$  are clearly first-class.

In [5] conditions are identified that guarantee unique solutions to a Hamiltonian system with constraints. In that paper the Dirac-Bergmann method is adopted, and the conditions involve the nondegeneracy of a matrix  $A(q, p)$  with  $A_{ij}(q, p) = \{\phi_j, \text{ad}_{\frac{\rho_i}{H}}\}^2$ . It is clear that this result cannot be translated immediately in a Lagrangian analogue, due to the absence of *primary constraints functions* (cfr. [2]): in the Lagrangian framework, the constraints are *implicit* in the degeneracy of the presymplectic form. Once the constraints (the gauge freedom) is made explicit, the *gauge vector field algorithm* in [10] provides the analogue to the results in [5]. We will briefly consider this algorithm now.

Vector fields  $X$  are said to be first-class if  $X|_N \in TN^\perp \cap TN$  and second-class if  $X|_N \in TN^\perp - TN$ . The reader can verify that the first-class vector fields generate an involutive distribution on  $N$ .

It can be proven that all gauge vector fields form a subalgebra  $\mathfrak{g}_G$  in the algebra of first-class vector fields: if  $V$  is a gauge vector field, and  $X$  solves the equations of motion on  $N$ , then  $[\overline{X}, V]$  is a gauge vector field as well<sup>3</sup>: by taking brackets between all gauge vector fields and between gauge vector fields and the solution vector field, one ends up with  $\mathfrak{g}_G$ . Any two points that can be connected along an integral curve of a vector field in  $\mathfrak{g}_G$  represent the same physical state. This characterisation of the gauge vector fields then leads to the Lagrangian – vector field – counterpart of [5].

<sup>2</sup> $\rho_k$  denotes the smallest non-negative integer such that for some  $(q, p)$  not all Poisson brackets in the columns  $A_{k*}$  vanish; the author assumes  $\rho_k < \infty$ , i.e. there are no primary first-class constraints.

<sup>3</sup> $\overline{X}$  denotes any extension of  $X$  to  $M$ .

## 4 LQ Singular optimal control

Let us now apply the above theory to linear-quadratic singular optimal control.

(12) becomes

$$P_{ij} [X_{x^i} - \dot{x}^i] = 0 \quad (18)$$

$$P_{ij} X_{\dot{x}^i} = [-PA + Q]_{[ij]} X_{x^i} + [A^T PA - A^T Q - Q^T A + R]_{ij} x^i \quad (19)$$

The first equation determines the components  $X_x = \dot{x} + v$  up to terms  $v \in \ker P$ ; note that for nontrivial  $v$ 's, the equations of motion will not be second-order equations.

In the regular case, it follows

$$\frac{dx^i}{dt} = \dot{x}^i, \quad (20)$$

$$\frac{d\dot{x}^i}{dt} = P_{ij}^{-1} \cdot \left( [-PA + Q]_{[ij]} \dot{x}^i + [A^T PA - A^T Q - Q^T A + R]_{ij} x^i \right); \quad (21)$$

these are the Euler-Lagrange equations, as is readily checked.

In the singular case, we apply the Gotay-Nester algorithm.

If by chance  $dE$  lies everywhere in the range of  $\flat$ , then the possibly non-unique solution will of course be  $\flat^{-1}(dE)$ . Generically however, this will not be the case. We assume that all points for which  $dE$  is in fact in the range of  $\flat$ , form a submanifold  $M_2$  of  $M_1 \equiv TQ$ , and we restrict the system to this submanifold.

Alternatively, we determine  $TM_1^\perp = \ker \Omega_L$  :

$Z \in TM_1^\perp$  iff

$$\begin{cases} Z_{x^i} \in \ker P \\ P_{ij} Z_{\dot{x}^i} = [-PA + Q]_{[ij]} Z_{x^i} \end{cases} \quad (22)$$

Then  $m \in M_2$  if the secondary constraint  $\langle TM_1^\perp, dE \rangle(m) = 0$  is satisfied, i.e.

$$Z_{x^j} \cdot \left( [A^T PA + R - A^T Q - Q^T A]_{ij} x^i \right) + Z_{\dot{x}^j} \cdot P_{ij} \dot{x}^i = 0, \quad (23)$$

and with  $Z \in TM_1^\perp$

$$Z_{x^j} \cdot \left( [A^T PA + R - A^T Q - Q^T A]_{ij} x^i - [-PA + Q]_{[ij]} \dot{x}^i \right) = 0. \quad (24)$$

$M_2$  is thus defined to be the submanifold of  $M_1$  where

$$\ker P \cdot \left( [A^T P A + R - A^T Q - Q^T A]_{ij} x^i - [-P A + Q]_{[ij]} \dot{x}^i \right) = 0. \quad (25)$$

By definition, the equations of motion will have algebraic solutions here. Yet for the equations to be consistent, their solution  $X$  should be tangent to  $M_2$  as well: if not, the flow of the system will evolve off  $M_2$ . This demand restricts the system possibly to a submanifold  $M_3$  of  $M_2$ , defined by a similar constraint: the algorithm iterates.

Finally we will end up with a final constraint manifold  $M_k$  and a vector field  $X$  tangent to it. Again,  $X$  need not be unique: it is only determined up to elements of  $\ker \Omega_L \cap TM_k$ .

*Example.* Consider the following numerical values for the matrices in a linear-quadratic singular optimal control problem:

$$P = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Q = \mathbf{0}_3, \quad R = \mathbf{I}_3$$

and

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}.$$

The equations of motion can only be solved if  $x_1 = 0$ . This defines  $M_2$ , as can be verified by calculating the constraint (25). Note that  $X_{x^1}$  is arbitrary due to the degeneracy of  $P$  ( $\ker P = \text{span} \{[1, 0, 0]^T\}$ ).

$M_3$  is then determined by the – differential – constraint that  $X$  should be tangent to  $M_2$ ; this restricts  $X_x$  further:  $X_{x^i} = \dot{x}^i + 0, \forall i$ .

All arbitrariness in  $X$  is now resolved and the restriction of the two-form to  $M_2 \equiv N$  is clearly seen to be symplectic; the motion is described by the resulting second-order equations in  $x^2$  and  $x^3$ .

## 5 Conclusions

In this paper we outlined a method for defining consistent equations of motion for systems with a degenerate Lagrangian (cost) function. The method is adapted from the Gotay-Nester-Hinds algorithm for the resolution of constraints in presymplectic systems. We set up the conditions under which the time evolution of the

system is unique, and illustrated the algorithm with a simple linear-quadratic singular optimal control problem.

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