A Differential Geometric Approach to Fault Detection and Isolation for State Affine Systems

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Abstract

In this paper, a geometric approach to the synthesis of a residual generator for fault detection and isolation (FDI) in state affine systems is considered. A necessary and sufficient condition to solve the so-called fundamental problem of residual generation is obtained. The proposed approach resorts to extensions of the notions of (C, A)-invariant and unobservability subspaces, and it yields a constructive design method.

Keywords: Fault detection and isolation, state affine system, observer

1 Introduction

In this paper we consider the design of a part of an advanced monitoring system, namely the residual generator. The latter is a filter that processes the measured plant outputs and the actuator commands in order to generate signals called residuals. These filter outputs are nominally equal to zero in the absence of fault, when the filter transient has vanished. Some of them become distinguishably different from zero upon occurrence of specific faults.

This problem has been extensively studied for linear systems. Different approaches have been proposed such as algebraic methods, frequency domain solutions ([1] and references therein) , and geometric approaches [2], [3]. In this paper, the latter approach is generalized to the design of residual generators for state affine systems. The problem has previously been considered in [4], but a general design algorithm was missing in that paper, as well as necessary and sufficient conditions for the existence of a solution. These missing elements are developed here.

Other papers related to our work are [5], [6], [7]. They are dealing with nonlinear systems that are not state affine. For such systems, a complete methodology to design a residual generator is difficult to obtain due to the fact that asymptotic observers can only be designed for specific classes of systems.

The plan of the paper is as follows. Observability and observer design for state affine systems is reviewed in section 2. The so-called fundamental problem of residual generation (FPRG) for state affine systems is stated in section 3 and a necessary and sufficient condition for the existence of a solution is given. This requires the extension to the state affine case of the geometric tools presented in [2] for linear systems.

2 Observability and observer synthesis for state affine systems up to output injection

Consider the following state affine system up to output injection:

$$\begin{cases} \dot{x} = A(u)x + \phi(u, y) \\ y = Cx \end{cases} \tag{1}$$

where $A(u)$ is a matrix whose entries are smooth functions of $u_i, i = 1, \cdots, m$; $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, and $u = (u_1, \ldots, u_m) \in \mathbb{R}^m$ are respectively the state vector, the output vector and the input vector. $\phi(u, y)$ is a smooth function w.r.t. its arguments.

2.1 Some notions on observability

In [8] , the authors show that the observability of (1) is equivalent to the observability of the state affine system:

$$\begin{cases} \dot{x} = A(u)x \\ y = Cx \end{cases} \tag{2}$$

This means that the output injection keeps the observability. More precisely, it is shown that an input u is universal on $[0, T]$ for system (1) if and only if it is universal on $[0, T]$ for system (2). Remember that an input u is universal on $[0, T]$ if it distinguishes (in the sense of [9], page 93) every pair of initial states $(x, \bar{x})$, $x \neq \bar{x}$ on $[0, T]$.

It is well-known that a linear system is observable if and only if the observability rank condition is fulfilled (Kalman rank condition requiring that the observability matrix be full rank). A similar rank condition can be defined for nonlinear systems :

$$\begin{cases} \dot{x} = f(x,u) \\ y = h(x) \end{cases} \tag{3}$$

with $x \in V$ an open subset of $\mathbb{R}^n$, $u \in U$ a measurable subset of $\mathbb{R}^m$, $y \in \mathbb{R}^p$. To this end, we first define the observation space $\mathcal{O}_3$ of (3) to be the smallest real vector space containing the components $h_1, \cdots, h_p$ of h, and closed under the Lie derivatives L_{f_u}, where u describes U and f_u is the vector field defined by $f_u(x) = f(x,u)$. Then (3) is said to be observable in the rank sense at some $x \in V$ if $dim d\mathcal{O}_3(x) = n$ where $d\mathcal{O}_3(x) = \{d\tau|_x, \tau \in \mathcal{O}_3\}$, d being the classical differential operator. For the state affine system (2), a simple calculation shows that $d\mathcal{O}_2$ can be identified with the vector space spanned by:

$$\{C_1, \ldots, C_p\} \cup \{C_i A(u^1) \ldots A(u^k),\ i = 1, \ldots, p,\ k \geq 1,\ u^1, \ldots, u^k \in \mathbb{R}^m\}$$

where $C_i, i = 1, \cdots, p$ denotes the i^{th} row of C. Moreover, it is easy to check that observability and rank observability are equivalent in the state affine case.

2.2 Observer synthesis for state affine systems

The following property is well-known: an input u is universal on $[t_1, t_2]$ for system (2) if and only if the observability grammian, $G(u, t_1, t_2) = \int_{t_1}^{t_2} \Phi_u^\top(\tau, t_1) C^\top C \Phi_u(\tau, t_1) d\tau$, is a symmetric positive definite matrix, where

$$\frac{d\Phi_u}{d\tau}(\tau, t_1) = A(u(\tau))\Phi_u(\tau, t_1)$$

$$\Phi_u(t_1, t_1) = I, \quad \text{the identity matrix}$$

We have to define a specific class of inputs to be able to state theorem 1 below.

Definition 1 *A bounded input u from $\mathbb{R}^+$ into $\mathbb{R}^m$ is said to be regularly persistent if there exist $T > 0$, $\alpha > 0$ and $t_0 \geq 0$ such that for all $t \geq t_0$, $\lambda_{min}[G(u, t, t+T)] \geq \alpha$, where $\lambda_{min}[G]$ stands for the smallest eigenvalue of the matrix G. In particular, a regularly persistent input is universal on every $[t, t+T]$.*

Theorem 1 *[10] Suppose that u is a regularly persistent input for system (1), then the following system:*

$$\begin{cases} \dot{\hat{x}} = A(u)\hat{x} + \phi(u,y) - S^{-1}C^\top(C\hat{x} - y) \\ \dot{S} = -\theta S - A^\top(u)S - SA(u) + C^\top C \end{cases} \tag{4}$$

with $\theta > 0$, $\hat{x}(0) \in \mathbb{R}^n$ and $S(0)$ any symmetric positive definite matrix, is an exponential observer for system (1). That is to say:

$$\|\hat{x}_u(t) - x_u(t)\|^2 \leq \lambda e^{-\theta t}\|\hat{x}(0) - x(0)\|^2$$

where λ is a positive constant which depends on θ, $S(0)$ and the upper bound on $\|u\|$.

3 The FPRG for state affine systems

3.1 Problem statement

Consider the continuous-time state affine system described by:

$$\begin{cases} \dot{x}(t) = A(u)x(t) + E_1(x(t))v_1(t) + E_2(x(t))v_2(t) \\ y(t) = Cx(t) \end{cases} \tag{5}$$

where $A(u)$ is given in (1), v_1 and v_2 are respectively ℓ_1 and ℓ_2-dimensional failure mode vectors, and $E_i(x)$ $(i = 1, 2)$ are $n \times \ell_i$ matrices depending smoothly on x. Moreover, we assume that $rank E_i(x) = \ell_i, i = 1, 2$ for all x in $\mathbb{R}^n$. The developments are restricted to the situation where two failure modes are considered but they can be generalised to an arbitrary number of failure modes as in [3].

In order to be able to define the FPRG for system (5), let us introduce the following filter with inputs u and y:

$$\begin{aligned} \dot{z}(t) &= \bar{A}(u)z(t) + \bar{D}(u)y(t) + \varphi(g(t))r(t) && (6) \\ \dot{g}(t) &= \psi(u(t), g(t)) && (7) \\ r(t) &= \bar{C}z(t) + Ly(t) && (8) \end{aligned}$$

where $z(t) \in \mathbb{R}^{\bar{n}}$, $g(t) \in \mathbb{R}^N$, $r(t) \in \mathbb{R}^q$, $\bar{A}(u)$ and $\bar{D}(u)$ are smooth functions of u. $\bar{C}$ and L are constant matrices of appropriate dimensions. φ and ψ are smooth functions of their arguments.

Clearly, the output r of the augmented system (5), (6), (7), (8) depends on the initial states $x(0), z(0), g(0)$ and on u, v_1 and v_2 $(r(t) = r(t, x(0), z(0), g(0), u, v_1, v_2))$. For this signal to qualify as a residual, it should not be affected by v_2 and u, and it should be affected by v_1. This is stated more precisely in the conditions i) and ii) of the definition below, which states the FPRG for state affine systems. In the sequel, $L^\infty(\mathbb{R}^+, \mathbb{R}^q)$ (resp. $L^\infty_{loc}(\mathbb{R}^+, \mathbb{R}^q)$) denotes the space of bounded (resp. locally bounded) Borelian functions from $\mathbb{R}^+$ into $\mathbb{R}^q$.

Definition 2 *System (6),(7),(8) is a residual generator which detects and isolates fault v_1 if there exists a subset $\mathcal{U}$ of $L^\infty(\mathbb{R}^+, \mathbb{R}^m)$ s.t.*
i) when $v_1 = 0$, r is not affected by v_2 (in the sense of [9] page 136) and it asymptotically decays to zero for all

u in $\mathcal{U}$ and for any initial conditions $x(0), z(0)$ and $g(0)$.
ii) when $v_2 = 0$, for all v_1 of the form $v_1(t) = \bar{v}_1\alpha(t)$, where $\bar{v}_1$ is any constant vector of $\mathbb{R}^{\ell_1}$ and $\alpha \in L^{\infty}_{loc}(\mathbb{R}^+, \mathbb{R})$, r is affected by α, namely:

$$\begin{aligned}&\forall \bar{v}_1 \in \mathbb{R}^{\ell_1}; \exists u \in \mathcal{U}; \exists x(0) \in \mathbb{R}^n; \exists z(0) \in \mathbb{R}^{\bar{n}};\\ &\exists g(0) \in \mathbb{R}^N; \exists \alpha, \bar{\alpha} \in L^{\infty}_{loc}(\mathbb{R}^+, \mathbb{R}); \exists t \geq 0 \quad \text{such that}\\ &r(t; x(0), z(0), g(0), u, \bar{v}_1\alpha, 0) \neq\\ &\qquad\qquad r(t; x(0), z(0), g(0), u, \bar{v}_1\bar{\alpha}, 0)\end{aligned}$$

We say that the FPRG admits a solution if there exists a system of the form (6),(7),(8) satisfying i) and ii) above.

As in the linear case, appropriate geometric tools have to be defined in order to state a necessary and sufficient condition for the existence of a solution to the FPRG.

3.2 Geometric tools:

Definition 3 *A subspace V is said to be $(C, A(u))$-invariant if one of the following equivalent conditions holds:*

1) $\sum_{u \in U} A(u)(V \cap \ker C) \subset V$,
2) *There exists a $(n \times p)$-matrix $D(u)$ of which the entries are smooth functions of u such that: $(A(u) + D(u)C)V \subset V$ for all $u \in U$.*

The equivalence between conditions 1) and 2) can be obtained in a similar way as in the linear case (see [11]). Let $\mathcal{J}(C, A(u), V)$ be the set of all $(C, A(u))$-invariant subspaces containing a given subspace of $\mathbb{R}^n$, V. The following result holds [12]:

Proposition 1 *i) $\mathcal{J}(C, A(u), V)$ admits a unique minimum (w.r.t. $\subset$) denoted by V^*.*

ii) moreover V^* is the limit of the following sequence:

$$\begin{cases} V_0 = V \\ V_{i+1} = V_i + \sum_{u \in U} A(u)(V_i \cap \ker C) \end{cases} \qquad (9)$$

In order to extend the result of [3], to state affine systems, we need to define the notion equivalent to $(C, A)-$unobservability subspace for such systems. To do so, let $\mathcal{O}(L, D(u))$ denote the observation space of the following system:

$$\begin{cases} \dot{x} = (A(u) + D(u)C)x = \tilde{A}(u)x \\ \tilde{y} = LCx = Ly \end{cases} \qquad (10)$$

namely the real vector space spanned by the linear maps:

$$L_iCx,\ L_iC(A(u^1) + D(u^1)C)x, \ldots,$$
$$L_iC\prod_{k=1}^{l}(A(u^k) + D(u^k)C)x, \quad l \geq 2,\ u^1, \cdots, u^k \in U$$

where L_i is the ith row of L. Clearly, $\mathcal{O}(L, D(u))$ is of dimension smaller or equal to n. Set $d\mathcal{O}(L, D(u)) = \{d\tau, \tau \in \mathcal{O}(L, D(u))\}$, the space of exact one differential forms.

Definition 4 *A $(C, A(u))$-unobservability distribution is any distribution of the form $kerd\mathcal{O}(L, D(u))$, the distribution spanned by the vector fields X on $\mathbb{R}^n$ satisfying:*

$$< d\tau(x), X(x) >= 0, \quad \forall x \in \mathbb{R}^n, \quad \forall \tau \in \mathcal{O}(L, D(u))$$

where $< d\tau, X >$ is the duality product defined by $< d\tau, X >= L_X(\tau)$.

Since τ is a linear map, a basis of such a distribution can be made of constant vector fields. As in subsection 2.1, we identify $d\mathcal{O}(L, D(u))$, as well as $kerd\mathcal{O}(L, D(u))$ with linear subspaces of $\mathbb{R}^n$. Thus, we shall use the following algebraic definition:

Definition 5 *A $(C, A(u))$-unobservability subspace is any subspace of $\mathbb{R}^n$ that can be identified with $kerd\mathcal{O}(L, D(u))$ for some matrices L and $D(u)$.*

Remark 1 *A $(C, A(u))$-unobservability subspace is also a $(C, A(u) + K(u)C)$-unobservability subspace, where $K(u)$ is any $n \times p$ constant matrix of which the entries are smooth functions of u.*

Remark 2 *A $(C, A(u))$-unobservability subspace is a $(C, A(u))$-invariant subspace.*

From the definition of $\ker d\mathcal{O}(L, D(u))$, we obtain the following claim.

Claim 1
The $(C, A(u))$-unobservability subspace $kerd\mathcal{O}(L, D(u))$ is the limit of the following decreasing sequence:

$$\begin{cases} W_0 = \ker LC \\ W_{j+1} = W_j \cap [\bigcap_{u \in U}(A(u) + D(u)C)^{-1}W_j] \end{cases} \qquad (11)$$

Now, let V be any vector subspace of $\mathbb{R}^n$ and denote by $\mathcal{U}(C, A(u); V)$ the set of all $(C, A(u))$-unobservability subspaces containing V. Consider the following decreasing sequence:

$$\begin{cases} \tilde{W}_0 = V^* + \ker C \\ \tilde{W}_{j+1} = V^* + (\bigcap_{u \in U}(A(u)^{-1}\tilde{W}_j)) \cap \ker C \end{cases} \qquad (12)$$

where $V^* = min\mathcal{J}(C, A(u); V)$. Clearly, this sequence is a stationary one. Let n^* be such that $\forall n \geq n^*$, $\tilde{W}_n = \tilde{W}_{n^*}$ and denote this subspace by V^{**}.

Proposition 2 *[12]*
V^{**} is the smallest $(C, A(u))$-unobservability subspace containing V ($V^{**} = min\mathcal{U}(C, A(u); V)$).

3.3 Main result

Theorem 2 *The FPRG w.r.t. v_1 for system (5) admits a solution if and only if $\forall \bar{v}_1 \in \mathbb{R}^{\ell_1}; \exists x \in \mathbb{R}^n$ such that*

$$E_1(x)\bar{v}_1 \notin \mathcal{E}_2^{**} \tag{13}$$

with $\mathcal{E}_2^{**} = min\mathcal{U}(C, A(u); \mathcal{E}_2)$ where $\mathcal{E}_2$ is the real vector space spanned by the family $\mathcal{E}_2 = span\{E_2^i(x); 1 \leq i \leq \ell_2, x \in \mathbb{R}^n\}$, $E_2^i(x)$ being the ith column of $E_2(x)$.

The proof of the theorem requires some preliminary results.

Consider the following systems:

$$(\Sigma) \left\{ \begin{array}{l} \dot{\xi} = B(u)\xi + ve(\xi) + \varphi(g)r_\Sigma \\ \dot{g} = G(u, g) \\ r_\Sigma = M\xi \end{array} \right. \tag{14}$$

$$(\tilde{\Sigma}) \left\{ \begin{array}{l} \dot{\xi} = B(u)\xi + ve(\xi) \\ r_{\tilde{\Sigma}} = M\xi \end{array} \right. \tag{15}$$

where $v(t) \in \mathbb{R}$, $\xi(t) \in \mathbb{R}^n$, $g(t) \in R^N$, $e(\cdot)$ is a smooth vector field on $\mathbb{R}^n$, $B(u)$ and $\varphi(g)$ are matrices of which the entries are smooth functions of u and g respectively, and $G(u, g)$ is a smooth vector field on $\mathbb{R}^m \times \mathbb{R}^N$. M is a constant matrix.

Let $\mathcal{O}_{\tilde{\Sigma}}$ denote the observation space of the state affine part of $\tilde{\Sigma}$ (i.e. when $v = 0$). The following result can be found in [9] (propositions 4.17).

Proposition 3 *[9] $r_{\tilde{\Sigma}}$ is not affected by v iff:*

$$\forall x, \quad e(x) \in kerd\mathcal{O}_{\tilde{\Sigma}}$$

namely $e(x) \in kerM$, and for any $k \geq 1$, and every $u^1, \cdots, u^k \in \mathbb{R}^m$, we have $MB(u^1) \cdots B(u^k)e(x) = 0$.

Lemma 1[12] r_Σ is not affected by v if and only if $r_{\tilde{\Sigma}}$ is not affected by v.

Before stating lemma 2, let us introduce the following notations. Consider the state affine system (2). An extension of (2) is given by:

$$\left\{ \begin{array}{lcl} \dot{x}_e & = & A_e(u)x_e \\ y_e & = & C_e x_e \end{array} \right. \tag{16}$$

where $x_e = \begin{pmatrix} x \\ z \end{pmatrix} \in \mathbb{R}^{n+\bar{n}}$, $y_e = \begin{pmatrix} Cx \\ z \end{pmatrix} = C_e x_e$, $A_e(u) = \begin{pmatrix} A(u) & 0 \\ 0 & 0 \end{pmatrix}$, and $A(u)$ is as defined in (1).

Lemma 2 Let Z_e be any subspace of $\mathbb{R}^{n+\bar{n}}$ such that $Z_e = Z \times \{0\}$ where Z is a subspace of $\mathbb{R}^n$. Then,

$$Z_e^* = Z^* \times \{0\} \tag{17}$$

$$Z_e^{**} = Z^{**} \times \{0\} \tag{18}$$

where $Z_e^*(Z^*)$ is the smallest $(C_e, A_e(u))$-invariant ($(C, A(u))$-invariant) subspace containing Z_e (Z), and $Z_e^{**}(Z^{**})$is the smallest $(C_e, A_e(u))$-unobservability ($(C, A(u))$-unobservability) subspace containing Z_e (Z).

Proof
Only the proof of (17) will be given here. A similar argument can be used to prove (18) (see [12]).

From proposition 1, Z_e^* and Z^* are the respective limits of the following sequences:

$$\left\{ \begin{array}{l} V_{e0} = Z_e \\ V_{e(i+1)} = V_{ei} + \sum_{u \in U} A_e(u)(V_{ei} \cap \ker C_e) \end{array} \right. \tag{19}$$

$$\left\{ \begin{array}{l} V_0 = Z \\ V_{(i+1)} = V_i + \sum_{u \in U} A(u)(V_i \cap \ker C) \end{array} \right. \tag{20}$$

Hence, to prove (17), we only need to show that, for every i, $V_{ei} = V_i \times \{0\}$. Notice that $V_0 \times \{0\} = V_{e0}$. Now assume that $V_{ei} = V_i \times \{0\}$, and let us prove that $V_{e(i+1)} = V_{i+1} \times \{0\}$.

From (19), we deduce:

$$\begin{array}{l} V_{e(i+1)} = V_i \times \{0\} + \{x_e \in \mathbb{R}^{n+\bar{n}} : \\ \qquad x_e = \sum_{u \in U} A_e(u)x_{e,u}^i with\ x_{e,u}^i \in V_{ei} \cap kerC_e\} \end{array}$$

.
Setting $x_{e,u}^i = \begin{pmatrix} x_u^i \\ z_u^i \end{pmatrix}$ and using the induction hypothesis ($V_{ei} = V_i \times \{0\}$), $x_{e,u}^i \in V_{ei} \cap kerC_e$ is then equivalent to $x_u^i \in V_i \cap kerC$ and $z_u^i = 0$. Hence

$$\begin{array}{rcl} V_{e(i+1)} & = & (V_i + \sum_{u \in U} A(u)(V_i \cap kerC)) \times \{0\} \\ & = & V_{i+1} \times \{0\} \end{array}$$

Proof of Theorem 2
Only if part
Assume that the FPRG admits a solution, namely the following system obtained by combining (5) and (6), (7), (8):

$$\begin{array}{rcl} \dot{x} & = & A(u)x + E_1(x)v_1 + E_2(x)v_2 \quad (21) \\ \dot{z} & = & \bar{A}(u)z + \bar{D}(u)Cx + \varphi(g)r \quad (22) \\ \dot{g} & = & \psi(u, g) \quad (23) \\ r & = & LCx + \bar{C}z \quad (24) \end{array}$$

fulfils condition (i) and (ii) of the definition of the FPRG (definition 2). Consider fault vectors v_2, with one single non-zero component, say the i^{th} one; thus $v_2 = [\ 0\ \ \cdots\ \ 0\ \ v_{2i}\ \ 0\ \ \cdots\ \ 0\ ]^T$. For such vectors, (21), (22), (23), (24) in which $v_1 = 0$ can be written:

$$\left\{\begin{array}{l} \dot{x}_e = \tilde{A}_e(u)x_e + v_{2i}E^i_{2e}(x_e) + \varphi_e(g)r \\ \dot{g} = \psi(u,g) \\ r = \tilde{C}_e x_e \end{array}\right. \tag{25}$$

where $x_e = \begin{pmatrix} x \\ z \end{pmatrix}$, $\tilde{A}_e(u) = \begin{pmatrix} A(u) & 0 \\ \bar{D}(u)C & \bar{A}(u) \end{pmatrix}$, $\tilde{C}_e = \begin{pmatrix} LC & \bar{C} \end{pmatrix}$, $E^i_{2e}(x) = \begin{pmatrix} E^i_2(x) \\ 0 \end{pmatrix}$, $E^i_2(x)$ is the i^{th} column of $E_2(x)$, and $\varphi_e(g) = \begin{pmatrix} 0 \\ \varphi(g) \end{pmatrix}$. Using lemma 1, system (25) is not affected by v_{2i} iff the following system is not affected by v_{2i} either:

$$\left\{\begin{array}{l} \dot{x}_e = \tilde{A}_e(u)x_e + E^i_{2e}(x_e)v_{2i} \\ r = \tilde{C}_e x_e \end{array}\right. \tag{26}$$

Let $\tilde{\mathcal{O}}_e$ denote the observation space of the state affine part of (26)(i.e. when $v_{2i} = 0$); from proposition 3, we know that $E^i_{2e}(x_e) \in kerd\tilde{\mathcal{O}}_e$. This holds for every $i, i = 1, \cdots, \ell_2$ and every x_e. Therefore, the real vector space $\mathcal{E}_{2e}$ spanned by $\{E^i_{2e}(x_e); i = 1, \cdots, \ell_2; x_e \in \mathrm{I\!R}^{n+\bar{n}}\}$, is contained in $kerd\tilde{\mathcal{O}}_e$(denoted $\mathcal{S}_e$ in the sequel). Now, by condition ii) in the definition of the FPRG, the following system is affected by α for any fixed vector $\bar{v}_1$ of $\mathrm{I\!R}^{\ell_1}$:

$$\left\{\begin{array}{l} \dot{x} = A(u)x + \alpha E_1(x)\bar{v}_1 \\ (22), (23), (24) \end{array}\right. \tag{27}$$

Using again lemma 1, it follows that:

$$\left\{\begin{array}{l} \dot{x}_e = \tilde{A}_e(u)x_e + \alpha E_{1e}(x_e)\bar{v}_1 \\ r = \tilde{C}_e x_e \end{array}\right. \tag{28}$$

is also affected by α, where $E_{1e}(x_e) = [\ E_1(x)^T\ \ \ 0\ ]^T$. Hence there exists x_e such that:

$$E_{1e}(x_e)\bar{v}_1 \notin \mathcal{S}_e = kerd\tilde{\mathcal{O}}_e \tag{29}$$

By construction, $\mathcal{S}_e$ is a $(\tilde{C}_e, \tilde{A}_e(u))$-unobservability subspace. Now, setting $C_e = \begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix}$ with C as above, and I the $\bar{n} \times \bar{n}$ identity matrix ($\bar{n}$ is the dimension of the space of variables z), clearly $\mathcal{S}_e$ is also a $(C_e, \tilde{A}_e(u))$-unobservability subspace. Finally, from remark 1, $\mathcal{S}_e$ is a $(C_e, A_e(u))$-unobservability subspace, where $A_e(u)$ is as defined in (16). Therefore, $\mathcal{S}_e$ contains the smallest $(C_e, A_e(u))$-unobservability subspace containing $\mathcal{E}_{2e} = \mathcal{E}_2 \times \{0\}$, namely $(\mathcal{E}_2 \times \{0\})^{**}$, which is equal to $\mathcal{E}_2^{**} \times \{0\}$ by lemma 2. Thus,

$$\mathcal{S}_e \supset \mathcal{E}_2^{**} \times \{0\} \tag{30}$$

By (29), there exists $x \in \mathrm{I\!R}^n$ such that $[\ E_1(x)^T\ \ \ 0\ ]^T\bar{v}_1 \notin \mathcal{S}_e$. Combining the latter expression with (30) yields (13).

If part

Consider matrices L and $D(u)$ such that $\mathcal{E}_2^{**} = kerd\mathcal{O}(L, D(u))$, then system (5) in which the output $y = Cx$ is replaced by $\bar{y} = LCx$ can be rewritten:

$$\begin{aligned} \dot{x} &= (A(u) + D(u)C)x - D(u)y + E_1(x)v_1 \\ &\qquad + E_2(x)v_2 \qquad (31) \\ \bar{y} &= LCx = Ly \qquad (32) \end{aligned}$$

Notice that the dynamics of (31) is identical to that of (5).

Now consider the observation space $\mathcal{O}(L, D(u))$ of the state affine system:

$$\begin{aligned} \dot{x} &= (A(u) + D(u)C)x \qquad (33) \\ \bar{y} &= LCx \qquad (34) \end{aligned}$$

From subsection 3.2, a basis of $\mathcal{O}(L, D(u))$ is made of linear maps $(T_1x, \cdots, T_dx)$. Let $(T_{n-d}, \cdots, T_n)$ be $(n-d)$ row vectors such that the matrix $T = \begin{pmatrix} T_1 \\ \vdots \\ T_n \end{pmatrix}$ is invertible, and set $\zeta = Tx = \begin{pmatrix} \zeta^1 \\ \zeta^2 \end{pmatrix}$ where $\zeta^{1T} = \begin{pmatrix} T_1x & \cdots & T_dx \end{pmatrix}$, $\zeta^{2T} = \begin{pmatrix} T_{d+1}x & \cdots & T_nx \end{pmatrix}$.

In the new system of coordinates $(\zeta_1, \cdots, \zeta_n)$, system (31), (32) can be written:

$$\dot{\zeta}^1 = \overline{A^{11}}(u)\zeta^1 - \overline{D^1}(u)y + \overline{E^1_1}(\zeta)v_1 \tag{35}$$

$$\dot{\zeta}^2 = \overline{A^{21}}(u)\zeta^1 + \overline{A^{22}}(u)\zeta^2 - \overline{D^2}(u)y + \overline{E^2_1}(\zeta)v_1 + \overline{E^2_2}(\zeta)v_2 \tag{36}$$

$$\bar{y} = \overline{C}\zeta^1 \tag{37}$$

where

$$T(A(u) + D(u)C)T^{-1} = \begin{pmatrix} \overline{A^{11}}(u) & 0 \\ \overline{A^{21}}(u) & \overline{A^{22}}(u) \end{pmatrix} \tag{38}$$

$$\begin{pmatrix} \overline{D^1}(u) \\ \overline{D^2}(u) \end{pmatrix} = TD(u), \begin{pmatrix} \overline{E^1_i} \\ \overline{E^2_i} \end{pmatrix} = TE_i(T^{-1}\zeta),\ i = 1, 2, \tag{39}$$

$$[\overline{C}\quad 0] = LCT^{-1} \tag{40}$$

$\overline{E^1_2}(\zeta) = 0$, since $\mathcal{E}_2 \subset \mathcal{E}_2^{**} = kerd\mathcal{O}(L, \mathcal{D})$. Moreover, from the hypothesis of the if part, we know that for every $\bar{v}_1 \in \mathrm{I\!R}^{\ell_1}$, there exists ζ such that $E_1(T^{-1}\zeta)\bar{v}_1 \notin \mathcal{E}_2^{**}$. Thus $\overline{E^1_1}(\zeta) \neq 0$.

By construction, the state affine system corresponding to (35), (37):

$$\left\{\begin{array}{rcl}\dot{\zeta}^1 & = & \overline{A^{11}}(u)\zeta^1 \\ \bar{y} & = & \overline{C}\zeta^1\end{array}\right. \tag{41}$$

is observable in the rank sense. Hence it is observable (see subsection 2.1). Let $\mathcal{U}$ denote the set of regularly persistent inputs for this system (this set is not empty, see [10]). Thus for every $u \in \mathcal{U}$ the following system

$$\left\{\begin{array}{l}\dot{\hat{\zeta}}^1 = \overline{A^{11}}(u)\hat{\zeta}^1 - S^{-1}\overline{C}^T(\overline{C}\hat{\zeta}^1 - \bar{y}) \\ \dot{S} = -\theta S - \overline{A^{11}}(u)^T S - S\overline{A^{11}}(u) + \overline{C}^T\overline{C}\end{array}\right.$$

is an exponential observer for (41) (see subsection 2.2), where $\hat{\zeta}^1(0) \in \mathbb{R}^d$ is arbitrary, $S(0)$ is any symmetric positive definite matrix and $\theta > 0$ is a design parameter. More precisely, set $e(t) = \hat{\zeta}^1(t) - \zeta^1(t)$, then we have:

$$\left\{\begin{array}{l}\dot{e} = (\overline{A^{11}}(u) - S^{-1}\overline{C}^T\overline{C})e \\ \dot{S} = -\theta S - \overline{A^{11}}(u)^T S - S\overline{A^{11}}(u) + \overline{C}^T\overline{C}\end{array}\right. \tag{42}$$

Moreover $\|e(t)\|$ exponentially converges to 0.

Our candidate residual generator is then of the form:

$$\left\{\begin{array}{l}\dot{\hat{\zeta}}^1 = \overline{A^{11}}(u)\hat{\zeta}^1 - \overline{D^1}(u)y - S^{-1}\overline{C}^T(\overline{C}\hat{\zeta}^1 - Ly) \\ \dot{S} = -\theta S - \overline{A^{11}}(u)^T S - S\overline{A^{11}}(u) + \overline{C}^T\overline{C} \\ r = \overline{C}\hat{\zeta}^1 - Ly\end{array}\right. \tag{43}$$

Let us check condition $i)$ of definition 2. Set $e(t) = \hat{\zeta}^1(t) - \zeta^1(t)$ where $\hat{\zeta}^1(t)$ is given by (43) and $\zeta^1(t)$ by (35). If $v_1 = 0$, then $(e(t), S(t))$ satistfies equation (42). Introducing (40) into (43), we get:

$$r = \overline{C}\hat{\zeta}^1 - Ly = \bar{C}\hat{\zeta}^1 - \bar{C}\zeta^1 = Ce$$

Hence $\|r(t)\|$ exponentially converges to 0 for every $u \in \mathcal{U}$.

The proof that condition ii) in definition 2 is also fulfilled can be found in [12].

4 Conclusion

The computation procedure for $\mathcal{E}_2^{**}$ and for the residual generator which detects and isolates v_1, when it exists, can be summarized as follows:
-**First step:** compute $\mathcal{E}_2^*$ the smallest $(C, A(u))$ invariant subspace containing $\mathcal{E}_2$.
-**Second step:** calculate L such that $\ker LC = \mathcal{E}_2^* + \ker C$.
-**Third step:** compute $\mathcal{E}_2^{**}$ by using (12) and $D(u)$ such that $(A(u) + D(u)C)\mathcal{E}_2^{**} \subset \mathcal{E}_2^{**}$.
-**Fourth step:** check that condition (13) is fulfilled and if so, build the residual generator given by (43).

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