

# Optimal Worst Case Estimation for LPV-FIR Models with Bounded Errors

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## Abstract

In this paper discrete time linear parameter varying (LPV) models with finite impulse response (FIR) dynamic structure are considered. Measurement errors are assumed to be bounded and in such condition the worst case parameter estimate errors are derived together with the input sequences that allow their determination.

The main result of the paper consists in showing that the optimal input design of LPV-FIR models is achieved combining the available results on optimal input design for invariant FIR models with the results on optimal input design for static blocks.

## 1 Introduction

In this paper discrete time linear parameter varying (LPV) models with finite impulse response (FIR) dynamic structure are considered.

In linear parameter varying models the parameters of the equations linking the inputs  $u$  to the outputs  $y$  are in general functions of a (possibly vector valued) variable  $p$ , assumed to be measurable, that determines the operating condition of the system.

Although such class of model structures is less general than generic linear time varying models, in several industrial applications it is perceived to be quite satisfactory for representing real systems and its connection to the industrial practice of gain scheduling makes it appealing for the solution of control design problems.

LPV models have in general a number of parameters that is quite bigger than the number of parameters in their corresponding time invariant models and therefore also the number of measurements required for parameter estimation is correspondingly large. This fact increases the interest for optimal experiment design aimed at getting better estimates with less measurements.

Since the  $p$  variable can be regarded as an auxiliary

input that, at least for identification purposes, could be under the designer's control, the optimal experiment design becomes an optimal input design that consists in finding the shortest sequences both for the input  $u$  and for the variable  $p$  that can ensure some kind of optimality for the derived parameter estimates.

In this paper bounded measurement errors and uncertainties are considered and therefore the optimality of the input sequences is evaluated in terms of the worst case estimate error they allow to achieve.

The main result of this paper consists in showing that the minimum worst case estimation error is obtained combining the results on optimal input design for time invariant FIR models as described in [2] with the results on optimal input design for static blocks described in [3], [5] and [4]. Namely the input sequence for  $u$  is the same as that for time invariant FIR models, but repeated a suitable number of times for each one of which the input  $p$  is kept constant at the different optimal input levels that allow optimal worst case identification of a suitable static system.

## 2 Notation

Let a linear parameter varying discrete-time model, whose dynamic is described by a FIR, be represented by

$$y_k = B(q, p_k)u_k + e_k \quad (1)$$

where  $y_k$ ,  $u_k$ ,  $p_k$  and  $e_k$  denote respectively the system output and input, the operating condition  $p$  and the measurement error  $e$  at sample  $k$ . The dynamic behavior is described by the FIR regressor

$$B(q, p_k) = b_0(p_k) + b_1(p_k)q^{-1} + \dots + b_{n_b-1}(p_k)q^{-n_b+1} \quad (2)$$

where  $q^{-1}$  represents the usual shift operator and  $b_i(p_k)$   $i = 0, \dots, n_b - 1$  are the FIR coefficients.

The  $n_b$  FIR coefficients are assumed to be linearly dependent on the parameter vector to be estimated ac-

cording to

$$b_i(p_k) = \sum_{j=1}^N b_{i,j} f_j(p_k) \quad (3)$$

where  $f_j(\cdot)$  are given functions and  $b_{i,j}$ ,  $i = 0, \dots, n_b - 1$ ,  $j = 1, \dots, N$ , are the parameter to be estimated that can be rearranged in one single vector  $\theta$  as

$$\theta = [ b_{0,1} \ b_{0,2} \ \dots \ b_{0,N} \ b_{1,1} \ \dots \ b_{n_b-1,N} ] . \quad (4)$$

When  $m$  consecutive measurements are collected from the system, they can be arranged in matrix form as

$$y = A(u, p)\theta + e \quad (5)$$

where  $A(u, p)$  is the regression matrix and  $y \in R^m$ ,  $u \in R^{m+n_b-1}$ ,  $e \in R^m$  and  $p \in R^m$  are the measurement vector, the input vector, the measurement error vector, and the operating condition vector respectively.

The regression matrix  $A(u, p)$  has the following structure

$$A(u, p) = \begin{bmatrix} V(u_1, p_1) & V(u_0, p_1) & \dots & V(u_{-n_b+2}, p_1) \\ \vdots & \vdots & & \vdots \\ V(u_k, p_k) & V(u_{k-1}, p_k) & \dots & V(u_{k-n_b+1}, p_k) \\ \vdots & \vdots & & \vdots \\ V(u_m, p_m) & V(u_{m-1}, p_m) & \dots & V(u_{m-n_b+1}, p_m) \end{bmatrix} \quad (6)$$

with

$$V(u_i, p_k) = [ u_i f_1(p_k) \ u_i f_2(p_k) \ \dots \ u_i f_N(p_k) ] . \quad (7)$$

In this paper the system input  $u_k$  and the operating condition parameter  $p_k$  are assumed to be bounded so that

$$|u_k| \leq U \quad \forall k \quad (8)$$

and

$$p_a \leq p_k \leq p_b \quad \forall k. \quad (9)$$

Also the measurement error is assumed to be component wise bounded so that the error vector  $e$  must belong to its membership set  $\Omega_e$  defined as

$$\Omega_e = \{ e \in R^m : |e_k| \leq E, k = 1, \dots, m \} . \quad (10)$$

With this assumption it is common practice to look for the parameter feasible set  $\mathcal{D}(u, p)$  defined as

$$\mathcal{D}(u, p) = \{ \theta \in R^{N \cdot n_b} : y = A(u, p)\theta + e, e \in \Omega_e \} \quad (11)$$

that collects all the parameter vectors  $\theta$  consistent with the model (5), the measurements  $y$  and the errors  $e$ . Any such vector could be the one that generated the data and therefore the “size” of  $\mathcal{D}(u, p)$  is a measure of

the estimate’s reliability. Such “size” is usually evaluated in terms of the width of the parameter uncertainty intervals  $W(PUI_{b_{i,j}})$  defined as

$$W(PUI_{b_{i,j}}) = b_{i,j}^M - b_{i,j}^m, \quad (12)$$

where

$$b_{i,j}^m = \min_{\theta \in \mathcal{D}(u, p)} b_{i,j}, \quad b_{i,j}^M = \max_{\theta \in \mathcal{D}(u, p)} b_{i,j}, \quad (13)$$

while

$$PUI_{b_{i,j}} = [ b_{i,j}^m, b_{i,j}^M ] . \quad (14)$$

An alternative measure of the “size” of  $\mathcal{D}(u, p)$  is its radius  $r(\mathcal{D}(u, p))$  that represents the maximum achievable estimation error in some suitable norm and is given by

$$r(\mathcal{D}(u, p)) = \inf_{\vartheta \in R^{N \cdot n_b}} \sup_{\theta \in \mathcal{D}(u, p)} \|\vartheta - \theta\| . \quad (15)$$

Throughout this paper the  $l_\infty$  norm radius will be considered.

Remark that  $\mathcal{D}(u, p)$ , beside being a function of the input vector  $u$  and the operating condition  $p$ , also depends on the output vector  $y$  and thus on the unknown error realization. In order to get rid of this last dependence it is common to refer to the maximum achievable “size” with respect to any possible error realization, referred to as the worst case estimation error.

The largest width of the parameter uncertainty intervals (denoted in the following as  $W(\hat{P}UI_{b_{i,j}})$ ) and the largest radius (denoted in the following as  $\hat{r}(\mathcal{D}(u, p))$ ) with respect to any error realization can be easily computed, for linear in the parameter systems, ([8], [10], [11]) from relations (12)-(15) in which however the set  $\mathcal{D}(u, p)$  is substituted by the set  $\hat{\mathcal{D}}(u, p)$  defined as

$$\hat{\mathcal{D}}(u, p) = \{ \theta \in R^{N \cdot n_b} : A(u, p)\theta = e, e \in \Omega_e \} . \quad (16)$$

### 3 Review of already available relevant results

In order to derive the main result of this paper it is necessary to recall some results about optimal experiment design in the worst case estimation error context.

First it is convenient to remind that the worst case estimation error can be achieved with a rather low number of measurements that is bounded between  $n_p$  and  $n_p^2$  where  $n_p$  is the total number of parameters to be estimated [1]. Remark that extra measurements cannot improve the worst case estimation error, although they can improve the actual error to an extent that is related to the obtained error realization.

Systems for which the worst case estimation error can be obtained with a set of  $n_p$  suitable measurements only are referred to as *minimal worst case* systems. Some

sufficient condition for a system to be minimal worst case can be found in literature [1] [9].

For the special case of time invariant FIR systems with  $n_b$  parameters, that can be represented by equations (1)-(5) assuming, in relation (3),  $N = 1$  and  $f_1(p) = \cos t$ , it has been proved [2] that such systems are *minimal worst case*, hence the worst case estimation error can be obtained with a set of  $n_b$  suitable measurements. Moreover it has been shown that suitable input sequences exist that allow to get  $n_b$  subsequent measurements leading to the worst case estimation error. One of these optimal input series is the one in which

$$\begin{aligned} u_i &= U & i &= 1, \dots, n_b \\ u_i &= -U & i &= n_b + 1, \dots, 2n_b. \end{aligned} \quad (17)$$

This sequence is also “cyclic” in the sense that  $u_{k+n_b} = -u_k$  and can therefore be cyclically applied to the input  $u$ . For such kind of optimal input series it turns out that  $2n_b - 1$  input samples must first be applied in order to collect  $n_b$  measurements (the FIR filter needs to be loaded) thereafter any new set of  $n_b$  subsequent inputs allow to collect a new set of  $n_b$  measurements that alone allows to determine again the worst case estimation error. Remark that in general there are several optimal “cyclic” input series that can be found by exhaustive search. Among these it is in general convenient to choose the one with the smallest volume of the corresponding  $\hat{\mathcal{D}}(u)$  set as discussed in [2].

For what concerns the worst case error radius  $\hat{r}(\mathcal{D}(u))$  of time invariant FIR systems it has been proved [2] that it results to be

$$\hat{r}(\mathcal{D}(u)) = E/U \quad (18)$$

and, always from the results in [2], although it is not explicitly stated therein, it can be easily shown that the worst case error parameter uncertainty interval width  $W(P\hat{U}I_{b_i})$  is

$$W(P\hat{U}I_{b_i}) = 2E/U \quad \forall i \quad (19)$$

For what concerns the optimal input design for static blocks, remark that such systems can be trivially represented by equations (1)-(5) forcing  $n_b = 1$  in relation (2). In such condition only the input  $p$  needs to be optimally chosen. Such choice consists in finding  $n_{opt}$  optimal levels  $\{p_1^{opt}, \dots, p_{n_{opt}}^{opt}\}$  that, applied to the input  $p$  allow the optimal worst case identification of the parameters  $b_{0,i}$ ,  $i = 1, \dots, N$ . Remind that the worst case width  $W(P\hat{U}I_{b_{0,j}})$  of each one of the parameter uncertainty intervals depends on  $N$  optimal inputs so that the total number  $n_{opt}$  of required inputs for the worst case estimation error evaluation is  $N \leq n_{opt} \leq N^2$ .

The choice of the optimal  $n_{opt}$  input levels  $\{p_1^{opt}, \dots, p_{n_{opt}}^{opt}\}$  can always be numerically performed, as discussed in [1], for any family of functions

$f_1(p) \dots f_N(p)$  that are linear independent on the interval  $[p_a, p_b]$ . There are also some special families of functions for which closed form analytical results are available. One such case is the one of complete Taylor polynomial bases in which

$$f_j(p_k) = p_k^{j-1} \quad j = 1, \dots, N. \quad (20)$$

For these functions, as shown in [3] and [4], the set of  $n_{opt} = N$  optimal input levels  $\{p_1^{opt} \dots p_{n_{opt}}^{opt}\}$  over the interval  $[p_a, p_b]$  consists of the following distinct levels

$$\begin{cases} p_1^{opt} &= p_a \\ p_i^{opt} &= p_a + \frac{1}{2}(p_b - p_a)(1 + \cos \frac{i-1}{N-1}\pi), \\ & i = 2, \dots, N-1 \\ p_N^{opt} &= p_b. \end{cases} \quad (21)$$

Finally remark that the general requirement of linear independence for the functions  $f_1(p) \dots f_N(p)$  on the interval  $[p_a, p_b]$  is not restrictive. Actually it is a requirement for the identifiability of the dynamic system described by relations (1)-(5). In fact if the  $f_1(\cdot), \dots, f_N(\cdot)$  are not linearly independent then there exist two different sets of parameters  $b_{i,j}$ ,  $j = 1, \dots, N$  satisfying relation (3) for the same  $b_i(p)$ .

#### 4 Optimal worst case experiment design

The following theorem, allowing to find optimal input sequences, in a worst parameter error sense both for the input  $u$  and for the operating condition  $p$  of a LPV-FIR system, represents the main result of this paper.

**Theorem 1** *Consider a LPV-FIR model as described in relations (1)-(5).*

*Assume that the optimal worst parameter error identification of a static block described by*

$$\tilde{y}(p) = \sum_{j=1}^N \beta_j f_j(p)$$

*with  $f_j(p)$  as in (3), consists in  $n_{opt}$  optimal input levels  $\{p_1^{opt}, \dots, p_{n_{opt}}^{opt}\}$  ensuring the worst error case widths of the parameter uncertainty intervals.*

*Let such widths be  $W(P\hat{U}I_j) = W_j^{opt}$ ,  $j = 1, \dots, N$ , when the measurement error bound is normalized to  $E = 1$ .*

*Then the worst error case width of the parameter uncertainty interval  $W(P\hat{U}I_{b_{i,j}})$  of each one of the LPV-FIR model parameter  $b_{i,j}$  is*

$$W(P\hat{U}I_{b_{i,j}}) = 2W_j^{opt} \frac{E}{U}, \quad i = 0, \dots, n_b - 1 \quad j = 1, \dots, N$$

and it is achieved collecting  $n_b \cdot n_{opt}$  measurements with the following sequences for input  $u$  and for parameter  $p$ :

- the input  $u$  is any "cyclic" sequence optimal for the time invariant FIR systems with  $n_b$  parameters

- the parameter  $p$  is kept constant for  $n_b$  measurements at one of the different optimal levels  $p_i^{opt}$ ; it varies step-wise every  $n_b$  measurements in such a way that over all the  $n_{opt}$  sets of  $n_b$  measurements it assumes all the values  $p_i^{opt}$ ,  $i = 1, \dots, n_{opt}$ .

Proof.

Each one of the  $n_{opt}$  sets of  $n_b$  measurements with  $p = p_j^{opt}$  allows to derive a set of  $n_b$  values  $b_i(p_j^{opt})$ ,  $i = 0, \dots, n_b - 1$ . Such values have, according to the previously recalled results of [2], an uncertainty interval of  $\pm E/U$ . In fact the  $b_i(p_j^{opt})$  can be regarded as the parameters of a time invariant FIR with optimal input sequence.

For each set of  $N$  parameters  $b_{i,j}$ ,  $j = 1, \dots, N$ , it is possible to derive the corresponding worst case width  $W(P\hat{U}I_{b_{i,j}})$  of the parameter uncertainty intervals using the  $n_{opt}$  relations

$$b_i(p_k^{opt}) = \sum_{j=1}^N b_{i,j} f_j(p) \quad k = 1, \dots, n_{opt}.$$

They lead, by hypotheses, to the worst error case width of the parameter uncertainty intervals

$$W(P\hat{U}I_{b_{i,j}}) = 2W_j^{opt} \frac{E}{U} \quad i = 0, \dots, n_b \quad j = 1, \dots, N$$

since the bounds on  $b_i(p_k^{opt})$  are equal to  $\pm E/U$  and the optimal worst error case parameter intervals of the static block in relation (1) are  $W_j^{opt}$ ,  $j = 1, \dots, N$ , for measurement errors bounded between  $\pm 1$ .

If any measure is added to the previously defined set of optimal measurements, it is easy to show that no improvement can be achieved. In fact, if the new added measure is collected with a  $p$  level equal to any of the optimal ones  $\{p_1^{opt}, \dots, p_{n_{opt}}^{opt}\}$ , then the new measure combined with the other ones with the same value of  $p = p_j^{opt}$  will lead, in the worst error context, to the same value  $b_i(p_j^{opt})$  and no improvement on the LPV-FIR parameters will be obtained.

If on the contrary the new added measure is collected with a  $p$  value different from the optimal ones its uselessness can be proved supposing to collect not only one, but  $n_b$  measurements with the same value of  $p$ . This set of measurements allows to derive a new set of  $n_b$  values  $b_i(p)$ ,  $i = 0, \dots, n_b - 1$  that have, according to the previously recalled results of [2], an uncertainty

interval of  $\pm E/U$ . This information however does not improve the worst case uncertainty of the parameters  $b_{i,j}$  since it only adds a set of  $n_b$  relations of the form

$$b_i(p_k) = \sum_{j=1}^N b_{i,j} f_j(p_k)$$

that, by definition, is useless with respect to the determination of the worst case width of the parameter uncertainty intervals.

Finally it is easy to see that if any of the previously recalled optimal  $n_b \cdot n_{opt}$  measurements is omitted the worst case parameter uncertainty interval width is not achieved. In fact if one of the optimal measurements is omitted there will no longer be a set of  $N$  measurements for one of the optimal levels  $p_j^{opt}$  so that the bounds of the corresponding parameters  $b_i(p_j^{opt})$   $i = 0, \dots, n_b - 1$  will be larger than  $\pm E/U$  thus leading to larger bounds on at least one set of the parameters  $b_{i,j}$ .  $\square$

The previous results hold indeed also for a vector valued  $p$  variable.

Finally remark that these results can be extended to the case in which (3) is substituted by

$$b_i(p_k) = \sum_{j=1}^N b_{i,j} f_{i,j}(p_k) \quad (22)$$

so that different sets of functions are used to represent the dependence on  $p$  of each one of the FIR parameters.

In this case the set of optimal levels of  $p$  consists of the union of the input sets that are optimal for the  $n_b$  static blocks that can be represented by relation (22) with  $i = 0, \dots, n_b - 1$ .

## 5 Conclusions

In this paper the problem of the worst case identification of discrete time linear parameter varying (LPV) models with finite impulse response (FIR) dynamic structure in the case of unknown but bounded errors is of concern. Worst case parameter uncertainties are derived together with the input sequences that allow their determination.

The problem of the optimal input design for the worst case identification of LPV-FIR models is addressed. It results that such optimal input is achieved combining the available results on optimal input design for invariant FIR models with the results on optimal input design for static blocks.

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