Normalized $\mathcal{H}_\infty$ Controller Reduction with *A Priori* Bounds

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Abstract

This paper describes an approach to the reduction of controllers for the normalized coprime factor robustness problem as well as the normalized $\mathcal{H}_\infty$ problem. It is shown that a relative error approximation of a coprime factor representation of any suboptimal controller leads to a stability guarantee and an upper bound on the performance degradation when the reduced order controller is implemented. When the approximation is performed on the controller generator, guaranteed *a priori* stability and performance bounds are obtained in terms of the synthesis Riccati equation solutions of the normalized $\mathcal{H}_\infty$ control problems.

1 Introduction

This paper considers the order reduction of controllers for the normalized $\mathcal{H}_\infty$ design techniques of [11] and [13]. We show that if the inverse-weighted balanced truncation technique of [15] is applied to the *generator* of all (suboptimal) normalized $\mathcal{H}_\infty$ controllers, one may induce a reduction on the coprime factors of the central controller. When this reduced order controller is implemented, we obtain guaranteed *a priori* stability and performance degradation bounds (*a priori* in the sense that they depend only on the performance level and the solutions of the synthesis Riccati equations). Despite the intense interest in obtaining reduced order $\mathcal{H}_\infty$ controllers with guaranteed *a priori* bounds (in the above sense) these efforts have hitherto only been partially successful.

Generally LQG and $\mathcal{H}_\infty$ controllers are of high order and are therefore impractical to implement and approximating such controllers is usually treated as an independent problem. The basic approach to the approximation step is to apply one of the well known model reduction techniques to the controller, and then infer *a posteriori* the impact of the reduction process on the closed-loop stability and performance. While the use of the synthesis Riccati equation solutions has already been suggested [13, 10], these ideas have not so far been translated into a method with an underpinning theoretical basis. For the normalized LQG and $\mathcal{H}_\infty$ problems, controller reduction based on balancing with respect to the synthesis Riccati equation solutions are known to work in practice, but no *a priori* bounds have been provided which justify fully the method.

This paper is a generalization of the controller reduction procedure in [9] where the filtering and control Riccati equation solutions for the normalized LQG problem are shown to be equal to the controllability and observability Gramians of the left and right coprime factorizations of the plant and controller, respectively. The approach is based on solving a normalized LQG problem and then using an *a posteriori* bound on the $\mathcal{H}_\infty$ norm of the normalized closed-loop transfer matrix to obtain a condition for closed-loop stability and an upper bound on the performance degradation. A major contribution of [9] is to emphasize the relevance of relative controller reduction to closed-loop controller approximation when the controller is represented in coprime factor form.

Motivated by [9], model and controller reduction procedures were developed in [4] for multi-objective synthesis problems using LMI techniques. Guaranteed *a priori* bounds were obtained using augmented coprime factorizations of the model and controller that utilized the closed-loop Riccati inequalities. These bounds were used to derive conditions on stability and performance degradation similar to the ones developed here.

Related to this work is that in [3] and [8] where the emphasis is on performance preserving controller reduction, rather than bounding the performance degradation resulting from a controller reduction step.

The coprime factor controller reduction problem was investigated in [16] where an additive reduction formulation was used to derive *a posteriori* conditions on the stability and performance degradation of the normalized closed-loop that are tighter than those in [11]. In contrast, our approach demonstrates that if the inverse-weighted balanced truncation technique is used, it is possible to obtain *a priori* conditions for stability and bounds on the resulting performance degradation. A full version of this paper can be found in [6].

The notation used is standard and follows that of [17], except that real rational transfer functions will be represented by boldface letters with the dependence on s mostly suppressed.

2 Preliminaries

In this section we review the relative error model reduction problem [15]. For a transfer matrix $\boldsymbol{G}$, the relative approximation of $\boldsymbol{G}$ is defined as $\hat{\boldsymbol{G}} = (I + \boldsymbol{\Delta})\boldsymbol{G}$ where $\boldsymbol{\Delta}$ is the relative error function. This problem is a special case of the frequency-weighted model reduction problem [7] where the model is frequency weighted using its own inverse. In the sequel we make extensive use of the following result, which is a generalization of the inverse-weighted model reduction method of [15], and gives *a priori* error bound for the relative model approximation.

Theorem 2.1 *[6] Let* $\boldsymbol{G} \overset{s}{=} (A, B, C, D) \in \mathcal{RH}_\infty$ *be square and minimum phase and assume also that* $\det D \neq 0$ *so that* $\boldsymbol{G}^{-1} \overset{s}{=} (A - BD^{-1}C, BD^{-1}, -D^{-1}C, D^{-1})$ *is defined. Let* $P = P' \geq 0$ *and* $Q = Q' \geq 0$ *be generalized controllability and observability Gramians of the realizations of* $\boldsymbol{G}^{-1}$ *and* $\boldsymbol{G}$, *respectively, so that*

$$(A - BD^{-1}C)P + P(A - BD^{-1}C)' + (BD^{-1})(BD^{-1})' \leq 0$$
$$A'Q + QA + C'C \leq 0.$$

Suppose that the realization for $\boldsymbol{G}$ *is inverse-weighted balanced, so that*

$$\begin{aligned} P = Q &= \operatorname{diag}(\Sigma_1, \Sigma_2) \\ &= \operatorname{diag}(\sigma_1 I_{s_1}, \ldots, \sigma_r I_{s_r}, \sigma_{r+1} I_{s_{r+1}}, \ldots, \sigma_N I_{s_N}) \end{aligned} \tag{1}$$

where $\sigma_1 > \cdots > \sigma_N \geq 0$, *and that it is partitioned compatibly with* Σ_1 *and* Σ_2 *as*

$$\boldsymbol{G} \overset{s}{=} \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right].$$

Then $\hat{\boldsymbol{G}} \overset{s}{=} (A_{11}, B_1, C_1, D) = (I + \boldsymbol{\Delta})\boldsymbol{G}$ *is stable and minimum phase where* $\boldsymbol{\Delta} \in \mathcal{RH}_\infty$ *satisfies* $\|\boldsymbol{\Delta}\|_\infty \leq \delta$ *with*

$$\delta := -1 + \prod_{i=r+1}^{N} (1 + 2\sigma_i\sqrt{1 + \sigma_i^2} + 2\sigma_i^2). \tag{2}$$

3 Normalized Coprime Factor $\mathcal{H}_\infty$ Controller Reduction

In this section we derive guaranteed *a priori* stability and performance bounds when a normalized coprime factor $\mathcal{H}_\infty$ controller [11] is replaced by a reduced-order approximation.

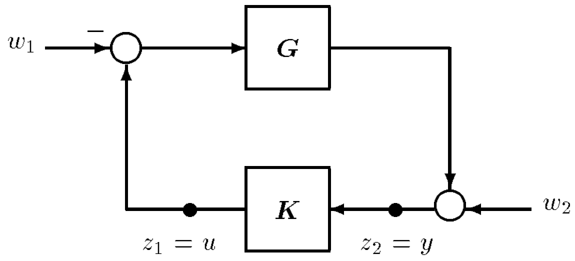

Figure 1: The normalized configuration $\mathcal{H}(\boldsymbol{G}, \boldsymbol{K})$.

Let $\boldsymbol{G}$ be a given plant and consider the normalized feedback configuration shown in Figure 1 [11, 17]. The closed-loop transfer matrix from $[w_1^T \;\; w_2^T]^T$ to $[z_1^T \;\; z_2^T]^T$ is given by

$$\mathcal{H}(\boldsymbol{G}, \boldsymbol{K}) = \left[\begin{array}{c} \boldsymbol{K} \\ I \end{array}\right](I - \boldsymbol{G}\boldsymbol{K})^{-1}\left[\begin{array}{cc} -\boldsymbol{G} & I \end{array}\right].$$

For any $\gamma > \gamma_{opt} = \inf\{\|\mathcal{H}(\boldsymbol{G}, \boldsymbol{K})\|_\infty : \boldsymbol{K}$ stabilizes $\boldsymbol{G}\}$, the suboptimal $\mathcal{H}_\infty$ control problem is to find all internally stabilizing controllers $\boldsymbol{K}$ such that $\|\mathcal{H}(\boldsymbol{G}, \boldsymbol{K})\|_\infty \leq \gamma$. Any such $\boldsymbol{K}$ is called a $\gamma-$suboptimal controller. The next lemma summarizes the solution of the suboptimal problem [11, 17].

Lemma 3.1 *[6] Let* $\boldsymbol{G}$ *have a minimal state-space realization* $\boldsymbol{G} \overset{s}{=} (A, B, C, 0)$. *Then there exist unique, stabilizing, positive definite solutions* X_2 *and* Y_2 *to the algebraic Riccati equations*

$$A'X_2 + X_2A - X_2BB'X_2 + C'C = 0 \tag{3}$$
$$AY_2 + Y_2A' - Y_2C'CY_2 + BB' = 0 \tag{4}$$

respectively, and $\gamma_{opt} = \sqrt{1 + \lambda_{\max}(X_2Y_2)} \geq 1$. *For any* $\gamma > \gamma_{opt} \geq 1$ *define* $\beta := \sqrt{1 - \gamma^{-2}}$ *so that* $0 < \beta \leq 1$. *Then all* $\gamma-$*suboptimal controllers are given by*

$$\boldsymbol{K} = (\boldsymbol{\Theta}_{11}\boldsymbol{Q} + \boldsymbol{\Theta}_{12})(\boldsymbol{\Theta}_{21}\boldsymbol{Q} + \boldsymbol{\Theta}_{22})^{-1}, \tag{5}$$

for $\boldsymbol{Q} \in \mathcal{RH}_\infty, \|\boldsymbol{Q}\|_\infty < \gamma$, *where*

$$\boldsymbol{\Theta} = \left[\begin{array}{cc} \boldsymbol{\Theta}_{11} & \boldsymbol{\Theta}_{12} \\ \boldsymbol{\Theta}_{21} & \boldsymbol{\Theta}_{22} \end{array}\right] \overset{s}{=} \left[\begin{array}{c|cc} A_\Theta & ZB & \beta^{-1}ZY_2C' \\ \hline -\beta^{-2}B'X_2 & I & 0 \\ \beta^{-2}C & 0 & \beta^{-1}I \end{array}\right]$$

$$\boldsymbol{\Theta}^{-1} \overset{s}{=} \left[\begin{array}{c|cc} A_{\Theta^{-1}} & ZB & ZY_2C' \\ \hline \beta^{-2}B'X_2 & I & 0 \\ -\beta^{-1}C & 0 & \beta I \end{array}\right]$$

with $\boldsymbol{\Theta}, \boldsymbol{\Theta}^{-1} \in \mathcal{RH}_\infty, A_\Theta = A - BB'X_2, A_{\Theta^{-1}} = Z(A - Y_2C'C)Z^{-1}$ *and* $Z = (I - \gamma^{-2}\beta^{-2}Y_2X_2)^{-1}$.

Proof: The proof can be found in a slightly different (but equivalent) form in [11]. The A-matrix for $\boldsymbol{\Theta}^{-1}$ is obtained from the realization of $\boldsymbol{\Theta}$ as

$$A - BB'X_2 + \beta^{-2}ZBB'X_2 - \beta^{-2}ZY_2C'C = Z(A - Y_2C'C)Z^{-1}$$

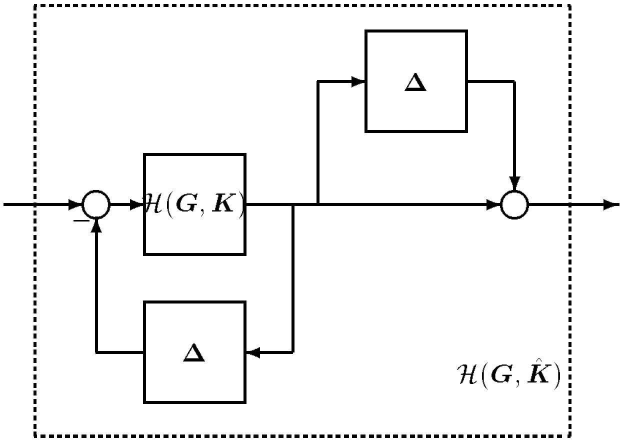

Figure 2: The normalized feedback configuration perturbation.

where the second expression follows from the fact that $A-Y_2C'C=(I+Y_2X_2)(A-BB'X_2)(I+Y_2X_2)^{-1}$ [2] and a simple manipulation of equations (3) and (4). ■

Remark 3.1 *Taking* $\boldsymbol{Q}=0$ *in (5) gives the central controller in the form of the right coprime factorization* $\boldsymbol{K}_{11}=\boldsymbol{\Theta}_{12}\boldsymbol{\Theta}_{22}^{-1}\overset{s}{=}(A-BB'X_2-\beta^{-2}ZY_2C'C, ZY_2C', -\beta^{-2}B'X_2, 0)$.

The following result shows that we can perform a relative reduction on a coprime factorization of any $\gamma-$suboptimal controller $\boldsymbol{K}$. It is similar to the one stated by [9], except that the coprime factorization of $\boldsymbol{G}$ is not required. It is also given in a form that allows a feedback interpretation of the relative error function.

Theorem 3.1 *[6] Let* $\boldsymbol{K}$ *be a* $\gamma-$*suboptimal controller for* $\boldsymbol{G}$. *Suppose that* $\boldsymbol{K}=\boldsymbol{U}\boldsymbol{V}^{-1}$ *is a right coprime factorization of* $\boldsymbol{K}$ *and let* $\begin{bmatrix}\hat{\boldsymbol{U}}\\ \hat{\boldsymbol{V}}\end{bmatrix}=(I+\boldsymbol{\Delta})\begin{bmatrix}\boldsymbol{U}\\ \boldsymbol{V}\end{bmatrix}$ *be a relative approximation of* $\begin{bmatrix}\boldsymbol{U}\\ \boldsymbol{V}\end{bmatrix}$. *Define* $\hat{\boldsymbol{K}}=\hat{\boldsymbol{U}}\hat{\boldsymbol{V}}^{-1}$. *Then*

$$\mathcal{H}(\boldsymbol{G},\hat{\boldsymbol{K}})=(I+\boldsymbol{\Delta})\mathcal{H}(\boldsymbol{G},\boldsymbol{K})[I+\boldsymbol{\Delta}\mathcal{H}(\boldsymbol{G},\boldsymbol{K})]^{-1}. \quad (6)$$

Furthermore,

$$\boldsymbol{\Delta}\in\mathcal{RH}_\infty \text{ and } \|\boldsymbol{\Delta}\|_\infty<\gamma^{-1}\Rightarrow\mathcal{H}(\boldsymbol{G},\hat{\boldsymbol{K}})\in\mathcal{RH}_\infty. \quad (7)$$

Finally, if $\|\boldsymbol{\Delta}\|_\infty<\gamma^{-1}$ *then*

$$\|\mathcal{H}(\boldsymbol{G},\hat{\boldsymbol{K}})\|_\infty<\frac{\gamma}{1-\|\boldsymbol{\Delta}\|_\infty\gamma}.$$

The effect of relative controller reduction on the closed-loop is shown in Figure 2. Note that the stability condition and the bound on performance degradations require the evaluation of $\|\boldsymbol{\Delta}\|_\infty$. The following result gives an *a priori* bound on $\|\boldsymbol{\Delta}\|_\infty$ by using the inverse-weighted balanced truncation method of Theorem 2.1 on the controller generator $\boldsymbol{\Theta}$.

Theorem 3.2 *[6] Let all variables be as defined in Lemma 3.1. Let* $P=P'\geq 0$ *be the controllability Gramian of the realization of* $\boldsymbol{\Theta}^{-1}$ *and let* $Q=Q'\geq 0$ *be the observability Gramian of the realization of* $\boldsymbol{\Theta}$ *so that*

$$\begin{aligned} A_{\Theta^{-1}}P+PA'_{\Theta^{-1}}+ZBB'Z'+ZY_2C'CY_2Z'=0\\ A'_\Theta Q+QA_\Theta+\beta^{-4}X_2BB'X_2+\beta^{-4}C'C=0.\end{aligned}$$

Then

1. *The Gramians* P *and* Q *are given by*

$$\begin{aligned} P &= ZY_2Z' \quad (8)\\ Q &= \beta^{-4}X_2. \quad (9)\end{aligned}$$

2. *Suppose that the realization for* $\boldsymbol{\Theta}$ *is inverse-weighted balanced, so that (1) holds, and that it is partitioned compatibly with* Σ_1 *and* Σ_2 *as*

$$\boldsymbol{\Theta}\overset{s}{=}\left[\begin{array}{cc|c} A_{11} & A_{12} & B_1\\ A_{21} & A_{22} & B_2\\ \hline C_1 & C_2 & D_\Theta\end{array}\right] \quad (10)$$

where $D_\Theta=\mathrm{diag}(I,\beta I)$. *Then*

$$\hat{\boldsymbol{\Theta}}=\begin{bmatrix}\hat{\boldsymbol{\Theta}}_{11} & \hat{\boldsymbol{\Theta}}_{12}\\ \hat{\boldsymbol{\Theta}}_{21} & \hat{\boldsymbol{\Theta}}_{22}\end{bmatrix}\overset{s}{=}\left[\begin{array}{c|c} A_{11} & B_1\\ \hline C_1 & D_\Theta\end{array}\right] \quad (11)$$

is stable and minimum phase and $\hat{\boldsymbol{\Theta}}=(I+\boldsymbol{\Delta})\boldsymbol{\Theta}$ *where* $\boldsymbol{\Delta}\in\mathcal{RH}_\infty$ *satisfies* $\|\boldsymbol{\Delta}\|_\infty\leq\delta$ *where* δ *is defined in (2) and*

$$\sigma_i^2=\frac{\beta^{-4}\nu_i^2}{(1-\gamma^{-2}\beta^{-2}\nu_i^2)^2}.$$

where $\nu_1^2,\ldots,\nu_N^2$ *are the (distinct) eigenvalues of* X_2Y_2.

Let $\hat{\boldsymbol{\Theta}}=(I+\boldsymbol{\Delta})\boldsymbol{\Theta}$, with $\|\boldsymbol{\Delta}\|_\infty\leq\delta$, be obtained using the relative reduction procedure of Theorem 3.2 so that $\begin{bmatrix}\hat{\boldsymbol{\Theta}}_{12}\\ \hat{\boldsymbol{\Theta}}_{22}\end{bmatrix}=(I+\boldsymbol{\Delta})\begin{bmatrix}\boldsymbol{\Theta}_{12}\\ \boldsymbol{\Theta}_{22}\end{bmatrix}$. If we take $\begin{bmatrix}\boldsymbol{U}\\ \boldsymbol{V}\end{bmatrix}=\begin{bmatrix}\boldsymbol{\Theta}_{12}\\ \boldsymbol{\Theta}_{22}\end{bmatrix}$ in Theorem 3.1, we can give a controller reduction procedure with guaranteed *a priori* stability and performance bounds when the reduced order controller is implemented.

Corollary 3.1 *Let all variables be as defined in Lemma 3.1 and let* $\boldsymbol{K}_{11}=\boldsymbol{\Theta}_{12}\boldsymbol{\Theta}_{22}^{-1}$ *be the central controller defined in Remark 3.1. Suppose that* T *is an inverse-weighted balancing transformation for* $\boldsymbol{\Theta}$ *so that* $T^{-1}ZY_2Z'(T^{-1})'=\beta^{-4}T'X_2T=\mathrm{diag}(\Sigma_1,\Sigma_2)$

$$=\mathrm{diag}(\sigma_1 I_{s_1},\ldots,\sigma_r I_{s_r},\sigma_{r+1}I_{s_{r+1}},\ldots,\sigma_N I_{s_N})$$

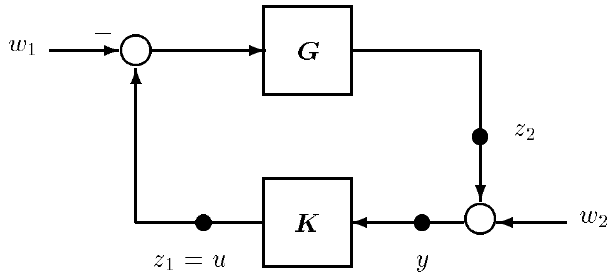

Figure 3: The normalized configuration $\breve{\mathcal{H}}(\boldsymbol{G}, \boldsymbol{K})$.

with $\sigma_1 > \cdots > \sigma_N \geq 0$. Apply the similarity transformation T to $\boldsymbol{K}_{11}$ and partition compatibly with Σ_1 and Σ_2 as

$$\boldsymbol{K}_{11} \overset{s}{=} \left[\begin{array}{c|c} T^{-1}(A_\Theta - \beta^{-2} Z Y_2 C' C)T & T^{-1} Z Y_2 C' \\ \hline -\beta^{-2} B' X_2 T & 0 \end{array}\right]$$
$$\overset{s}{=}: \left[\begin{array}{cc|c} \hat{A}_{11} & \hat{A}_{12} & \hat{B}_1 \\ \hat{A}_{21} & \hat{A}_{22} & \hat{B}_2 \\ \hline \hat{C}_1 & \hat{C}_2 & 0 \end{array}\right].$$

Define the reduced order controller $\hat{\boldsymbol{K}}_{11} := \hat{\boldsymbol{\Theta}}_{12}\hat{\boldsymbol{\Theta}}_{22}^{-1} \overset{s}{=} (\hat{A}_{11}, \hat{B}_1, \hat{C}_1, 0)$. Then $\mathcal{H}(\boldsymbol{G}, \hat{\boldsymbol{K}}_{11})$ is stable if $\delta < \gamma^{-1}$, in which case

$$\|\mathcal{H}(\boldsymbol{G}, \hat{\boldsymbol{K}}_{11})\|_\infty < \frac{\gamma}{1 - \delta\gamma}.$$

4 Normalized $\mathcal{H}_\infty$ Controller Reduction

In this section we derive guaranteed *a priori* stability and performance bounds when a normalized $\mathcal{H}_\infty$ controller [13, 12] is replaced by a reduced-order approximation. This is a slightly modified normalized $\mathcal{H}_\infty$ control problem compared to the one in Section 3 and is shown in Figure 3.

Let $\boldsymbol{G}$ be a given plant. The closed-loop transfer matrix from $[w_1^T \;\; w_2^T]^T$ to $[z_1^T \;\; z_2^T]^T$ is given by

$$\begin{aligned} \breve{\mathcal{H}}(\boldsymbol{G}, \boldsymbol{K}) &= \mathcal{H}(\boldsymbol{G}, \boldsymbol{K}) + J \\ &= \begin{bmatrix} -\boldsymbol{K}(I - \boldsymbol{G}\boldsymbol{K})^{-1}\boldsymbol{G} & \boldsymbol{K}(I - \boldsymbol{G}\boldsymbol{K})^{-1} \\ -(I - \boldsymbol{G}\boldsymbol{K})^{-1}\boldsymbol{G} & \boldsymbol{G}\boldsymbol{K}(I - \boldsymbol{G}\boldsymbol{K})^{-1} \end{bmatrix} \end{aligned}$$

where $J := \begin{bmatrix} 0 & 0 \\ 0 & -I \end{bmatrix}$. The next lemma summarizes the solution of the suboptimal problem [13, 17].

Lemma 4.1 *[6] Let $\boldsymbol{G}$ have a minimal state-space realization $\boldsymbol{G} \overset{s}{=} (A, B, C, 0)$. For any $\gamma > \breve{\gamma}_{opt} := \inf\{\|\breve{\mathcal{H}}(\boldsymbol{G}, \boldsymbol{K})\|_\infty : \boldsymbol{K}$ stabilizes $\boldsymbol{G}\}$ there exist unique, stabilizing, positive definite solutions X_∞ and Y_∞ to the algebraic Riccati equations*

$$\begin{aligned} A'X_\infty + X_\infty A + C'C - (1 - \gamma^{-2})X_\infty BB'X_\infty &= 0 \\ AY_\infty + Y_\infty A' + BB' - (1 - \gamma^{-2})Y_\infty C'CY_\infty &= 0 \end{aligned}$$

respectively, such that $\lambda_{\max}(X_\infty Y_\infty) < \gamma^2$. Furthermore, all $\gamma-$suboptimal controllers are given as (5) where

$$\boldsymbol{\Theta} = \begin{bmatrix} \boldsymbol{\Theta}_{11} & \boldsymbol{\Theta}_{12} \\ \boldsymbol{\Theta}_{21} & \boldsymbol{\Theta}_{22} \end{bmatrix} \overset{s}{=} \left[\begin{array}{c|cc} A_\Theta & Z_\infty B & Z_\infty Y_\infty C' \\ \hline -B'X_\infty & I & 0 \\ C & 0 & I \end{array}\right]$$

$$\boldsymbol{\Theta}^{-1} \overset{s}{=} \left[\begin{array}{c|cc} A_{\Theta^{-1}} & Z_\infty B & Z_\infty Y_\infty C' \\ \hline B'X_\infty & I & 0 \\ -C & 0 & I \end{array}\right]$$

with $\boldsymbol{\Theta}, \boldsymbol{\Theta}^{-1} \in \mathcal{RH}_\infty, A_\Theta = A - (1 - \gamma^{-2})BB'X_\infty, A_{\Theta^{-1}} = Z_\infty(A - (1 - \gamma^{-2})Y_\infty C'C)Z_\infty^{-1}$ and $Z_\infty = (I - \gamma^{-2}Y_\infty X_\infty)^{-1}$.

The implications of performing a relative reduction on a coprime factorization of any $\gamma-$suboptimal controller $\boldsymbol{K}$ is analyzed in the following theorem.

Theorem 4.1 *[6] Let $\boldsymbol{K}$ be any $\gamma-$suboptimal controller for $\boldsymbol{G}$. Suppose that $\boldsymbol{K} = \boldsymbol{U}\boldsymbol{V}^{-1}$ is a right coprime factorization of $\boldsymbol{K}$ and let $\begin{bmatrix} \hat{\boldsymbol{U}} \\ \hat{\boldsymbol{V}} \end{bmatrix} = (I + \boldsymbol{\Delta}) \begin{bmatrix} \boldsymbol{U} \\ \boldsymbol{V} \end{bmatrix}$ be a relative approximation of $\begin{bmatrix} \boldsymbol{U} \\ \boldsymbol{V} \end{bmatrix}$. Define $\hat{\boldsymbol{K}} = \hat{\boldsymbol{U}}\hat{\boldsymbol{V}}^{-1}$. Then*

$$\breve{\mathcal{H}}(\boldsymbol{G}, \hat{\boldsymbol{K}}) = [\breve{\mathcal{H}}(\boldsymbol{G}, \boldsymbol{K}) - \breve{J}\boldsymbol{\Delta}\mathcal{H}(\boldsymbol{G}, \boldsymbol{K})][I + \boldsymbol{\Delta}\mathcal{H}(\boldsymbol{G}, \boldsymbol{K})]^{-1} \tag{12}$$

where $\breve{J} = \begin{bmatrix} -I & 0 \\ 0 & 0 \end{bmatrix}$. Furthermore,

$$\|\boldsymbol{\Delta}\|_\infty < (1+\gamma)^{-1} \text{ and } \boldsymbol{\Delta} \in \mathcal{RH}_\infty \Rightarrow \breve{\mathcal{H}}(\boldsymbol{G}, \hat{\boldsymbol{K}}) \in \mathcal{RH}_\infty.$$

Finally, if $\|\boldsymbol{\Delta}\|_\infty < (\gamma + 1)^{-1}$ then

$$\|\breve{\mathcal{H}}(\boldsymbol{G}, \hat{\boldsymbol{K}})\|_\infty < \frac{\gamma + (1 + \gamma)\|\boldsymbol{\Delta}\|_\infty}{1 - (1 + \gamma)\|\boldsymbol{\Delta}\|_\infty}. \tag{13}$$

For the remainder of this section, we assume that $\gamma > \max\{1, \breve{\gamma}_{opt}\}$. As argued in [12], this is not a severe restriction since $\breve{\gamma}_{opt} < 1$ if and only if $\boldsymbol{G}$ is asymptotically stable and has Hankel norm less than 1. The following result gives an *a priori* bound on $\|\boldsymbol{\Delta}\|_\infty$ by using the inverse-weighted balanced truncation method of Theorem 2.1 on the controller generator $\boldsymbol{\Theta}$ and is similar to Theorem 3.2.

Theorem 4.2 *[6] Let all variables be as defined in Lemma 4.1. Suppose that $\gamma > \max\{1, \breve{\gamma}_{opt}\}$ and define $\beta := \sqrt{1 - \gamma^{-2}}$ so that $0 < \beta \leq 1$. Let $P = P' \geq 0$ be a generalized controllability Gramian of the realization of $\boldsymbol{\Theta}^{-1}$ and $Q = Q' \geq 0$ be a generalized observability Gramian of the realization of $\boldsymbol{\Theta}$ so that*

$$\begin{gathered} A_{\Theta^{-1}}P + PA'_{\Theta^{-1}} + Z_\infty BB'Z'_\infty + Z_\infty Y_\infty C'CY_\infty Z'_\infty \leq 0 \\ A'_\Theta Q + QA_\Theta + X_\infty BB'X_\infty + C'C \leq 0. \end{gathered}$$

Then

1. *The generalized Gramians P and Q can be taken as*

$$\begin{aligned} P &= \beta^{-2} Z_\infty Y_\infty Z'_\infty && (14) \\ Q &= \beta^{-2} X_\infty . && (15) \end{aligned}$$

2. *Suppose that the realization for $\boldsymbol{\Theta}$ is inverse-weighted balanced, so that (1) holds, and that it is partitioned compatibly with Σ_1 and Σ_2 as (10). Then $\hat{\boldsymbol{\Theta}}$ defined in (11) is stable and minimum phase and $\hat{\boldsymbol{\Theta}} = (I + \boldsymbol{\Delta})\boldsymbol{\Theta}$ where $\boldsymbol{\Delta} \in \mathcal{RH}_\infty$ satisfies $\|\boldsymbol{\Delta}\|_\infty \le \delta$ where δ is defined in (2) with*

$$\sigma_i^2 = \frac{\beta^{-4}\nu_i^2}{(1-\gamma^{-2}\nu_i^2)^2}$$

where $\nu_1^2, \ldots, \nu_N^2$ are the (distinct) eigenvalues of $X_\infty Y_\infty$.

Similar to Corollary 3.1, we use Theorems 3.1 and 4.2 to give a controller reduction procedure with guaranteed *a priori* stability and performance bounds.

Corollary 4.1 *Let all variables be as define in Theorem 4.2 and let $\boldsymbol{K}_{11} = \boldsymbol{\Theta}_{12}\boldsymbol{\Theta}_{22}^{-1}$ be the central controller. Suppose that T is an inverse-weighted balancing transformation for $\boldsymbol{\Theta}$ so that $\beta^{-2}T^{-1}Z_\infty Y_\infty Z'_\infty (T^{-1})' = \beta^{-2}T' X_\infty T = \operatorname{diag}(\Sigma_1, \Sigma_2)$*

$$= \operatorname{diag}(\sigma_1 I_{s_1}, \ldots, \sigma_r I_{s_r}, \sigma_{r+1} I_{s_{r+1}}, \ldots, \sigma_N I_{s_N})$$

with $\sigma_1 > \cdots > \sigma_N \ge 0$. Apply the similarity transformation T to $\boldsymbol{K}_{11}$ and partition compatibly with Σ_1 and Σ_2 as

$$\begin{aligned} \boldsymbol{K}_{11} &\overset{s}{=} \left[\begin{array}{c|c} T^{-1}(A_\Theta - Z_\infty Y_\infty C'C)T & T^{-1}Z_\infty Y_\infty C' \\ \hline -B'X_\infty T & 0 \end{array}\right] \\ &\overset{s}{=}: \left[\begin{array}{cc|c} \hat{A}_{11} & \hat{A}_{12} & \hat{B}_1 \\ \hat{A}_{21} & \hat{A}_{22} & \hat{B}_2 \\ \hline \hat{C}_1 & \hat{C}_2 & 0 \end{array}\right]. \end{aligned}$$

Define the reduced controller $\hat{\boldsymbol{K}}_{11} = \hat{\boldsymbol{\Theta}}_{12}\hat{\boldsymbol{\Theta}}_{22}^{-1} \overset{s}{=} (\hat{A}_{11}, \hat{B}_1, \hat{C}_1, 0)$. Then $\breve{\mathcal{H}}(\boldsymbol{G}, \hat{\boldsymbol{K}}_{11}) \in \mathcal{RH}_\infty$ if $\delta < (1+\gamma)^{-1}$ in which case

$$\|\breve{\mathcal{H}}(\boldsymbol{G}, \hat{\boldsymbol{K}}_{11})\|_\infty < \frac{\gamma + (1+\gamma)\delta}{1-(1+\gamma)\delta}.$$

Remark 4.1 *There are two sources of conservatism in the stability condition and performance degradation bound for $\breve{\mathcal{H}}(\boldsymbol{G}, \hat{\boldsymbol{K}})$ compared to $\mathcal{H}(\boldsymbol{G}, \hat{\boldsymbol{K}})$. First, the bounds that are derived are related to $(\gamma + 1)$ rather than γ. Second, the* a priori *bound on $\|\boldsymbol{\Delta}\|_\infty$ uses the generalized Gramians P and Q which satisfy the inequalities of Theorem 4.2. However, in practice, that conservatism does not reflect on the quality of the controller approximation.*

Remark 4.2 *It is interesting to note that in the limit $\gamma \to \infty$, we have $X_\infty \to X_2, Y_\infty \to Y_2$ and $Z_\infty \to I$ [12]. Thus the normalized $\mathcal{H}_\infty$ control problem reduces to the normalized LQG problem. That is, $\boldsymbol{K}_{11}$ is the unique stabilizing controller that minimizes $\|\breve{\mathcal{H}}(\boldsymbol{G}, \boldsymbol{K})\|_2$. When this controller is approximated using the method given in this section, we can use (12) to write the stability condition as $1 + \|\breve{\mathcal{H}}(\boldsymbol{G}, \boldsymbol{K}_{11})\|_\infty < \delta^{-1}$ and the performance bound as*

$$\|\breve{\mathcal{H}}(\boldsymbol{G}, \hat{\boldsymbol{K}}_{11})\|_2 < \frac{\|\boldsymbol{H}\|_2 + \|\boldsymbol{\Delta}\|_2(1+\|\boldsymbol{H}\|_\infty)}{1-(1+\|\boldsymbol{H}\|_\infty)\delta)}$$

where $\boldsymbol{H} = \breve{\mathcal{H}}(\boldsymbol{G}, \boldsymbol{K}_{11})$, provided that $1 + \|\breve{\mathcal{H}}(\boldsymbol{G}, \boldsymbol{K}_{11})\|_\infty < \delta^{-1}$, where $\|\breve{\mathcal{H}}(\boldsymbol{G}, \boldsymbol{K}_{11})\|_\infty$ and $\|\boldsymbol{\Delta}\|_2$ need to be calculated a posteriori. *This can be considered as a justification for the LQG-based balance-and-truncate method of [10]. If $\|\mathcal{H}(\boldsymbol{G}, \boldsymbol{K}_{11})\|_\infty$ is calculated instead, then it is a simple exercise to show that the stability condition becomes $\|\mathcal{H}(\boldsymbol{G}, \boldsymbol{K}_{11})\|_\infty < \delta^{-1}$ and the performance bound becomes*

$$\|\breve{\mathcal{H}}(\boldsymbol{G}, \hat{\boldsymbol{K}}_{11})\|_2 < \frac{\|\breve{\mathcal{H}}(\boldsymbol{G}, \boldsymbol{K}_{11})\|_2 + \|\boldsymbol{\Delta}\|_2 \|\mathcal{H}(\boldsymbol{G}, \boldsymbol{K}_{11})\|_\infty}{1-\|\mathcal{H}(\boldsymbol{G}, \boldsymbol{K}_{11})\|_\infty \delta}$$

which reproduces the controller reduction results of [9] (See Theorem 17 in [9]).

5 Example

To demonstrate the effectiveness of the controller reduction procedure we consider synthesizing a reduced-order $\mathcal{H}_\infty$ controller for the HIMAT vehicle taken from the μ-Analysis and Synthesis Toolbox [1]. The model consists of two-input, two-output and four states that represents a scaled, linearized plant of a remotely piloted vehicle version of an advanced fighter. We used the same weighting functions for the loop shaping design suggested in [1].

For the normalized coprime factor design case we used $\gamma = 2.2952$ to synthesize a 6-th order $\mathcal{H}_\infty$ γ-suboptimal controller which is reduced using several methods and compared to the approach given here and the results are listed in Table 1 (see [17, 3] for details of these methods). Table 1 indicates that our approach seems to perform better than any of the other procedures.

For the same example, Table 2 gives $\|\boldsymbol{\Delta}\|_\infty$, its upper bound δ, estimates for $\|\mathcal{H}(\boldsymbol{G}, \hat{\boldsymbol{K}})\|_\infty$ using $\|\boldsymbol{\Delta}\|_\infty$ and δ, and compares the estimates with the actual value of $\|\mathcal{H}(\boldsymbol{G}, \hat{\boldsymbol{K}})\|_\infty$. The table demonstrates that the bounds suggested are good indicators for the quality of the truncation when the controller is moderately reduced. The higher the level of truncation, the more conservative the bound on performance degradation.

Order of $\hat{\boldsymbol{K}}$	5	4	3	2
$\lVert \boldsymbol{K} - \hat{\boldsymbol{K}} \rVert_\infty$	2.296	2.293	2.371	1363
$\left\lVert \begin{bmatrix} \boldsymbol{\Theta}_{12} \\ \boldsymbol{\Theta}_{22} \end{bmatrix} - \begin{bmatrix} \hat{\boldsymbol{U}} \\ \hat{\boldsymbol{V}} \end{bmatrix} \right\rVert_\infty$	2.295	2.297	2.413	16.857
$\lVert [\tilde{\boldsymbol{\Theta}}_{21}\ \tilde{\boldsymbol{\Theta}}_{22}] - [\hat{\tilde{\boldsymbol{U}}}\ \hat{\tilde{\boldsymbol{V}}}] \rVert_\infty$	2.295	2.296	2.378	14.797
$\lVert (I - \boldsymbol{G}\boldsymbol{K})^{-1} \boldsymbol{G}(\boldsymbol{K} - \hat{\boldsymbol{K}}) \rVert_\infty$	2.296	2.294	***U***	***U***
$\left\lVert [-\tilde{N}\ \tilde{M}]\left(\begin{bmatrix} \boldsymbol{\Theta}_{12} \\ \boldsymbol{\Theta}_{22} \end{bmatrix} - \begin{bmatrix} \hat{\boldsymbol{U}} \\ \hat{\boldsymbol{V}} \end{bmatrix}\right) \right\rVert_\infty$	2.295	2.297	11.779	7.884
$\left\lVert \left([\tilde{\boldsymbol{\Theta}}_{21}\ \tilde{\boldsymbol{\Theta}}_{22}] - [\hat{\tilde{\boldsymbol{U}}}\ \hat{\tilde{\boldsymbol{V}}}]\right) \begin{bmatrix} -N \\ M \end{bmatrix} \right\rVert_\infty$	2.295	2.296	2.379	35.759
$\left\lVert W \boldsymbol{\Theta}^{-1} \left(\begin{bmatrix} \boldsymbol{\Theta}_{12} \\ \boldsymbol{\Theta}_{22} \end{bmatrix} - \begin{bmatrix} \hat{\boldsymbol{U}} \\ \hat{\boldsymbol{V}} \end{bmatrix}\right) \right\rVert_\infty$	2.295	2.296	U	U
$\lVert ([\tilde{\boldsymbol{\Theta}}_{21}\ \tilde{\boldsymbol{\Theta}}_{22}] - [\hat{\tilde{\boldsymbol{U}}}\ \hat{\tilde{\boldsymbol{V}}}]) \tilde{\boldsymbol{\Theta}}^{-1} W \rVert_\infty$	2.295	2.296	U	U
$\lVert (\hat{\boldsymbol{\Theta}} - \boldsymbol{\Theta}) \boldsymbol{\Theta}^{-1} \rVert_\infty$	2.295	2.296	U	U
$\lVert \boldsymbol{\Theta}^{-1} (\hat{\boldsymbol{\Theta}} - \boldsymbol{\Theta}) \rVert_\infty$ (Our algorithm)	**2.296**	**2.293**	**2.359**	**7.564**

Table 1: $\|\mathcal{H}(\boldsymbol{G}, \hat{\boldsymbol{K}})\|_\infty$: U–unstable, $W = \mathrm{diag}(\gamma^{-1}I, I)$.

Order of $\hat{\boldsymbol{K}}$	5	4	3	2
$\lVert \boldsymbol{\Delta} \rVert_\infty$	0.001	0.002	0.196	0.996
δ	0.001	0.003	0.220	1.934
Est. $\lVert \mathcal{H}(\boldsymbol{G}, \hat{\boldsymbol{K}}) \rVert_\infty$ using $\lVert \boldsymbol{\Delta} \rVert_\infty$	2.298	2.307	4.174	$\lVert \boldsymbol{\Delta} \rVert_\infty^{-1} < \gamma$
Est. $\lVert \mathcal{H}(\boldsymbol{G}, \hat{\boldsymbol{K}}) \rVert_\infty$ using δ	2.298	2.309	4.638	$\delta > \gamma^{-1}$
Actual $\lVert \mathcal{H}(\boldsymbol{G}, \hat{\boldsymbol{K}}) \rVert_\infty$	2.296	2.293	2.359	7.564

Table 2:

Remark 5.1 *An inspection of Table 1 shows that, apart from providing* a priori *stability and error bounds, the quality of our approximation is at least as good as any of the other techniques. Using the energy propagation concept introduced by [14], and applied to the closed-loop model reduction algorithm of [5], it can be shown that the eigenvalues of the Gramians used in our controller balancing and truncation have a controller input/output energy interpretation.*

6 Conclusions

It is shown that the Gramians (or generalized Gramians) for the normalized controller generator $\boldsymbol{\Theta}$ are given in terms of the synthesis Riccati equation solutions. This is exploited to give an *a priori* upper bound on the relative error when the inverse-balanced truncation technique is used to approximate $\boldsymbol{\Theta}$. This approximation induces a relative reduction on the coprime factors of the normalized central controller. Combined with the bound on the relative approximation error, this gives guaranteed *a priori* bounds on closed-loop stability and the resulting performance degradation. A full theoretical justification for controller reduction techniques based on balancing with respect to the synthesis Riccati solutions is thus established. The technique also provides an intimate link between the controller synthesis and controller reduction problems.

References

[1] G.J. Balas, J.C. Doyle, K. Glover, A. Packard, and R. Smith. *μ-Analysis and Synthesis Toolbox (μ-Tools).* Mathworks, Natick, MA, 1995.

[2] R. Bucy. The Riccati equation and its bounds. *J. Comput. Systems Sci.*, 6:343–353, 1972.

[3] X. Chen and K. Zhou. $\mathcal{H}_2$ and $\mathcal{H}_\infty$ controller reduction with guaranteed performance. In *Proc. IEEE Conf. Dec. & Control*, Kobe, Japan, 1996.

[4] H.M.H El-Zobaidi. *Model and Controller Reduction in Linear Robust Control.* PhD thesis, Imperial College of Science, Technology and Medicine, University of London, 1998.

[5] H.M.H. El-Zobaidi and I. Jaimoukha. Robust control and model and controller reduction of linear parameter varying systems. In *Proc. IEEE Conf. Dec. & Control*, Tampa, FL, 1998.

[6] H.M.H. El-Zobaidi, I.M. Jaimoukha, and D.J.N. Limebeer. Normalized $\mathcal{H}_\infty$ controller reduction with a priori bounds. Under review. *IEEE Trans. Automatic Control*, 2000.

[7] D. Enns. Model reduction with balanced realizations: An error bound and a frequency weighted generalization. In *Proc. IEEE Conf. Dec. & Control*, pages 127–132, Las Vegas, NV, Dec. 1984. IEEE Press, New York.

[8] P.J. Goddard and K. Glover. Controller approximation: approaches for preserving $\mathcal{H}_\infty$ performance. *IEEE Trans. Automatic Control*, 43(7):858–871, 1998.

[9] G. Gu. Model reduction with relative/multiplicative error bounds and relations to controller reduction. *IEEE Trans. Automatic Control*, 40(8):1478–1485, 1995.

[10] E. A. Jonkheere and L. M. Silverman. A new set of invariants for linear systems-application to reduced order compensator design. *IEEE Trans. Automatic Control*, 28(10):953–964, 1983.

[11] D. McFarlane and K. Glover. *Robust Controller Design Using Normalized Coprime Factor Plant Description*, volume 138 of *Lecture Notes in Control and Information Sciences.* Springer-Verlag, 1990.

[12] D. Mustafa and K. Glover. Controller reduction by $\mathcal{H}_\infty$-balanced truncation. *IEEE Trans. Automatic Control*, 36(6):668–682, 1991.

[13] D. Mustafa and K. Glover. *Minimum Entropy $\mathcal{H}_\infty$ Control.* Number 146 in Lecture Notes in control and Information Sciences. Springer-Verlag, 1991.

[14] S. Weiland. Balanced representations and approximation of linear systems. In *Proc. IEEE Conf. Dec. & Control*, pages 1334–1336, 1989.

[15] K. Zhou. Frequency-weighted L_∞ norm and optimal Hankel norm model reduction. *IEEE Trans. Automatic Control*, 40(10):1687–1699, 1995.

[16] K. Zhou and J. Chen. Performance bounds for coprime factor controller reductions. *Systems & Control Letts.*, 26(2):119–127, 1995.

[17] K. Zhou, J.C. Doyle, and K. Glover. *Robust and Optimal Control.* Prentice-Hall, Inc., Upper Saddle River, NJ, 1996.