Fliess Operators on L_p Spaces: Convergence and Continuity

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Abstract

Fliess operators as input-output mappings are particularly useful in a number of fundamental problems concerning nonlinear realization theory. In the classical analysis of these operators, certain growth conditions on the coefficients in their series representations insure uniform and absolute convergence, provided each admissible input is uniformly bounded by some fixed upperbound. In some emerging applications of this theory, however, it is more natural to consider other classes of inputs. In this paper, L_p function spaces are considered. In particular, growth conditions are developed which provide sufficient conditions for convergence and continuity, and insure that any realization of the operator yields a well defined state space model on the input space.

1. Introduction

The class of operators known as Fliess operators are particularly useful in a number of fundamental problems in realization theory for nonlinear input-output maps [1, 2, 3, 7, 14, 15]. They are characterized by a certain convergent functional series whose coefficients can be explicitly computed in terms of any corresponding state space realization, when one exists. In the classical analysis of these operators, growth conditions on the coefficients are normally assumed in order to guarantee uniform and absolute convergence of the series, provided each admissible input is uniformly bounded in absolute value by some fixed upperbound [2, 3, 4, 8, 9, 12]. In some emerging applications of this theory, however, it is more natural to consider other classes of inputs, for example, finite energy inputs [5, 6, 10, 11]. Hence the question arises as to whether there exists suitable growth conditions which at least insure the necessary convergence properties for these alternative input sets. In this paper, the function spaces $L_p[t_0, t_0+T]$, $p \geq 1$, are considered when T is both bounded and unbounded. In particular, growth conditions are developed which provide sufficient conditions for convergence and continuity, and insure that any realization of the operator yields a well defined state space model on the input space. It is demonstrated by example, however, that these conditions are not necessary.

Consider the set of input-output operators having m inputs and ℓ outputs. Let $I = \{0, 1, \ldots, m\}$, and let I^k be the set of all sequences $(i_k i_{k-1} \cdots i_1)$, where $i_r \in I$ for $1 \leq r \leq k$. For $k = 0$, I^0 denotes the set whose only element is the empty sequence ϕ. Let $I^* = \bigcup_{k\geq 0} I^k$. For any $\eta \in I^*$, $|\eta|$ is the number of symbols in η. A formal power series in the $m+1$ noncommutative variables $\{\theta_0, \theta_1, \ldots, \theta_m\}$ has the form:

$$c = \sum_{\eta \in I^*} c(\eta)\, w_\eta,$$

where $w_\eta = \theta_{i_k}\theta_{i_{k-1}} \ldots \theta_{i_1}$ if $\eta = (i_k i_{k-1} \cdots i_1)$, and $c(\eta) \in \mathbb{R}^\ell$. Observe that such a series is nothing but a mapping from I^* to $\mathbb{R}^\ell$. Given a set of smooth vector fields $\{g_0, g_1, \ldots, g_m\}$, a smooth $\mathbb{R}^\ell$-valued function h, and any $x_0 \in \mathbb{R}^n$, define the iterated Lie derivative of h for $\eta \in I^*$ as $L_{g_\eta} h(x_0) = L_{g_{i_1}} L_{g_{i_2}} \ldots L_{g_{i_k}} h(x_0)$.

Let $p \geq 1$ and $a < b$ be given. For a measurable function $u : [a, b] \to \mathbb{R}^m$, we write $\|u\|_p$ for $\max\{\|u_i\|_p : 1 \leq i \leq m\}$, where $\|u_i\|_p$ is the usual L_p norm for a measurable real-valued function, u_i, defined on $[a, b]$. Let $L_p^m[a, b]$ denote the set of all measurable functions defined on $[a, b]$ having a finite $\|\cdot\|_p$ norm. Clearly, a function $u = (u_1, \ldots, u_m) \in L_p^m[a, b]$ if and only if $u_i \in L_p[a, b]$ for each i, and $\|u\|_p \geq \|u_i\|_p$ for all i. For $r > 0$, define

$$B_p^m(r)[a, b] = \{u \in L_p^m[a, b], \ \ \|u\|_p \leq r\}.$$

For $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, define $|x| = \max\{|x_1|, |x_2|, \ldots, |x_n|\}$. Let $t_0 \in \mathbb{R}$ and $T > 0$ be given. For each $\eta \in I^*$, define inductively the mapping $E_\eta : L_1^m[t_0, t_0+T] \to \mathcal{C}[t_0, t_0+T]$ by $E_\phi = 1$, and

$$E_{i_k i_{k-1} \cdots i_1}[u](t) = \int_{t_0}^{t} u_{i_k}(\tau) E_{i_{k-1} \cdots i_1}[u](\tau)\, d\tau,$$

where $u_0(t) \equiv 0$. Then for each power series c, one can formally associate a corresponding Fliess operator defined by

$$F_c[u](t) = \sum_{\eta \in I^*} c(\eta)\, E_\eta[u](t).$$

In next section various local convergence and continuity theorems for such an operator are presented.

2. Convergence and Continuity Theorems: The Local Case

Two numbers p and q in $[1,\infty]$ are called conjugate exponents if

$$\frac{1}{p} + \frac{1}{q} = 1.$$

We adopt the convention that $1/\infty = 0$. Thus, 1 and ∞ are conjugate exponents.

Lemma 2.1 *Assume $p > 1$. Let p and q be conjugate exponents. Suppose for a given power series c there exists real numbers $K > 0$ and $M > 0$ such that*

$$|c(\eta)|^q \le K|\eta|!\, M^{|\eta|}, \qquad \forall\, \eta \in I^*. \tag{1}$$

Then for any $r > 0$, there exists a $T > 0$ such that for each $u \in B_p^m(r)[t_0, t_0+T]$, the series

$$y(t) = \sum_{\eta \in I^*} c(\eta) E_\eta[u](t)$$

converges absolutely and uniformly on $[t_0, t_0+T]$. Furthermore, $y(\cdot)$ is absolutely continuous on $[t_0, t_0+T]$.

Proof. The case of $p = \infty$ is available in the literature, see for example [3, 8, 9, 12]. Below we assume that $1 < p < \infty$. To prove the lemma, it is sufficient to consider only the situation where $\ell = 1$.

Fix $r, T > 0$ and pick any $u \in B_p^m(r)[t_0, t_0+T]$. Let $r_1 = \max\{r,\ T^{1/p}\}$ $(= \max\{r,\ \|1\|_p\})$, and $b = r_1^q$. By the Hölder inequality, it follows for any $i = 0, 1, \ldots, m$ and $t \in [t_0, t_0+T]$ that

$$\begin{aligned}
\left|E_i[u](t)\right|^q &= \left|\int_{t_0}^t u_i(\tau)\, d\tau\right|^q \le \left(\int_{t_0}^t |u_i(\tau)|\ d\tau\right)^q \\
&\le \left[\left(\int_{t_0}^t |u_i(\tau)|^p\ d\tau\right)^{\frac{1}{p}} \cdot (t-t_0)^{\frac{1}{q}}\right]^q \\
&\le b\,(t-t_0).
\end{aligned}$$

Suppose it is known for any $(i_k \cdots i_1) \in I^k$ with k fixed that

$$|E_{i_k\cdots i_1}[u](t)|^q \le b^k \frac{(t-t_0)^k}{k!}, \qquad t \in [t_0, t_0+T].$$

Then it follows that for any $t \in [t_0, t_0+T]$,

$$\begin{aligned}
\left|E_{i_{k+1}i_k\cdots i_1}[u](t)\right|^q &= \left|\int_{t_0}^t u_{i_{k+1}}(\tau) E_{i_k\cdots i_1}[u](\tau)\, d\tau\right|^q \\
&\le \left(\int_{t_0}^t \left|u_{i_{k+1}}(\tau)\right|^p\ d\tau\right)^{\frac{q}{p}} \\
&\qquad \cdot \int_{t_0}^t |E_{i_k\cdots i_1}[u](\tau)|^q\ d\tau \\
&\le b \cdot \int_{t_0}^t b^k \frac{(\tau - t_0)^k}{k!}\, d\tau \\
&= b^{k+1} \frac{(t-t_0)^{k+1}}{(k+1)!}.
\end{aligned}$$

So, we have shown by induction that

$$|E_{i_k\cdots i_1}[u](t)|^q \le b^k \frac{(t-t_0)^k}{k!}, \qquad \forall\, t \in [t_0, t_0+T], \tag{2}$$

holds for all $(i_k \cdots i_1) \in I^*$. Now since

$$|c(i_k \cdots i_1)|^q \le K\, k!\, M^k,$$

then for any given $k \ge 1$,

$$\begin{aligned}
|c(i_k \cdots i_1) E_{i_k\cdots i_1}[u](t)|^q &= |c(i_k \cdots i_1)|^q \cdot |E_{i_k\cdots i_1}[u](t)|^q \\
&\le K\, k!\, M^k \cdot b^k \frac{(t-t_0)^k}{k!} \\
&\le K\, M^k b^k T^k.
\end{aligned}$$

Thus, for any $k \ge 1$, the function

$$a_k(t) := \sum_{i_k,\ldots,i_1=0}^m c(i_k \cdots i_1) E_{i_k\cdots i_1}[u](t)$$

has the following upper bound estimate:

$$\begin{aligned}
|a_k(t)| &\le \sum_{i_k,\ldots,i_1=0}^m \left[|c(i_k \cdots i_1) E_{i_k\cdots i_1}[u](t)|^q\right]^{1/q} \\
&\le \sum_{i_k,\ldots,i_1=0}^m (K\, M^k b^k T^k)^{1/q} \\
&= (m+1)^k\, K^{1/q} [(b\, M\, T)^{1/q}]^k \\
&= \max\Big\{K^{1/q}\, [r\, M^{1/q} T^{1/q}\, (m+1)]^k, \\
&\qquad K^{1/q}\, [T^{1/p}\, M^{1/q} T^{1/q}\, (m+1)]^k\Big\} \\
&= \max\Big\{K^{1/q}\, [r\, M^{1/q} T^{1/q}\, (m+1)]^k, \\
&\qquad K^{1/q}\, [T M^{1/q}\, (m+1)]^k\Big\}.
\end{aligned}$$

Let

$$T_0 = \min\left\{\frac{1}{r^q (m+1)^q M},\ \frac{1}{(m+1) M^{1/q}}\right\}. \tag{3}$$

Then, for any $T \in [0,\ T_0)$, there is some $\delta \in [0, 1)$ such that

$$\begin{aligned}
\max\Big\{&K^{1/q}\, [r\, M^{1/q} T^{1/q}\, (m+1)]^k, \\
&K^{1/q}\, [T M^{1/q}\, (m+1)]^k\Big\} \le K^{1/q}\, \delta^k.
\end{aligned}$$

Thus, the series $\sum |a_k(t)|$ converges uniformly on $[t_0,\ t_0+T]$. This shows that, for any $T \le T_0$, the series

$$y(t) = c(\phi) + \sum_{k=1}^\infty a_k(t) \tag{4}$$

converges absolutely and uniformly on $[t_0, t_0+T]$.

The absolute continuity property of $y(\cdot)$ follows from the fact that each $E_{i_k\cdots i_1}[u]$ is absolutely continuous on

$[t_0, t_0 + T]$ and the fact that the series (4) converges uniformly on the same interval. ∎

The above proof shows that for $p > 1$, if c satisfies the growth condition (1), then the series defines a Fliess operator from $B_p^m(r)[t_0,\, t_0+T]$ to a bounded subset of $\mathcal{C}[t_0,\, t_0+T]$ for any $T \in [0,\, T_0)$, where T_0 is defined as in (3). Of course since T is finite, this implies then that in general such operators map also $B_p^m(r)[t_0,\, t_0+T]$ into $B_q^\ell(s)[t_0,\, t_0+T]$ for some $s > 0$. The next result addresses the case when $p = 1$. Note that this is a stricter growth condition on the coefficients of c since the factorial term is absent.

Lemma 2.2 *Suppose for a given power series c there exists real numbers $K > 0$ and $M > 0$ such that*

$$|c(\eta)| \le K\, M^{|\eta|}, \qquad \forall\, \eta \in I^*. \tag{5}$$

Then for any $0 < r < \frac{1}{M(m+1)}$ and $0 < T < \frac{1}{M(m+1)}$ it follows that for each $u \in B_1^m(r)[t_0, t_0+T]$, the series

$$y(t) = \sum_{\eta \in I^*} c(\eta) E_\eta[u](t)$$

converges absolutely and uniformly on $[t_0, t_0+T]$. Furthermore, $y(\cdot)$ is absolutely continuous on $[t_0, t_0+T]$.

Proof. Let $r, T \in \left(0, \frac{1}{M(m+1)}\right)$, and let $\delta = \max\{rM(m+1), TM(m+1)\} < 1$. Pick any $u \in B_1^m(r)[t_0, t_0+T]$. Let $r_1 = \max\{r, T\} (= \max\{r, \|1\|_1\})$. For any $t \in [t_0, t_0+T]$, $|E_i[u](t)| \le r_1$. By induction, one can show that for any $(i_k \cdots i_1) \in I^k$ and any $t \in [t_0, t_0+T]$,

$$|E_{i_k\cdots i_1}[u](t)| \le r_1^k.$$

Combining this with estimate (5), one has:

$$|c(i_k \cdots i_1) E_{i_k\cdots i_1}[u](t)| \;\le\; K M^k r_1^k.$$

Thus, for any $k \ge 1$, the function

$$a_k(t) = \sum_{i_k,\ldots,i_1=0}^{m} c(i_k \cdots i_1) E_{i_k\cdots i_1}[u](t)$$

has the following upper bound estimate:

$$\begin{aligned} |a_k(t)| &\le \sum_{i_k,\ldots,i_1=0}^{m} |c(i_k \cdots i_1) E_{i_k\cdots i_1}[u](t)| \\ &\le (m+1)^k\, K M^k r_1^k \\ &= \max\left\{K(m+1)^k M^k r^k,\ K(m+1)^k M^k T^k\right\} \\ &\le K\delta^k. \end{aligned}$$

From this it follows that the series

$$y(t) = c(\phi) + \sum_{k=1}^{\infty} a_k(t)$$

converges absolutely and uniformly on $[t_0, t_0+T]$.

The absolute continuity of the function $y(\cdot)$ follows again from the fact that the convergence is uniform and the fact that each $a_k(\cdot)$ is absolutely continuous on $[t_0, t_0+T]$. ∎

3. Convergence and Continuity Theorems: The Global Case

For any fixed $t_0 \in \mathbb{R}$ and any $p \ge 1$, define the function space $L_{p,e}^m(t_0)$ by

$$L_{p,e}^m(t_0) = \{u :\; u \in L_p^m[t_0, t_1]\ \forall\, t_0 < t_1 < \infty\}.$$

Lemma 3.1 *Suppose for a given power series c there exists real numbers $K > 0$ and $M > 0$ such that*

$$|c(\eta)| \le K\, M^{|\eta|}, \qquad \forall\, \eta \in I^*. \tag{6}$$

Then for any $u \in L_{1,e}^m(t_0)$, the series

$$y(t) = \sum_{\eta \in I^*} c(\eta) E_\eta[u](t)$$

converges absolutely and uniformly on $[t_0, t_0+T]$ for any $T > 0$. In particular, $y(t)$ converges absolutely on $[t_0, \infty)$.

For any fixed $(i_k \cdots i_1) \in I^k$ and any $0 \le j \le m$, let $\omega^j_{i_k\cdots i_1}$ denote the cardinality of the set $\{i_s : i_s = j,\ 1 \le s \le k\}$, i.e., the number of indices in $(i_k \cdots i_1)$ that take the value j. Clearly $0 \le \omega^j_{i_k\cdots i_1} \le k$. To prove Lemma 3.1, we will first prove the following:

Lemma 3.2 *Suppose for some $k > 0$, $(i_k \cdots i_1) \in I^k$ and $0 \le j \le m$ that $\omega^j_{i_k\cdots i_1} = l$. Then, for any $T > 0$, and any $u \in L_1^m[t_0, t_0+T]$ with $\|u\|_1 = r$, it holds that*

$$|E_{i_k\cdots i_1}[u](t)| \le r_1^{k-l}\, \frac{\left(\int_{t_0}^t |u_j(\sigma)|\, d\sigma\right)^l}{l!},\ \forall t \in [t_0, t_0+T], \tag{7}$$

where $r_1 = \max\{r, T\}$ $(= \max\{r, \|u_0\|_1\})$. In particular,

$$|E_{i_k\cdots i_1}[u](t)| \le \frac{r_1^k}{l!},\ \forall t \in [t_0, t_0+T].$$

Proof. We will prove the result by induction on k. First observe that the result follows directly from the definition in the case when $k = 1$. Now suppose that the result holds for some given k. Pick any $(i_{k+1} i_k \cdots i_1) \in I^{k+1}$ and any $0 \le j \le m$. Let $l = \omega^j_{i_{k+1} i_k\cdots i_1}$. Suppose $l = 0$. Then it holds that $\omega^j_{i_k\cdots i_1} = 0$, and hence,

$$\begin{aligned} |E_{i_{k+1} i_k\cdots i_1}[u](t)| &\le \int_{t_0}^T |u_{i_{k+1}}(\tau)| \cdot |E_{i_k\cdots i_1}(\tau)|\, d\tau \\ &\le \left(\int_{t_0}^T |u_{i_{k+1}}(\tau)|\, d\tau\right) \cdot r_1^k \le r_1^{k+1}. \end{aligned}$$

If $l = k+1$, then $i_{k+1} = i_k = \cdots = i_1 = j$, and thus,

$$\begin{aligned} \left|E_{i_{k+1}i_k\cdots i_1}[u](t)\right| &\leq \int_{t_0}^{t} |u_j(\tau)| \cdot |E_{i_k\cdots i_1}(\tau)|\, d\tau \\ &\leq \int_{t_0}^{t} |u_j(\tau)| \frac{\left(\int_{t_0}^{\tau} |u_j(\tau_1)|\, d\tau_1\right)^k}{k!}\, d\tau \\ &= \frac{\left(\int_{t_0}^{t} |u_j(\tau)|\, d\tau\right)^{k+1}}{(k+1)!}. \end{aligned}$$

Next assume that $1 \leq l \leq k$.

Case I. $i_{k+1} = j$. In this case, $\omega^j_{i_k\cdots i_1} = l-1$. Hence,

$$\begin{aligned} \left|E_{i_{k+1}i_k\cdots i_1}[u](t)\right| &\leq \int_{t_0}^{t} |u_j(\tau)| \cdot |E_{i_k\cdots i_1}(\tau)|\, d\tau \\ &\leq r_1^{k-(l-1)} \int_{t_0}^{t} |u_j(\tau)| \\ &\qquad \cdot \frac{\left(\int_{t_0}^{\tau} |u_j(\tau_1)|\, d\tau_1\right)^{l-1}}{(l-1)!}\, d\tau \\ &= r_1^{k+1-l} \frac{\left(\int_{t_0}^{t} |u_j(\tau)|\, d\tau\right)^{l}}{l!}. \end{aligned}$$

Case II. $i_{k+1} \neq j$. In this case, $\omega^j_{i_k\cdots i_1} = l$. Hence,

$$\begin{aligned} \left|E_{i_{k+1}i_k\cdots i_1}[u](t)\right| &\leq \int_{t_0}^{t} \left|u_{i_{k+1}}(\tau)\right| \cdot |E_{i_k\cdots i_1}(\tau)|\, d\tau \\ &\leq r_1^{k-l} \int_{t_0}^{t} \left|u_{i_{k+1}}(\tau)\right| \\ &\qquad \cdot \frac{\left(\int_{t_0}^{\tau} |u_j(\tau_1)|\, d\tau_1\right)^{l}}{l!}\, d\tau \\ &\leq r_1^{k-l} \int_{t_0}^{t} \left|u_{i_{k+1}}(\tau)\right| \\ &\qquad \cdot \frac{\left(\int_{t_0}^{t} |u_j(\tau_1)|\, d\tau_1\right)^{l}}{l!}\, d\tau \\ &= r_1^{k-l} \frac{\left(\int_{t_0}^{t} |u_j(\tau_1)|\, d\tau_1\right)^{l}}{l!} \\ &\qquad \cdot \int_{t_0}^{t} \left|u_{i_{k+1}}(\tau)\right|\, d\tau \\ &\leq r_1^{k+1-l} \frac{\left(\int_{t_0}^{t} |u_j(\tau_1)|\, d\tau_1\right)^{l}}{l!}. \end{aligned}$$

Since we have shown that the result is true for every case of $k+1$, Lemma 3.2 is proved by induction. ∎

Observe that for any $k > 0$, if $k = n(m+1)+s$ for some integers $n \geq 0$, $0 \leq s < m+1$, then there is at least one $j \in \{0, 1, \ldots m\}$ such that $\omega^j_{i_k\cdots i_1} \geq n$. Thus, we have the following:

Corollary 3.1 *For any $k > 0$, let $n \geq 0$ and $0 \leq s < m+1$ be integers such that $k = n(m+1)+s$. Let $T > 0$ be given. Then for any $(i_k \cdots i_1) \in I^k$ and any $u \in L_1^m[t_0, t_0+T]$ with $r = \|u\|_1$, it holds that*

$$|E_{i_k\cdots i_1}[u](t)| \leq \frac{r_1^k}{n!}, \ \forall t \in [t_0, t_0+T], \tag{8}$$

where $r_1 = \max\{r, T\}$.

We are now ready to prove Lemma 3.1.

Proof of Lemma 3.1. Suppose c satisfies the growth condition (6) for some $K > 0$ and $M > 0$. Fix $T > 0$. Pick any $u \in L^m_{1,e}[t_0, \infty)$. Let $r = \|u_{[t_0,t_0+T]}\|_1$, where $u_{[t_0,t_0+T]}$ denotes the restriction of u to $[t_0, t_0+T]$. Let $r_1 = \max\{r,\ T\}$.

Fix $k > 0$. Let $n \geq 0$ and $0 \leq s < m+1$ be integers such that $k = n(m+1)+s$. Then (8) together with (6) implies that for any $(i_k \cdots i_1) \in I^k$:

$$|c(i_k\cdots i_1)E_{i_k\cdots i_1}[u](t)| \leq K\,M^k \frac{r_1^k}{n!}, \qquad \forall\, t \in [t_0, t_0+T].$$

Hence, for

$$a_k(t) = \sum_{i_k,\ldots,i_1=0}^{m} c(i_k\cdots i_1)E_{i_k\cdots i_1}[u](t),$$

it holds that

$$\begin{aligned} |a_k(t)| &\leq \sum_{i_k,\ldots,i_1=0}^{m} |c(i_k\cdots i_1)E_{i_k\cdots i_1}[u](t)| \\ &\leq K\,M^k\,(m+1)^k \frac{r_1^k}{n!} \\ &= KM^{n(m+1)+s}(m+1)^{n(m+1)+s}\, \frac{r_1^{n(m+1)+s}}{n!} \end{aligned}$$

for all $t \in [t_0, t_0+T]$. It then follows that (defining $a_0(t) = 0$)

$$\begin{aligned} \sum_{k=0}^{\infty} |a_k(t)| &= \sum_{s=0}^{m}\sum_{n=0}^{\infty} \left|a_{n(m+1)+s}(t)\right| \\ &\leq \sum_{s=0}^{m}\sum_{n=0}^{\infty} KM^s\,(m+1)^s\, r_1^s \\ &\qquad \cdot \frac{\left[(M\,(m+1)r_1)^{m+1}\right]^n}{n!} \\ &= K\left(\sum_{s=0}^{m}[M\,(m+1)\,r_1]^s\right) \\ &\qquad \cdot \left(\sum_{n=0}^{\infty} \frac{\left[(M\,(m+1)r_1)^{m+1}\right]^n}{n!}\right), \end{aligned}$$

for all $t \in [t_0, t_0+T]$. This shows that the series

$$y(t) = c(\phi) + \sum_{k=1}^{\infty} a_k(t)$$

converges absolutely and uniformly on $[t_0, t_0+T]$. ∎

Observe that for any $p \geq 1$, $L^m_{p,e} \subseteq L^m_{1,e}$. As a consequence of Lemma 3.1, we have the following:

Corollary 3.2 *Suppose for a given power series c, the coefficients satisfies the growth condition (6) for some $K > 0$ and $M > 0$. Then for any $u \in L^m_{p,e}(t_0)$, $p \geq 1$, the series*

$$y(t) = \sum_{\eta \in I^*} c(\eta) E_\eta[u](t)$$

converges on $[t_0, \infty)$. Furthermore, $y(\cdot)$ is continuous on $[t_0, \infty)$.

The above corollary says that if c satisfies the growth condition (6), then F_c defines an operator from $L^m_{p,e}(t_0)$ to $\mathcal{C}[t_0, \infty)$ for any $p \geq 1$.

4. Local L_p-Systems

In this section we consider Fliess operators that have a corresponding state space realization. Specifically, given the formal power series c, it is assumed that there is at least one analytic state space realization $(g, g_1, \ldots, g_m, h, x_0)$ such that $L_{g_\eta} h(x_0) = c(\eta)$ for all $\eta \in I^*$. In this setting, consider the following definition.

Definition 4.1 *An analytic state space system realization $(f, g_1, \ldots, g_m, h, x_0)$ of a Fliess operator F_c is a **local L_p-system** if for some $r > 0$, there exists some $T > 0$ such that the output function $y(t) = h(x(t))$ with*

$$\dot{x}(t) = f(x(t)) + \sum_{i=1}^{m} g_i(x(t)) u_i(t), \quad x(0) = x_0$$

is defined for all $t \in [0, T]$ and all $u \in B^m_p(r)[0, T]$.

Clearly, Lemma 2.1 provides a sufficient condition for any system being locally L_p when $p > 1$. In certain special cases even stronger statements can be made:

Lemma 4.1 *A single input driftless system, i.e., a system (f, g_1, h, x_0) with $f \equiv 0$, is a locally L_p-system for any $p \geq 1$.*

Proof. Consider an analytic state space system

$$\dot{x}(t) = g(x(t))u(t), \quad y(t) = h(x(t)), \quad x(0) = x_0. \quad (9)$$

Observe that the corresponding generating series is

$$c = h(x_0) + \sum_{k=1}^{\infty} L^k_g h(x_0)\, \theta^k_1,$$

where $\theta^k_1 = \underbrace{\theta_1 \cdots \theta_1}_{k}$. The Fliess operator is given by:

$$F_c[u](t) = h(x_0) + \sum_{k=1}^{\infty} L^k_g h(x_0) E_{\theta^k_1}[u](t). \qquad (10)$$

Observe that

$$\int_{t_0}^{t} u(s) \int_{t_0}^{s} u(s_1) ds_1 ds = \int_{t_0}^{t} U(s) \left(\frac{d}{ds} U(s) \right) ds = \frac{[U(t)]^2}{2},$$

where $U(t) = \int_{t_0}^{t} u(s) ds$. Hence, by induction,

$$E_{\theta^k_1}[u](t) = \frac{[U(t)]^k}{k!}.$$

Since g_1 and h are analytic, there exist some $L > 0$ and $M > 0$ such that, for any $k \geq 0$,

$$\left| L^k_{g_1} h(x_0) \right| \leq L M^k k!,$$

(c.f. [13]). Let $r_1 = \frac{1}{M}$. It can then be seen that for any $0 < r < r_1$, and for any $T > 0$, if $u \in B^1_1(r)[0, T]$, the series in (10) converges absolutely and uniformly on $[0, T]$.

On the other hand, by virtual of [7, Theorem 3.1.5], one sees that for any $r > 0$, there exists some $T_1 > 0$, such that on $[t_0, t_0 + T_1]$, the solution of system (9) has the series expansion defined by F_c. It follows that for any $u \in B^1_1(r)[t_0, t_0 + \hat{T}]$, the corresponding solution $y(t)$ of (9) is defined on $[t_0, t_0 + \hat{T}]$. That is, system (9) is a local L_1-system. To be more precise, Theorem 3.1.5 in [7] was proved for the class of piecewise continuous input functions. But the same proof also applies to the class of L_1 input functions, as long as there exist some interval $[t_0, t_0 + b]$ on which the series (10) converges absolutely and uniformly.

Finally, for any $1 < p < \infty$ and any $t > t_0$, the Hölder inequality yields

$$\int_{t_0}^{t} |u(s)|\ ds \leq \left(\int_{t_0}^{t} |u(s)|^p\ ds \right)^{1/p} \left(\int_{t_0}^{t} ds \right)^{1/q},$$

where q is conjugate to p. Thus, $L^1_p(r_p)[0, T] \subseteq L^1_1(r)[0, T]$, where $r = r_p T^{1/q}$. Thus, for any $1 < p < \infty$, system (9) is a local L_p-system. The case $p = \infty$ is proven in [7, Chapter 3]. ∎

In the next section, it is shown by example that the growth conditions of Lemma 2.1 are *not* necessary for a system to be L_p.

5. Examples

In this section a variety of examples are presented to illustrate the concepts from the previous sections. The first example shows that all bilinear systems are L_p-systems in the sense of Section 2 for all $p \geq 1$. In fact, such system satisfy the more restrictive growth condition (6), which insures the global convergence and continuity properties for the corresponding Fliess operator. A similar analysis holds for linear systems. The second example is one that fails the growth condition (1) for all $1 \leq p < \infty$, even though by Lemma 4.1 it is known to be locally L_p for all $p \geq 1$. Hence, such growth conditions are only sufficient conditions for a system to be locally L_p. The final example is in a sense *in between* the previous two. It satisfies the growth condition (1) for a finite $p > 1$, but not for all finite p values.

Example 5.1 Consider a bilinear system

$$\begin{aligned} \dot{x} &= Ax + \sum_{i=1}^{m} N_i x \, u_i \\ y &= Cx, \end{aligned}$$

where $C \in \mathbb{R}^{\ell \times n}$. Let x_0 be any nonzero vector in $\mathbb{R}^n$. Using the induced matrix norm, define $K = \|C\| \, |x_0|$, and $M = \max_{0 \leq i \leq m} \|N_i\|$, where $N_0 = A$.
Then observe that for any $k \geq 0$ it follows directly that

$$|c(i_k \ldots i_1)| = |CN_{i_k} \ldots N_{i_1} x_0| \leq \|C\| \, M^k \, |x_0| = K \, M^k,$$

Trivially, $|c(\phi)| = 0 \leq K$. Hence, the growth condition (6) is satisfied.

Example 5.2 Consider the scalar system

$$\begin{aligned} \dot{x} &= x^2 u \\ y &= x, \end{aligned}$$

in which case the effective index set is $I^* = \{(\underbrace{1 \ldots 1}_{k}) : k \geq 0\}$. The coefficients for $x_0 = 1$ are easily shown to be

$$c(\underbrace{1 \ldots 1}_{k}) = L_g^k h(1) = k!, \quad k \geq 0.$$

Hence, $|c(\underbrace{1 \ldots 1}_{k})|^q = (k!)^q$ satisfies the growth condition (1) only when $q = 1$ $(p = \infty)$. If one applies the local coordinate transformation $z = \phi(x) = (x-1)/x$ in a neighborhood of $x_0 = 1$, then the resulting Wiener system

$$\begin{aligned} \dot{z} &= u \\ y &= \frac{1}{1-z} \end{aligned}$$

has identical coefficients for $z_0 = 0$.

Example 5.3 Consider the Wiener system

$$\begin{aligned} \dot{x} &= u \\ y &= e^{x^2/2} \end{aligned}$$

near $x_0 = 0$. The coefficients for $x_0 = 0$ are $c(\emptyset) = 1$ and

$$c(\underbrace{1 \ldots 1}_{k}) = \begin{cases} 1 \cdot 3 \cdot 5 \cdot \cdots \cdot k-1 & : \; k \text{ even} \\ 0 & : \; k \text{ odd}. \end{cases}$$

In which case,

$$|c(\eta)|^q \leq |\eta|!, \quad \forall \eta \in I^*$$

for $q = 1$ $(p = \infty)$ and $q = 2$ $(p = 2)$. Thus, for any $r > 0$ there exists $T > 0$ such that the input-output system can be viewed as a mapping of the form $F_c : B_\infty^1(r)[0,T] \mapsto L_1^1[0,T]$ or $F_c : B_2^1(r)[0,T] \mapsto L_2^1[0,T]$. No other conclusions of this type can be drawn since the growth conditions in this paper are only proven to be sufficient conditions. But of course the state equation has a well defined solution for all inputs in $L_p^1[0,T]$, $p \geq 1$, and as with the previous example, the system is locally L_p for all $p \geq 1$.

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