MAXIMALLY ROBUST CONTROLLERS FOR MULTIVARIABLE SYSTEMS

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Abstract

The set of all optimal controllers which maximize a robust stability radius for unstructured additive perturbations may be obtained using Hankel-norm approximation methods. These controllers guarantee robust stability for all perturbations which lie inside an open ball in the uncertainty space (say of radius r_1). Necessary and sufficient conditions are obtained for a perturbation lying on the boundary of this ball to be destabilizing for all maximally robust controllers. It is thus shown that a "worst-case direction" exists along which all boundary perturbations are destabilizing. By imposing a parametric constraint such that the permissible perturbations cannot have a "projection" of magnitude larger than $(1-\delta)r_1, 0<\delta\leq 1$, in the most critical direction, the uncertainty region guaranteed to be stabilized by a subset of all maximally robust controllers can be extended beyond the ball of radius r_1. The choice of the "best" maximally robust controller - in the sense that the uncertainty region guaranteed to be stabilized becomes as large as possible - is associated with the solution of a superoptimal approximation problem. Expressions for the improved stability radius are obtained and some links with μ-analysis are pursued.

1 Notation

$\mathcal{R}$ and $\mathcal{C}$ denote the sets of real and complex numbers, respectively. $\mathcal{C}_+$ ($\bar{\mathcal{C}}_+$), $\mathcal{C}_-$ ($\bar{\mathcal{C}}_-$) denote the open (closed) right half-plane and the open (closed) left half-plane, respectively. For a complex matrix A, A^T denotes the transpose and A' denotes the complex-conjugate transpose. $\sigma_i(A)$ denotes the ith largest singular value with the smallest and largest denoted by $\underline{\sigma}(A)$ and $\bar{\sigma}(A)$, respectively. The norm of A is defined as $\|A\|=\bar{\sigma}(A)$.

$\mathcal{L}_\infty^{p\times m}$ denotes the space of all $p\times m$ matrix functions with entries uniformly bounded on the $j\omega$-axis. $\mathcal{H}_\infty^{p\times m}$ and $\mathcal{H}_\infty^{-p\times m}$ denote the subspaces of $\mathcal{L}_\infty^{p\times m}$ consisting of all matrix functions whose entries are analytic in $\bar{\mathcal{C}}_+$ and $\bar{\mathcal{C}}_-$, respectively. $\|.\|_\infty$ denotes the $\mathcal{L}_\infty$ norm of matrices in $\mathcal{L}_\infty$ or the $\mathcal{H}_\infty$ norm of matrices in $\mathcal{H}_\infty$ depending on context. $\gamma\mathcal{BH}_\infty^{p\times m}=\{G\in\mathcal{H}_\infty^{p\times m}:\|G\|_\infty\leq\gamma\}$ is the γ ball of $\mathcal{H}_\infty^{p\times m}$. The prefix $\mathcal{R}$ before a set symbol means that the elements of the set are real-rational. Matrix dimensions of spaces are occasionally suppressed. $G(s)^\sim := G'(-\bar{s})$ denotes the para-hermitian conjugate of $G(s)$. The Hankel operator with symbol $G\in\mathcal{H}_\infty$ is denoted by Γ_G. The Hankel norm of G is written as $\|\Gamma_G\|$ and the smallest Hankel singular value as $\underline{\sigma}(\Gamma_G)$.

Matrix (scalar and vector) transfer functions will be represented by uppercase (lowercase) boldface letters and with the dependence on s mostly suppressed. If $\boldsymbol{G}^{-1}=\gamma^{-2}\boldsymbol{G}^\sim$, then $\boldsymbol{G}$ is called γ-allpass (or simply allpass if $\gamma=1$). A matrix function $\boldsymbol{G}\in\mathcal{RH}_\infty$ which satisfies $\boldsymbol{G}^\sim\boldsymbol{G}=I$ is called inner. A matrix function $\boldsymbol{G}(s)\in\mathcal{RH}_\infty$ which has full column rank for all $s\in\bar{\mathcal{C}}_+$ is called outer. If $\boldsymbol{U}$ and $\boldsymbol{H}=\begin{bmatrix}\boldsymbol{H}_{11} & \boldsymbol{H}_{12}\\ \boldsymbol{H}_{21} & \boldsymbol{H}_{22}\end{bmatrix}$ are matrix functions with compatible dimensions, we define the lower linear fractional map $\mathcal{F}_l(\boldsymbol{H},\boldsymbol{U})=\boldsymbol{H}_{11}+\boldsymbol{H}_{12}\boldsymbol{U}(I-\boldsymbol{H}_{22}\boldsymbol{U})^{-1}\boldsymbol{H}_{21}$, provided that $I-\boldsymbol{H}_{22}(\infty)\boldsymbol{U}(\infty)$ is invertible. If $\mathcal{U}$ is a set, then $\mathcal{F}_l(\boldsymbol{H},\mathcal{U})$ denotes the set $\{\mathcal{F}_l(\boldsymbol{H},\boldsymbol{U}):\boldsymbol{U}\in\mathcal{U}\}$ and if $\boldsymbol{G}_1,\boldsymbol{G}_2,\boldsymbol{G}_3\in\mathcal{L}_\infty$ have appropriate dimensions, then $\boldsymbol{G}_1+\boldsymbol{G}_2\mathcal{U}\boldsymbol{G}_3$ denotes the set $\{\boldsymbol{G}_1+\boldsymbol{G}_2\boldsymbol{U}\boldsymbol{G}_3:\boldsymbol{U}\in\mathcal{U}\}$.

If $\boldsymbol{G}\in\mathcal{L}_\infty$ we define, for each i, $s_i^\infty(\boldsymbol{G})=\sup\{\sigma_i(\boldsymbol{G}(j\omega)):\omega\in\mathcal{R}\}$. Clearly, $s_1^\infty(\boldsymbol{G})=\|\boldsymbol{G}\|_\infty$. Suppose that $\mathcal{T}$ is a set of matrix functions. $\boldsymbol{T}\in\mathcal{T}$ is called a kth level superoptimal function if it minimizes the sequence $\{s_1^\infty(\boldsymbol{T}),s_2^\infty(\boldsymbol{T}),\ldots,s_k^\infty(\boldsymbol{T})\}$ with respect to lexicographic ordering among all $\boldsymbol{T}\in\mathcal{T}$. The minimized sequence is denoted by $\{s_1(\mathcal{T}),\ldots,s_k(\mathcal{T})\}$, and the $s_i(\mathcal{T})$'s are called the superoptimal levels of $\mathcal{T}$.

2 Introduction

The work presented here is related to the problem of maximizing the robust stability radius for systems subject to unstructured additive perturbations [14], [3], [12], [13]. In [3] it was shown that this problem is equivalent to a Nehari approximation and a parametrization was obtained for all controllers which guarantee a robust stabilization radius $r<r_1$. A parametrization of all maximally robust controllers ($r=r_1$) is also implicit in [3], [2]. Optimal interpolation theory is used in [12] to give a solution for single input/single output systems.

In the multi-input/single output or single input/multi-output case, the optimal controller is unique. In the matrix case, however, a continuum of optimal controllers typically exists. It is therefore natural to ask whether a subset of these controllers offers improved robust stability properties, in the sense that it guarantees closed-loop stability for a larger class of uncertainties, compared to those offered by the optimal solution set considered in total. More specifically, we seek to identify the set of all controllers which guarantees robust stability for the largest possible region of the uncertainty space containing the open ball of radius r_1 as a

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subset. Clearly, this can only be achieved by imposing a structure on the set of admissible uncertainties.

Our approach is as follows: From the work in [14], [13] and [3] the maximum robust stability radius r_1 is the inverse of the smallest achievable $\mathcal{H}_\infty$ norm among all interpolating functions $\mathcal{T}=\{\boldsymbol{K}(I-\boldsymbol{GK})^{-1}\}$, as $\boldsymbol{K}$ varies over the set of all internally stabilizing compensators of $\boldsymbol{G}$. Using an allpass dilation technique, the set of all optimal interpolating functions $\mathcal{T}_1 = \{\boldsymbol{T} \in \mathcal{T} : \|\boldsymbol{T}\|_\infty = r_1^{-1}\} \subseteq \mathcal{T}$ has the form $\mathcal{T}_1 = \boldsymbol{Y}\mathrm{diag}(r_1^{-1}\boldsymbol{a}, \hat{\boldsymbol{R}} + \mathcal{Q})\boldsymbol{X}$, where $\hat{\boldsymbol{R}} \in \mathcal{RH}_\infty^-$, $\boldsymbol{X}$ and $\boldsymbol{Y}$ are square inner matrices, $\boldsymbol{a}$ is a scalar allpass function and $\mathcal{Q}$ is the set of all r_1^{-1} suboptimal Nehari extensions of $\hat{\boldsymbol{R}}$, i.e. $\mathcal{Q}=\{\boldsymbol{Q}\in\mathcal{H}_\infty : \|\hat{\boldsymbol{R}}+\boldsymbol{Q}\|_\infty \le r_1^{-1}\}$.

Every optimal controller corresponding to an interpolating function in $\mathcal{T}_1$ stabilizes all perturbations in the open ball $\mathcal{D}_{r_1} = \{\boldsymbol{\Delta} \in \mathcal{L}_\infty : \|\boldsymbol{\Delta}\|_\infty < r_1, \eta(\boldsymbol{G}+\boldsymbol{\Delta}) = \eta(\boldsymbol{G})\}$, where $\eta(\cdot)$ denotes the number of poles in $\mathcal{C}_+$. Next, it is shown that perturbations $\boldsymbol{\Delta}$ on the boundary of $\mathcal{D}_{r_1}$ are uniformly destabilizing (i.e. they destabilize the closed-loop system for every optimal controller) if and only if $|\boldsymbol{x}^T(j\omega)\boldsymbol{\Delta}(j\omega)\boldsymbol{y}(j\omega)| = r_1$, for some $\omega\in\mathcal{R}$, and where $\boldsymbol{x}^T$ and $\boldsymbol{y}$ are the first row and column of $\boldsymbol{X}$ and $\boldsymbol{Y}$, respectively. Moreover, all frequencies ω are equally critical, in that destabilizing boundary perturbations can be constructed for every ω. This shows that it is futile to attempt to extend the uncertainty set guaranteed to be stabilized by a subset of all optimal controllers in the (frequency-dependent) direction defined by $\boldsymbol{x}^T$ and $\boldsymbol{y}$. By imposing a parametric constraint (uniform in ω) such that the permissible perturbations cannot have a "projection" of magnitude larger than $(1-\delta)r_1$ $(0<\delta\le 1)$ in this direction, the uncertainty region guaranteed to be stabilized by a subset of all optimal controllers can be extended beyond $\mathcal{D}_{r_1}$. Using a result in [8], it is shown that for each $\delta\in(0,1]$ the corresponding robust-stability radius is maximized by the set of controllers which minimize the first two superoptimal levels of $\mathcal{T}$. An expression of the improved stability radius is also obtained which involves δ and the first two superoptimal levels of $\mathcal{T}$. This work is related to the results in [10] which also uses superoptimization to extend the allowable perturbation set.

An alternative interpretation of our results gives interesting connections with the problem of robust stabilization of systems subject to structured perturbations and μ-synthesis [11]. By suitably defining δ, robust stabilization problems for a number of uncertainty structures can be formulated, and bounds on the achievable robust stability radius can be obtained. An upper bound on μ for the constant complex case is derived in Section 5. A full version of this work will appear in [5].

3 Robust stabilization for unstructured additive perturbations

Let $\boldsymbol{G}\in\mathcal{RL}_\infty$. When $\boldsymbol{\Delta}=0$, the closed-loop system in Figure 1 is internally stable if and only if it is well-posed, i.e. $\det(I-\boldsymbol{G}(\infty)\boldsymbol{K}(\infty))\neq 0$ and the four transfer functions $(u_1,u_2)\to(y_1,y_2)$ are in $\mathcal{H}_\infty$. In this case, we write $(\boldsymbol{G},\boldsymbol{K})\in\mathcal{S}$ and $\boldsymbol{K}\in\mathcal{K}$. Consider the set of

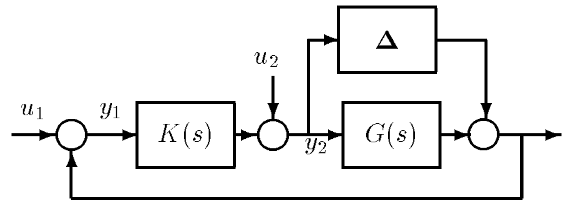

Figure 1: Closed-loop System

additively perturbed systems $\boldsymbol{G}+\boldsymbol{\Delta}$, $\boldsymbol{\Delta}\in\mathcal{D}_r(\boldsymbol{G})$, where

$$\mathcal{D}_r(\boldsymbol{G})=\{\boldsymbol{\Delta}\in\mathcal{L}_\infty : \|\boldsymbol{\Delta}\|_\infty<r, \eta(\boldsymbol{G})=\eta(\boldsymbol{G}+\boldsymbol{\Delta})\} \quad (1)$$

where $\eta(\cdot)$ denotes the number of poles in $\mathcal{C}_+$. The system $(\boldsymbol{G},\boldsymbol{K})$ is said to be $r-$robustly stable if $(\boldsymbol{G}+\boldsymbol{\Delta},\boldsymbol{K})\in\mathcal{S}$ for all $\boldsymbol{\Delta}\in\mathcal{D}_r(\boldsymbol{G})$ and is maximally robustly stable if $(\boldsymbol{G},\boldsymbol{K})$ is r_1-robustly stable and there exists $\boldsymbol{\Delta}\in\partial\mathcal{D}_{r_1}(\boldsymbol{G})=\{\boldsymbol{\Delta}\in\mathcal{L}_\infty : \|\boldsymbol{\Delta}\|_\infty = r_1, \eta(\boldsymbol{G})=\eta(\boldsymbol{G}+\boldsymbol{\Delta})\}$, such that $(\boldsymbol{G}+\boldsymbol{\Delta},\boldsymbol{K})\notin\mathcal{S}$.

Let $\boldsymbol{G}$ have left and right coprime factorizations $\boldsymbol{G}=\boldsymbol{N}\boldsymbol{M}^{-1}=\tilde{\boldsymbol{M}}^{-1}\tilde{\boldsymbol{N}}$, respectively, with $\boldsymbol{N},\boldsymbol{M},\tilde{\boldsymbol{N}},\tilde{\boldsymbol{M}}\in\mathcal{RH}_\infty$ and let $\boldsymbol{U},\boldsymbol{V},\tilde{\boldsymbol{U}},\tilde{\boldsymbol{V}}\in\mathcal{RH}_\infty$ satisfy the Bezout identities $\tilde{\boldsymbol{V}}\boldsymbol{M}-\tilde{\boldsymbol{U}}\boldsymbol{N}=I$ and $\tilde{\boldsymbol{M}}\boldsymbol{V}-\tilde{\boldsymbol{N}}\boldsymbol{U}=I$. Then the set of all stabilizing controllers of $\boldsymbol{G}$ is

$$\mathcal{K}=\{(\boldsymbol{U}+\boldsymbol{M}\boldsymbol{Q})(\boldsymbol{V}+\boldsymbol{N}\boldsymbol{Q})^{-1} : \boldsymbol{Q}\in\mathcal{H}_\infty\}. \quad (2)$$

Let $\mathcal{T}=\{\boldsymbol{K}(I-\boldsymbol{GK})^{-1} : \boldsymbol{K}\in\mathcal{K}\}$ be the set of all interpolating functions. Using (2) gives

$$\mathcal{T}=\{(\boldsymbol{U}+\boldsymbol{M}\boldsymbol{Q})\tilde{\boldsymbol{M}} \ : \ \boldsymbol{Q}\in\mathcal{H}_\infty\}, \quad (3)$$

with $\boldsymbol{M}$ and $\tilde{\boldsymbol{M}}$ inner [1]. The next result gives necessary and sufficient conditions for robust stabilization in the presence of additive perturbations and the maximal robust stability radius. We assume, without loss of generality, that $\boldsymbol{G}^{\sim}\in\mathcal{RH}_\infty$ and $\boldsymbol{G}(\infty)=0$ [3].

Theorem 3.1 *[14], [3], [13] Let $\boldsymbol{G}^{\sim}\in\mathcal{RH}_\infty, \boldsymbol{G}(\infty)=0$ and suppose that $(\boldsymbol{G},\boldsymbol{K})\in\mathcal{S}$. Then $(\boldsymbol{G},\boldsymbol{K})$ is $r-$robustly stable if and only if $\|\boldsymbol{K}(I-\boldsymbol{GK})^{-1}\|_\infty\le r^{-1}$. Furthermore, the maximum stability radius r_1 for which $(\boldsymbol{G},\boldsymbol{K})$ is r_1-robustly stable for some $\boldsymbol{K}\in\mathcal{K}$ is $r_1=\underline{\sigma}(\Gamma_{\boldsymbol{G}(-s)})=\|\Gamma_{\boldsymbol{M}^{\sim}\boldsymbol{U}(-s)}\|^{-1}$.*

4 Main results

The set of all maximally robust controllers may be characterized in terms of the set of all optimal Nehari extensions of $\boldsymbol{M}^{\sim}\boldsymbol{U}$, i.e. all $\boldsymbol{Q}\in\mathcal{H}_\infty$ which achieve

$$\|\boldsymbol{M}^{\sim}\boldsymbol{U}+\boldsymbol{Q}\|_\infty = r_1^{-1}. \quad (4)$$

This set can be parametrized as a linear fractional map of the set of all r_1 stable contractions.

Remark 4.1 *To avoid a messy indexing system we assume that the largest Hankel singular values of $\boldsymbol{R}(-s)$ and $\hat{\boldsymbol{R}}(-s)$ defined below in Theorems 4.1 and 4.2 are non-repeated. This is equivalent to the assumption that the first two superoptimal levels of $\mathcal{T}$ are distinct.*

Theorem 4.1 *[7], [4],[2] Let $\boldsymbol{R} = \boldsymbol{M}^{\sim}\boldsymbol{U} \in \mathcal{RH}_\infty^{-p\times m}$ and define $r_1 = \|\Gamma_{\boldsymbol{R}(-s)}\|^{-1}$ (see Theorem 3.1). Then there exists an embedding of $\boldsymbol{R}$ of the form,*

$$\boldsymbol{H}=\begin{bmatrix} \boldsymbol{H}_{11} & \boldsymbol{H}_{12} \\ \boldsymbol{H}_{21} & \boldsymbol{H}_{22} \end{bmatrix}=\begin{bmatrix} \boldsymbol{R}+\boldsymbol{Q}_{11} & \boldsymbol{Q}_{12} \\ \boldsymbol{Q}_{21} & \boldsymbol{Q}_{22} \end{bmatrix}:=\begin{bmatrix} \boldsymbol{R} & 0 \\ 0 & 0 \end{bmatrix}+\boldsymbol{Q}_a$$

with $\boldsymbol{Q}_a \in \mathcal{RH}_\infty^{(p+m-1)\times(m+p-1)}$, such that $\boldsymbol{H}\boldsymbol{H}^{\sim} = \boldsymbol{H}^{\sim}\boldsymbol{H} = r_1^{-2}I$ and $\|\boldsymbol{H}_{22}\|_\infty = \|\boldsymbol{Q}_{22}\|_\infty < r_1^{-1}$. Further, the set of all $\boldsymbol{Q}\in\mathcal{H}_\infty^{p\times m}$ such that $\|\boldsymbol{R}+\boldsymbol{Q}\|_\infty = r_1^{-1}$ is given by, $\mathcal{S}_1 = \mathcal{F}_l(\boldsymbol{Q}_a, r_1\mathcal{BH}_\infty^{(p-1)\times(m-1)})$.

Let $\mathcal{K}_1$ denote the set of all maximally-robust controllers of $\boldsymbol{G}$, and let $\mathcal{T}_1 = \{\boldsymbol{K}(I-\boldsymbol{G}\boldsymbol{K})^{-1} : \boldsymbol{K} \in \mathcal{K}_1\} \subseteq \mathcal{T}$ denote the set of all optimal interpolating functions. In view of Theorems 3.1 and 4.1, these may be parametrized as $\mathcal{K}_1 = \{(\boldsymbol{U}+\boldsymbol{M}\boldsymbol{Q})(\boldsymbol{V}+\boldsymbol{N}\boldsymbol{Q})^{-1} : \boldsymbol{Q} \in \mathcal{F}_l(\boldsymbol{Q}_a, r_1\mathcal{BH}_\infty^{(p-1)\times(m-1)})\}$ and

$$\mathcal{T}_1 = \{(\boldsymbol{U}+\boldsymbol{M}\boldsymbol{Q})\tilde{\boldsymbol{M}} : \boldsymbol{Q}\in\mathcal{F}_l(\boldsymbol{Q}_a, r_1\mathcal{BH}_\infty^{(p-1)\times(m-1)})\}, \quad (5)$$

respectively. The next theorem gives an alternative parametrization of $\mathcal{T}_1$ and shows that $\mathcal{T}_1$ can be diagonalized by rational allpass transformations.

Theorem 4.2 *[6] Let all variables be as defined in Theorem 4.1. Then,*

1. *There exists an r_1^{-1}-allpass completion of $\boldsymbol{H}_{22} = \boldsymbol{Q}_{22}$ of the form $\bar{\boldsymbol{H}} =$*

   $$\begin{bmatrix} \bar{\boldsymbol{H}}_{11} & \bar{\boldsymbol{H}}_{12} \\ \bar{\boldsymbol{H}}_{21} & \boldsymbol{H}_{22} \end{bmatrix}=\begin{bmatrix} \hat{\boldsymbol{R}}+\bar{\boldsymbol{Q}}_{11} & \bar{\boldsymbol{Q}}_{12} \\ \bar{\boldsymbol{Q}}_{21} & \boldsymbol{Q}_{22} \end{bmatrix}:=\begin{bmatrix} \hat{\boldsymbol{R}} & 0 \\ 0 & 0 \end{bmatrix}+\bar{\boldsymbol{Q}}_a$$

   with $\bar{\boldsymbol{Q}}_a, \bar{\boldsymbol{Q}}_{12}^{-1}, \bar{\boldsymbol{Q}}_{21}^{-1} \in \mathcal{RH}_\infty$ such that $\bar{\boldsymbol{H}}\bar{\boldsymbol{H}}^{\sim} = \bar{\boldsymbol{H}}^{\sim}\bar{\boldsymbol{H}} = r_1^{-2}I_{p+m-2}$ and $\hat{\boldsymbol{R}}\in\mathcal{RH}_\infty^{-(p-1)\times(m-1)}$.
2. *The set of all $\bar{\boldsymbol{Q}} \in \mathcal{H}_\infty^{(p-1)\times(m-1)}$ such that $\|\hat{\boldsymbol{R}}+\bar{\boldsymbol{Q}}\|_\infty \le r_1^{-1}$ is given by*

   $$\bar{\mathcal{S}}_1 = \mathcal{F}_l(\bar{\boldsymbol{Q}}_a, r_1\mathcal{BH}_\infty^{(p-1)\times(m-1)}).$$
3. *There exist inner matrices $\boldsymbol{X}$ and $\boldsymbol{Y}$ and a scalar allpass function $\boldsymbol{a}$ such that,*

   $$\mathcal{T}_1 = \boldsymbol{Y}\,\mathrm{diag}(r_1^{-1}\boldsymbol{a}, \mathcal{F}_l(\bar{\boldsymbol{H}}, r_1\mathcal{BH}_\infty^{(p-1)\times(m-1)}))\boldsymbol{X}. \quad (6)$$

Further,

$$\mathcal{F}_l(\bar{\boldsymbol{H}}, r_1\mathcal{BH}_\infty^{(p-1)\times(m-1)}) = \{\hat{\boldsymbol{R}}+\bar{\boldsymbol{Q}} : \bar{\boldsymbol{Q}}\in\mathcal{H}_\infty^{(p-1)\times(m-1)}, \|\hat{\boldsymbol{R}}+\bar{\boldsymbol{Q}}\|_\infty \le r_1^{-1}\}.$$

The theorem shows that every optimal interpolating function $\boldsymbol{T} \in \mathcal{T}_1$ has a partial pseudo-singular value decomposition with largest "singular value" r_1^{-1} and corresponding real-rational left and right "singular vectors", respectively.

In the sequel we give improved robust stability properties for the set of controllers which minimize the pair $\{s_1(\mathcal{T}), s_2(\mathcal{T})\}$ with respect to lexicographic ordering. We denote the set of interpolating functions with this property by $\mathcal{T}_2$ and the corresponding set of controllers by $\mathcal{K}_2$. Clearly, $\mathcal{T}_2 \subseteq \mathcal{T}_1 \subseteq \mathcal{T}$ and $\mathcal{K}_2 \subseteq \mathcal{K}_1 \subseteq \mathcal{K}$. We call $\mathcal{T}_2$ ($\mathcal{K}_2$) the superoptimal set of interpolating functions (controllers) with respect to the first two levels. The following result gives a parametrization of $\mathcal{T}_2$.

Lemma 4.1 *[5] $\mathcal{T}_2$ may be parametrized as*

$$\mathcal{T}_2 = \boldsymbol{Y}_1\,\mathrm{diag}(s_1\boldsymbol{a}, s_2\boldsymbol{b}, \check{\boldsymbol{R}} + \mathcal{S}_2)\boldsymbol{X}_1$$

where s_1 and s_2 are the first two superoptimal levels of $\mathcal{T}$ with $s_1 = r_1^{-1}$, $\boldsymbol{Y}_1$ and $\boldsymbol{X}_1$ are square inner matrices, $\boldsymbol{a}$ and $\boldsymbol{b}$ are scalar rational allpass functions, $\check{\boldsymbol{R}} \in \mathcal{RH}_\infty^{-(p-2)\times(m-2)}$ and $\mathcal{S}_2 = \{\check{\boldsymbol{Q}} \in \mathcal{H}_\infty : \|\check{\boldsymbol{R}}+\check{\boldsymbol{Q}}\|_\infty \le s_2\}$. Further, the first column (row) of $\boldsymbol{Y}_1$ ($\boldsymbol{X}_1$) is equal to the first column (row) of $\boldsymbol{Y}$ ($\boldsymbol{X}$) (see Theorem 4.2).

In the next part of the section we identify the set of all $\boldsymbol{\Delta} \in \partial\mathcal{D}_{r_1}(\boldsymbol{G})$ which destabilize $(\boldsymbol{G}, \boldsymbol{K})$ for every $\boldsymbol{K} \in \mathcal{K}_1$. We refer to such $\boldsymbol{\Delta}$'s as uniformly destabilizing.

Lemma 4.2 *[5] There exists a real rational $\boldsymbol{\Delta} \in \partial\mathcal{D}_{r_1}(\boldsymbol{G})$ such that $(\boldsymbol{G}+\boldsymbol{\Delta}, \boldsymbol{K}) \notin \mathcal{S}$ for every $\boldsymbol{K}\in\mathcal{K}_1$.*

Remark 4.2 *The proof is an adaptation of a result in [14]. Indeed, it is not surprising that (real-rational) destabilizing perturbations exist on $\partial\mathcal{D}_{r_1}(\boldsymbol{G})$. The new information supplied by Lemma 4.2 is that (real-rational) boundary perturbations exist which are destabilizing for* every *maximally robust controller $\boldsymbol{K} \in \mathcal{K}_1$.*

Denote by $\boldsymbol{x}^T$ and $\boldsymbol{y}$ the first row and column of $\boldsymbol{X}$ and $\boldsymbol{Y}$, respectively, defined in Theorem 4.2. Then, all uniformly destabilizing perturbations constructed in Lemma 4.2 have the property that $|\boldsymbol{x}^T\boldsymbol{\Delta}\boldsymbol{y}(j\omega)| = r_1$ for some $\omega \in \mathcal{R}$. Moreover, such perturbations can be constructed for every $\omega \in \mathcal{R}$. The next result shows that condition $|\boldsymbol{x}^T\boldsymbol{\Delta}\boldsymbol{y}(j\omega)| = r_1$ is necessary for a $\boldsymbol{\Delta} \in \partial\mathcal{D}_{r_1}(\boldsymbol{G})$ to be destabilizing for every $\boldsymbol{K}\in\mathcal{K}_1$.

Lemma 4.3 *[5] Let $\boldsymbol{\Delta} \in \partial\mathcal{D}_{r_1}(\boldsymbol{G})$ be a destabilizing perturbation of $\boldsymbol{G}$ for every $\boldsymbol{K} \in \mathcal{K}_1$. Then there exists an $\omega \in \mathcal{R}$, such that*

$$|\boldsymbol{x}^T(j\omega)\boldsymbol{\Delta}(j\omega)\boldsymbol{y}(j\omega)| = r_1. \quad (7)$$

Lemma 4.3 above shows that every $\boldsymbol{\Delta} \in \partial\mathcal{D}_{r_1}(\boldsymbol{G})$ which is destabilizing for all $\boldsymbol{K} \in \mathcal{K}_1$ satisfies $|\boldsymbol{x}^T(j\omega_o)\boldsymbol{\Delta}(j\omega_o)\boldsymbol{y}(j\omega_o)| = r_1$ for some $\omega_o \in \mathcal{R}$. Define the inner product of two matrices of compatible dimensions A and B as $\langle A, B\rangle = trace(A'B)$. Then, (7) says that every $\boldsymbol{\Delta} \in \partial\mathcal{D}_{r_1}(\boldsymbol{G})$ which is destabilizing for all $\boldsymbol{K} \in \mathcal{K}_1$ satisfies $|\langle \boldsymbol{y}(j\omega_o)\boldsymbol{x}^T(j\omega_o), \boldsymbol{\Delta}(j\omega_o)\rangle| = r_1$, i.e. that it has projection of magnitude r_1 in the "most critical direction" $\boldsymbol{y}(j\omega_o)\boldsymbol{x}^T(j\omega_o)$ for some $\omega_o \in \mathcal{R}$. Moreover, the proof of Lemma 4.2 shows that all frequencies $\omega \in \mathcal{R}$ are "equally critical", in the sense that the generalized Nyquist criterion can be violated at any $\omega \in \mathcal{R}$. This implies that it is futile to attempt to extend the uncertainty set guaranteed to be stabilized by a subset of $\mathcal{K}_1$ in the (frequency-dependent) direction $\boldsymbol{y}(j\omega)\boldsymbol{x}^T(j\omega)$, $\omega \in \mathcal{R}$. Suppose now that we impose a "structure" on the perturbation set of the form,

$$|\,\boldsymbol{x}^T(j\omega)\boldsymbol{\Delta}(j\omega)\boldsymbol{y}(j\omega)\,| \le r_1(1-\delta) \quad \forall\, \omega \in \mathcal{R}$$

for some (fixed) $\delta \in (0, 1]$. Note in view of Lemmas 4.2 and 4.3 that this bound is uniform in ω. In other words, we constrain the perturbation set so that $\boldsymbol{\Delta}$ cannot have a projection of magnitude larger that $r_1(1-\delta)$ in the most critical direction for all $\omega\in\mathcal{R}$. Formally, define

$$\mathcal{E}(\delta,\mu)=\{\boldsymbol{\Delta}\in\mathcal{D}_\mu(\boldsymbol{G}):\|\boldsymbol{x}^T\boldsymbol{\Delta}\boldsymbol{y}\|_\infty\le r_1(1-\delta)\} \quad (8)$$

where $\mathcal{D}_\mu(\boldsymbol{G})$ is defined in (1). Then, for each $\delta\in(0,1]$ we seek the set of controllers $\mathcal{K}_\delta\subseteq\mathcal{K}_1$ which maximize $\mu(\delta)$ under the constraint that $\boldsymbol{G}+\boldsymbol{\Delta}$ is stable for all $\boldsymbol{\Delta}\in\mathcal{D}_{r_1}(\boldsymbol{G})\cup\mathcal{E}(\delta,\mu)$. Suppose that the maximum is attained and is given by $\mu^*(\delta)$. It is clear that $\mu^*(\delta)$ is non-decreasing in $\delta\in(0,1]$. It is shown below that the sets $\mathcal{K}_\delta$ are identical for every $\delta\in(0,1]$ and equal to $\mathcal{K}_2$. A closed-form expression of $\mu^*(\delta)$ is also obtained which involves the first two superoptimal levels of $\mathcal{T}$.

The problem formulation above is motivated by a related problem in [8]: Suppose that $A\in\mathcal{C}^{n\times n}$ is nonsingular. We know that if $\bar\sigma(E)=\underline{\sigma}(A)$, then $A-E$ is singular if and only if $\langle u_nv_n',E\rangle=u_n'Ev_n=\underline{\sigma}(A)$, where u_n and v_n denote the singular vectors of A corresponding to $\underline{\sigma}(A)$. Also, if $\bar\sigma(E)<\underline{\sigma}(A)$, then $A-E$ is nonsingular. Suppose that $\bar\sigma(E)=\underline{\sigma}(A)$ and E is constrained to have a projection of magnitude (strictly) less than $\underline{\sigma}(A)$ in the direction u_nv_n'. This means that $A-E$ cannot become singular and so $\bar\sigma(E)$ must increase for $A-E$ to lose rank. To find how much $\bar\sigma(E)$ can increase before singularity occurs, we formulate the problem:

$$d(\phi)=\min\{\,\|E\|:\det(A-E)=0,|\langle u_nv_n',E\rangle|\le\phi\} \quad (9)$$

for $\phi<\underline{\sigma}(A):=\sigma_n(A)$. The solution is provided next.

Theorem 4.3 *[8] Let A be a square nonsingular complex matrix which has a singular value decomposition $A=U\Sigma V'$ where $\Sigma=\mathrm{diag}(\sigma_1,\ldots,\sigma_{n-1},\sigma_n)$ with $\sigma_1\ge\ldots\ge\sigma_{n-2}\ge\sigma_{n-1}>\sigma_n>0$ and denote by u_n and v_n the last columns of U and V, respectively. Then all E which minimize (9) are given by,*

$$E=U\begin{bmatrix}P_s & 0 & 0\\ 0 & -\phi & \nu\\ 0 & \nu' & \phi\end{bmatrix}V'$$

where P_s is arbitrary except for the constraint,

$$\|P_s\|\le\sqrt{\sigma_n\sigma_{n-1}+\phi(\sigma_n-\sigma_{n-1})} \quad (10)$$

and $\nu=\sqrt{(\phi+\sigma_{n-1})(\sigma_n-\phi)}e^{j\theta}$, $\theta\in[0,2\pi)$. The minimum value of $d(\phi)$ in (9) is given by the RHS of (10).

Theorem 4.3 shows that, provided $|\langle u_nv_n',E\rangle|\le\phi<\sigma_n$, $\|E\|$ can increase from σ_n to $\sqrt{\sigma_n\sigma_{n-1}+\phi(\sigma_n-\sigma_{n-1})}$ before $A-E$ loses rank. In [8] this elegant result is exploited to derive robust-stability bounds for a class of additive, multiplicative and inverse-multiplicative perturbations. These results are *a-posteriori*, i.e. they can only be used after a compensator has been designed. In our case, the results in [8] can be applied *a-priori* in that they are used to characterize the subset of all maximally robust controllers which maximize the "radius" $\mu(\delta)$ of the uncertainty set $\mathcal{E}(\mu,\delta)$ defined in (8). This *a-priori* character is a consequence of the alternative parametrization of the set of all optimal interpolation functions given in Theorem 4.2, which shows that there exists a (frequency-dependent) worst case direction (defined by $\boldsymbol{y}=\boldsymbol{M}\boldsymbol{v}$ and $\boldsymbol{x}^T=\boldsymbol{w}^\sim\tilde{\boldsymbol{M}}$ in Lemma 4.3 which is identical for all maximally robust controllers $\boldsymbol{K}\in\mathcal{K}_1$. The vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ are associated with the maximal Schmidt pair of the Hankel operator $\Gamma_{\boldsymbol{R}(-s)}$ [9].

In the sequel, we use Theorem 4.3 to characterize the subset of all optimal controllers $\mathcal{K}_1$ which maximize $\mu^*(\delta)$. We first need a version of Theorem 4.3 which allows us to treat the non-square and the singular cases.

Theorem 4.4 *[5] Suppose that $T\in\mathcal{C}^{p\times m}$ has a singular value decomposition, $T=U\,\mathrm{diag}(\Sigma,0)V'$, with $\Sigma=\mathrm{diag}(\sigma_1,\sigma_2,\ldots,\sigma_t)$, $\sigma_1>\sigma_2\ge\sigma_3\ge\ldots\ge\sigma_t>0$. Let v and u be the first columns of V and U, respectively, and let $\phi<\sigma_1^{-1}$ be given. Define $\mathcal{B}_d^{m\times p}=\{E\in\mathcal{C}^{m\times p}:\|E\|<d\}$, $\mathcal{P}(\phi)=\{E\in\mathcal{C}^{m\times p}:|v'Eu|\le\phi\}$, and*

$$d(\phi)=\sup\{d:\det(I-ET)\ne0,\forall E\in\mathcal{B}_d^{m\times p}\cap\mathcal{P}(\phi)\}.$$

Then $d(\phi)=\sqrt{\sigma_1^{-1}\sigma_2^{-1}-\phi(\sigma_2^{-1}-\sigma_1^{-1})}$. Also, all $E\in\mathcal{P}(\phi)$ such that $\det(I-ET)=0$ and $\|E\|=d(\phi)$ are given by

$$E=V\begin{bmatrix}\phi & \nu & 0\\ \nu' & -\phi & 0\\ 0 & 0 & P_s\end{bmatrix}U'$$

where $\nu=e^{j\theta}\sqrt{\left(\frac{1}{\sigma_2}+\phi\right)\left(\frac{1}{\sigma_1}-\phi\right)}$, $\theta\in[0,2\pi)$ and P_s is arbitrary except from the constraint $\|P_s\|\le d(\phi)$.

Remark 4.3 *Note that $d(\phi)$ depends only on σ_1,σ_2, and ϕ (and hence on u and v) and that it is decreasing in σ_2. Since all optimal interpolating functions have the same largest singular value s_1 (for all frequencies) and share the same left and right singular vectors corresponding to s_1, Theorem 4.4 suggests a link between the maximization of $\mu^*(\delta)$ and the minimization of the second largest singular value of the elements of $\mathcal{T}_1$.*

The next theorem, which is our main result, shows that $\mu^*(\delta)$ is maximized uniquely by the set of all superoptimal controllers with respect to the first two levels.

Theorem 4.5 *[5] Let $\boldsymbol{x}^T$ and $\boldsymbol{y}$ be the first row and column of $\boldsymbol{X}$ and $\boldsymbol{Y}$, respectively, and define $\mathcal{D}_r(\boldsymbol{G})$ and $\mathcal{E}(\delta,\mu)$ as in (1) and (8), respectively, for some fixed $\delta\in(0,1]$. Let $\mu^*(\delta)$ be the supremum of μ such that there exist a $\boldsymbol{K}$ for which $(\boldsymbol{G}+\boldsymbol{\Delta},\boldsymbol{K})\in\mathcal{S}$ for every $\boldsymbol{\Delta}\in\mathcal{D}_{r_1}(\boldsymbol{G})\cup\mathcal{E}(\delta,\mu)$. Then*

1. *For each δ, $\mu^*(\delta)=\sqrt{s_1^{-1}\left(\delta s_2^{-1}+(1-\delta)s_1^{-1}\right)}\ge r_1$.*
2. *$(\boldsymbol{G}+\boldsymbol{\Delta},\boldsymbol{K})\in\mathcal{S}$ for every $\Delta\in\mathcal{D}_{r_1}(\boldsymbol{G})\cup\mathcal{E}(\delta,\mu^*(\delta))$ if and only if $\boldsymbol{K}\in\mathcal{K}_2$.*
3. *(a) $\mathcal{E}(0,\mu^*(0))=\mathcal{D}_{r_1}(\boldsymbol{G})$. (b) For each $\boldsymbol{K}\in\mathcal{K}_2$, $(\boldsymbol{G}+\boldsymbol{\Delta},\boldsymbol{K})\in\mathcal{S}$ for every $\boldsymbol{\Delta}\in\bigcup_{\delta\in(0,1]}\mathcal{E}(\delta,\mu^*(\delta))$.*
4. *Let σ_n and σ_{n-1} denote the two smallest Hankel singular values of $\boldsymbol{G}(-s)$ with $\sigma_{n-1}>\sigma_n$. Then,*

$$\mu^*(\delta)\ge\sqrt{\delta\sigma_n\sigma_{n-1}+(1-\delta)\sigma_n^2}.$$

Figure 2 illustrates the set $\bigcup_{\delta\in(0,1]}\mathcal{E}(\delta,\mu^*(\delta))$ in the 2-dimensional case. Here, $s_1=1$ and $s_2=0.25$. The

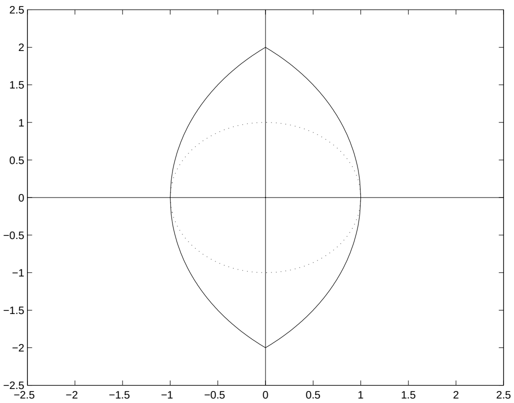

Figure 2: Extended permissible uncertainty set

worst direction is the horizontal axis. The (open) unit disc represents the set of uncertainties guaranteed to be stabilized by optimal controllers. The area bounded by the solid curve represents the set of uncertainties guaranteed to be stabilized by (second-level) superoptimal controllers. Note the increase in the stability radius in all direction other than the worst direction.

5 An upper bound on μ

Our results have so far been restricted to unstructured uncertainties. A different interpretation of our method allows us to tackle structured uncertainty models as well. We use the definitions and notation of [11]. Let $T \in \mathcal{C}^{n\times n}$ have a singular value decomposition

$$T = U\Sigma V', \ \Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n), \ U, V \in \mathcal{C}^{n\times n}, \quad (11)$$

with $U'U = V'V = I_n$ and assume that

$$\sigma_1 > \sigma_2 \geq \sigma_3 \geq \cdots \geq \sigma_n > 0. \quad (12)$$

Define the structured uncertainty set

$$\begin{aligned} \mathbf{\Delta} &= \{\mathrm{diag}(\delta_1 I_{r_1}, \ldots, \delta_S I_{r_S}, \Delta_1, \ldots, \Delta_F) : \\ &\qquad \delta_1, \ldots, \delta_S \in \mathcal{C}, \Delta_j \in \mathcal{C}^{m_j \times m_j}, j = 1, \ldots, F\} \end{aligned}$$

with $\sum_{i=1}^{S} r_i + \sum_{j=1}^{F} m_j = n$ and let $\mathbf{B_\Delta} = \{\Delta \in \mathbf{\Delta} : \|\Delta\| \leq 1\}$. Then the structured singular value of T is defined as

$$\mu_{\mathbf{\Delta}}(T)^{-1} = \min_{\substack{\Delta\in\mathbf{\Delta} \\ \det(I-\Delta T)=0}} \|\Delta\| = \min_{\substack{\Delta\in\mathbf{\Delta} \\ \det(\Sigma^{-1}-V'\Delta U)=0}} \|\Delta\|$$

(if there exists no $\Delta \in \mathbf{\Delta}$ such that $\det(I - \Delta T) = 0$, we define $\mu_{\mathbf{\Delta}}(T) = 0$). Let $u, v \in \mathcal{C}^{n\times 1}$ be the first columns of U and V, respectively. Partition u and v compatibly with $\mathbf{\Delta}$ as follows:

$$\begin{aligned} u &= [u_1^T, \cdots, u_S^T, u_{S+1}^T, \cdots, u_{S+F}^T]^T, \\ v &= [v_1^T, \cdots, v_S^T, v_{S+1}^T, \cdots, v_{S+F}^T]^T, \end{aligned} \quad (13)$$

with $u_i, v_i \in \mathcal{C}^{r_i}$, $u_{S+j}, v_{S+j} \in \mathcal{C}^{m_j}$, for $i = 1, \ldots, S$ and $j = 1, \ldots, F$. Then it is straightforward to verify that

$$\begin{aligned} \alpha_0 &:= \max_{\Delta\in\mathbf{B_\Delta}} |v'\Delta u| \\ &= \sum_{i=1}^{S} |v_i' u_i| + \sum_{j=1}^{F} \|v_{S+j}\|\|u_{S+j}\| \leq 1. \end{aligned} \quad (14)$$

Define $\mathbf{\Delta}_{\alpha_0} = \{\Delta \in \mathcal{C}^{n\times n} : |v'\Delta u| \leq \alpha_0 \|\Delta\|\}$. Then

$$\mu_{\mathbf{\Delta}}(T)^{-1} \geq \min_{\substack{\Delta\in\mathbf{\Delta}_{\alpha_0} \\ \det(\Sigma^{-1}-V'\Delta U)=0}} \|\Delta\| =: \bar{\mu}_{\mathbf{\Delta}}(T)^{-1},$$

since $\mathbf{\Delta}_{\alpha_0} \supseteq \mathbf{\Delta}$. Thus $\bar{\mu}_{\mathbf{\Delta}}(T)$ is an upper bound on $\mu_{\mathbf{\Delta}}(T)$. The next result uses Theorem 4.4 to give an expression for $\bar{\mu}_{\mathbf{\Delta}}(T)$ that is increasing in σ_2.

Theorem 5.1 *[5] Let all variables be as defined above and assume that (12) is satisfied. Let*

$$d = (\sigma_1 - \sigma_2)\alpha_0/2 + \sqrt{[(\sigma_1 - \sigma_2)\alpha_0/2]^2 + \sigma_1\sigma_2}. \quad (15)$$

Then

1. $\bar{\mu}_{\mathbf{\Delta}}(T) := \left(\min_{\substack{\det(\Sigma^{-1}-V'\Delta U)=0 \\ |v'\Delta u| \leq \alpha_0\|\Delta\|}} \|\Delta\|\right)^{-1} = d.$
2. *For all* $\alpha_0 \in [0, 1]$, *we have,* $\mu_{\mathbf{\Delta}}(T) \leq \bar{\mu}_{\mathbf{\Delta}}(T) \leq \sigma_1$.
3. *If* $\alpha_0 = 1$, *then*
$$\mu_{\mathbf{\Delta}}(T) = \bar{\mu}_{\mathbf{\Delta}}(T) = \sigma_1 \quad (16)$$
4. *If* $\alpha_0 < 1$, *then* $\mu_{\mathbf{\Delta}}(T) \leq \bar{\mu}_{\mathbf{\Delta}}(T) < \sigma_1$, *with* $\mu_{\mathbf{\Delta}}(T) = \bar{\mu}_{\mathbf{\Delta}}(T)$ *if and only if there exists* $\Delta \in \mathbf{\Delta}$ *such that*
$$V'\Delta U = d^{-1}\begin{bmatrix} \alpha_0 & e^{j\theta}\sqrt{1-\alpha_0^2} & 0 \\ e^{-j\theta}\sqrt{1-\alpha_0^2} & -\alpha_0 & 0 \\ 0 & 0 & \Delta_{22} \end{bmatrix}$$
for arbitrary θ *and any* $\Delta_{22} \in \mathcal{C}^{(n-2)\times(n-2)}$ *satisfying* $\|\Delta_{22}\| \leq 1$.

Remark 5.1 *Note that* $\bar{\mu}_{\mathbf{\Delta}}(T)$ *depends only on* σ_1, σ_2 *and* α_0 *and* α_0 *in turn depends only on* u, v *and* $\mathbf{\Delta}$ *(see (14)). In the context of the robust stabilization of systems with unstructured additive perturbations, all optimal interpolating functions share the same largest singular value* s_1 *and the same singular vectors corresponding to* s_1 *(vectors* $\boldsymbol{x}$ *and* $\boldsymbol{y}$*). Thus the only free parameter that can be used to minimize* $\bar{\mu}_{\mathbf{\Delta}}(\cdot)$ *within* $\mathcal{T}_1$ *is the second largest singular value. Noting that* $\bar{\mu}_{\mathbf{\Delta}}(\cdot)$ *is nonincreasing in* σ_2 *(see (15)) suggests that within* $\mathcal{T}_1$, $\bar{\mu}_{\mathbf{\Delta}}(\cdot)$ *is minimized by* $\mathcal{T}_2$.

Remark 5.2 *The bound* $\bar{\mu}_{\mathbf{\Delta}}$, *although in general tighter than* σ_1 *is less tight than the upper bound of the D-iteration [11]. In fact, it is shown in [11] that at the end of the D-iteration, either*

1. $\sigma_1(T) = \sigma_2(T)$, *in which case our results are not applicable (see (12)), or*
2. $\sigma_1(T) > \sigma_2(T)$, *in which case* $\mu_{\mathbf{\Delta}}(T) = \sigma_1(T)$. *It can be shown that this corresponds to* $\alpha_0 = 1$.

Our main purpose is to illustrate the improved robustness properties of superoptimal controllers, rather than attempting to improve the D-iteration bound.

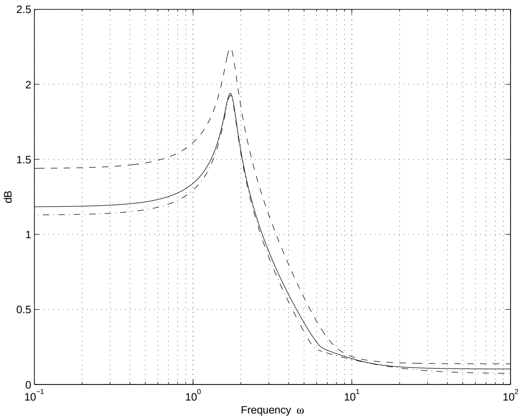

Figure 3: $\sigma_1(\boldsymbol{T})$ (dashed), D-iteration upper bound (dash-dot) and upper bound $\bar{\mu}_{\boldsymbol{\Delta}}(\boldsymbol{T})$ (solid)

Example 5.1 *Figure 3 illustrates the upper bound* $\bar{\mu}_{\boldsymbol{\Delta}}(\boldsymbol{T})$ *for a randomly generated* $\boldsymbol{T} \stackrel{s}{=} \left[\begin{array}{cc} A & B \\ C & D \end{array}\right] =$

$$\left[\begin{array}{rrr|rrrr} -5.91 & -11.49 & 6.03 & 0.59 & -1.90 & 0.19 & 1.35 \\ -1.85 & -5.62 & 1.63 & -0.31 & 1.11 & 0.63 & 0.12 \\ -7.71 & -17.40 & 7.07 & 0.97 & 0.72 & -0.35 & -0.58 \\ \hline -0.37 & 0.49 & 0.52 & 0.01 & 0.22 & 0.71 & 0.97 \\ 1.43 & -0.09 & 1.36 & 0.60 & 0.70 & 0.23 & 0.36 \\ 0.07 & 0.37 & -0.41 & 0.82 & 0.52 & 0.45 & 0.05 \\ -0.23 & -0.15 & 0.66 & 0.98 & 0.93 & 0.17 & 0.76 \end{array}\right]$$

with A stable. The computation is carried out pointwise across the frequency grid. The uncertainty structure $\boldsymbol{\Delta}$ is taken to be diagonal ($S = 0, F = 4, m_j = 1, j = 1, \ldots, 4$).

6 Conclusions

We have analyzed the robust stabilization problem subject to unstructured additive perturbations. We have shown that a critical direction exists in the uncertainty space, along which all maximum-norm boundary perturbations are destabilizing for every optimal controller.

We have shown that by imposing a parametric constraint in the most critical direction, the set of uncertainties guaranteed to be stabilized by a subset of all optimal controllers can be further extended. We have shown that the optimal solution to this problem is associated with the set of superoptimal controllers with respect to the first two levels, and have obtained a closed-form expression for the improved robust stability radius which involves the first two superoptimal levels.

By adapting out results to the structured uncertainty case, we have obtained an easily computable upper bound on the structured- singular value (which is tighter than the largest singular value), without the need to carry out a D-iteration. We have further shown that the minimization of this bound is equivalent to the minimization of the second largest singular value.

For purposes of clarity, our technique has been restricted to unstructured additive uncertainty. There is no conceptual difficulty, however, in extending our method to other types of uncertainty (multiplicative, coprime, etc.) or to include frequency weightings. Rather than analyzing each case individually, we intend to investigate general linear fractional transformation uncertainty models. This is likely to involve a general-distance superoptimal approximation problem, the solution of which is already in place [15], [9], [7].

Our method relies on Theorem 4.4 which generalizes a result in [8]. Section 5 suggests that generalizing this theorem should be useful in robust stability analysis of systems subject to structured uncertainty. We have derived some results in this direction which will be reported in a future publication.

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