Robust Optimization-based Control: an LMI Approach

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Abstract

An optimization-based control technique is developed to defining feasible input trajectories with respect to operating constraints. The proposed approach is based on the construction of a semidefinite programming (SDP) problem equivalent to the constrained control problem. When the system is subject to structured uncertainty, the nominal SDP problem is extended to account for uncertain parameters evolving in a certain ellipsoidal set.

1 Introduction

The intend in this paper is to define feasible input trajectories that satisfy operating constraints. The proposed approach consists in defining an optimization problem equivalent to the constrained control problem. One possible way is to cast this problem in an SDP problem written in terms of Linear Matrix Inequalities that can be efficiently solved using available softwares (see [5, 1, 13]). An alternative way is to use model predictive control which has been widely used for tracking problems subject to constraints (see [6] for a comprehensive survey and [8] for technical details). This technique is also referred to as moving horizon control or receding horizon control. More particularly, it involves a two step-procedure: (1) a sequence of future control actions is chosen according to a prediction of the future evolution of the system and applied to the plant until new measurements are available, (2) a new sequence is determined which replaces the previous one. Although more than one control move is generally calculated, only the first one is implemented. The proposed approach is very similar except the sequence is computed only once over the whole horizon of prediction, thus all calculated control moves are implemented. In other words, the proposed approach is an *open-loop prediction* comparable to the first step of predictive control. In that respect, it could be referred to as predictive control using a *long range horizon*. Since the sequence of control does not take into account new measurements, it is assumed there is a feedback controller that guarantees stability along the trajectory. This assumption is fairly standard in the model predictive approach and has found to be very convenient from a practical point of view.

Given a nonlinear continuous-time closed loop system, linearized discrete-time system can be derived around an operating point. One way to proceed is to use a standard forward-Euler discretization scheme. This approximation leads to modelling errors that can be represented as structured uncertainties. Moreover, the original nonlinear system may be not well known, thus increasing the discrepancy between the actual controlled plant and the linear model. Such errors can be captured by an uncertain system. The proposed work is extended to cope with structured uncertainty. An alternative way is to use classical approachs like the robust receding horizon control (see e.g. [9] and [11]).

Section 2 is devoted to the nominal SDP while Section 3 extends the nominal SDP problem to account for uncertainty. In each section, the control approach employed is presented in distinct steps: system considered and associated SDP problem, construction of an equivalent SDP-problem, illustration using a second order system. Concluding remarks end the paper.

2 Nominal SDP-problem

2.1 A discrete-time system without uncertainty

The problem considered in this paper consists in the computation of a control sequence $u = (u_0, \ldots, u_T)'$ for a discrete-time system of the form:

$$\forall k = 0, \ldots, T, \left\{ \begin{array}{rcl} x_{k+1} & = & Ax_k + Bu_k \\ y_k & = & Cx_k + Du_k \end{array} \right. \quad (1)$$

such that the outputs satisfy the constraints:

$$\forall k = 0, \ldots, T, \ \underline{y}_k \leq y_k \leq \overline{y}_k \ , \quad (2)$$

and the control satisfy the following constraints:

$$\forall k = 0, \ldots, T,\ \underline{u}_k \leq u_k \leq \overline{u}_k\ , \tag{3}$$

with the objective:

$$\texttt{minimize}\ c'u\ . \tag{4}$$

where the time horizon T, the initial state x_0, the constraints $\underline{y}_k, \overline{y}_k, \underline{u}_k, \overline{u}_k$ and the objective c are given.

2.2 Semi-definite programming problem

We consider a semi-definite programming problem (SDP) of the form:

$$\texttt{min} c'_{\mathcal{F}}\xi \text{ subject to } \mathcal{F}(\xi) = \mathcal{F}_0 + \sum_{i=1}^{m} \xi_i \mathcal{F}_i \geq 0 \tag{5}$$

where $c_{\mathcal{F}} \in \mathbb{R}^m - \{0\}$, and the symmetric matrices $\mathcal{F}_i = \mathcal{F}'_i \in \mathbb{R}^{n\times n}, i = 0, \ldots, m$ are given. SDPs are convex optimization problems and can be solved in polynomial-time with e.g. primal-dual interior-point methods [10, 13, 12, 7, 2]. SDPs include linear programs and convex constrained programs, and arise in a wide range of engineering applications, see e.g. [5, 1, 13].

2.3 Construction of an equivalent SDP-problem

We can build an SDP problem which solves the problem of computing u such that (2)-(4) are satisfied using the following:

Theorem 2.1 *We can build a vector $c_{\mathcal{F}}$ and an affine function $\mathcal{F}$ such that the SDP problem (5) and the discrete-time problem (1)-(4) are equivalent.*

Proof:

Let for all $0 \leq k \leq T$, $z_k = (y'_k\ u'_k)'$, $\overline{z}_k = (\overline{y}'_k\ \overline{u}'_k)'$, $\underline{z}_k = (\underline{y}'_k\ \underline{u}'_k)'$ and the matrices $\tilde{C} = (C'0)'$, $\tilde{D} = (D'I)'$. We can rewrite the problem as an equivalent form:

Find a control sequence $u_0, \ldots, u_T$ for the discrete-time dynamical system:

$$\forall k = 0, \ldots, T, \left\{ \begin{array}{rcl} x_{k+1} & = & Ax_k + Bu_k \\ z_k & = & \tilde{C}x_k + \tilde{D}u_k \end{array} \right.$$

such that the output z satisfies the constraint:

$$\forall k = 0, \ldots, T,\ \underline{z}_k \leq z_k \leq \overline{z}_k\ . \tag{6}$$

Therefore we obtain a discrete-time dynamical system such that all constraints have being translated into the output z_k. Suppose that the vectors $(z_0, \ldots, z_T)$ are in the vector space $\mathbb{R}^{n_z}$. Define the integer $n_Z := (T+1)n_z$. We can write the constraints (6) as:

$$\texttt{diag}\left(\left(\begin{array}{cc} z^i_{rk} & z^i_k - z^i_{ck} \\ z^i_k - z^i_{ck} & z^i_{rk} \end{array} \right) \begin{array}{c} 1 \leq i \leq n_z \\ 0 \leq k \leq T \end{array} \right) \geq 0$$

where z^i_k is the i-th component of z_k and z_{rk} and z_{ck} define respectively the radius and the center of the polytope for the constraints. More precisely they are defined, for all by $0 \leq k \leq T$, by $z_{rk} = \frac{\overline{z}_k - \underline{z}_k}{2}$, $z_{ck} = \frac{\overline{z}_k + \underline{z}_k}{2}$. Let the generalized output vector Z, the generalized radius vector, the generalized center vector, the generalized state vector X and the generalized input vector U be defined respectively by $Z = (z_0, \ldots, z_T)'$, $Z_r = (z_{r0}, \ldots, z_{rT})'$, $Z_c = (z_{c0}, \ldots, z_{cT})'$, $X = (x_1, \ldots, x_{T+1})'$, $U = (u_0, \ldots, u_T)'$. We obtain $Z = M * U + Nx_0$, where M and N are defined respectively by $M = P * \texttt{diag}(B) + \texttt{diag}(\tilde{D})$, $N = (\tilde{C}', 0, \ldots, 0)' + P * (A', 0, \ldots, 0)'$, with

$$P = \left(\begin{array}{cccc} 0 & 0 & & \\ \tilde{C} & \ddots & \ddots & \\ & \ddots & \ddots & 0 \\ & & \tilde{C} & 0 \end{array} \right) \left(\begin{array}{cccc} I & 0 & & \\ -A & \ddots & \ddots & \\ & \ddots & \ddots & 0 \\ & & -A & I \end{array} \right)^{-1} .$$

The generalized output vector Z evolves in the vector space $\mathbb{R}^{nz}$ and define $(e_i)_{i\in[1,n_Z]}$ the canonical basis of the vector space $\mathbb{R}^{nz}$. So the problem (2)-(4) is equivalent to the following one:

Find U such that: $\mathcal{F}(U) \geq 0$, with the objective minimize $c'_{\mathcal{F}}U$, where $c_{\mathcal{F}} = (c', \ldots, c')'$ and $\mathcal{F}(U)$ is defined by

$$\mathcal{F}(U) = \texttt{diag}_{i\in[1,n_Z]} \left(\begin{array}{cc} e'_i Z_r & e'_i(M * U + Nx_0 - Z_c) \\ e'_i(M * U + Nx_0 - Z_c) & e'_i Z_r \end{array} \right) \tag{7}$$

So we have rewritten the problem (2)-(4) in a classical SDP-problem (5) in variable U; we can compute the generalized input-vector U solving only one SDP. □

Remark 2.2 We may use an algorithm which uses the structure of the affine function $\mathcal{F}$. Indeed $\mathcal{F}$ has a sparse form.

Moreover it is preferable to compute the control sequence $(u_0, \ldots, u_T)$ in one step, like proposed in the proof, rather than to compute the control step by step, i.e. to build a SDP problem to compute u_0 and then another SDP problem to compute u_1 and so on. Indeed few number of steps can lead to an infeasible problem even though the global problem may be feasible. ◇

2.4 An example

Consider a second-order, continuous-time controlled system. We suppose that the data of this system have nominal values:

$$\left\{ \begin{array}{rcl} x'' + a^{nom}_1 x' + a^{nom}_2 x & = & a^{nom}_2 u \\ y & = & x \end{array} \right. \tag{8}$$

where the real parameters have the nominal value a_i^{nom}, $i = 1, 2$. We want to find $u(t)$ such that, for all $t \geq 0$, we have $\underline{u}(t) \leq u(t) \leq \overline{u}(t)$ and such that the output of the system (8) satisfies, for all $t \geq 0$, $\underline{y}(t) \leq y(t) \leq \overline{y}(t)$. Note that we could take a higher order of constraint like a constraint on the derivative of x. By discretizing this system using a forward-Euler scheme with sampling frequency h, we obtain a system of the form (1)-(3):

Find a control sequence $(u_0, \cdots, u_T)$ such that, for all $0 \leq t \leq T$, $\underline{u}_k \leq u_k \leq \overline{u}_k$ and such that the output $(y_0, \cdots y_T)$ of the two dimensional discrete-time dynamical system defined by (1) verifying the inequalities, for all $0 \leq k \leq T$, $\underline{y}_k \leq y_k \leq \overline{y}_k$, where

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \left(\begin{array}{cc|c} 1 & h & 0 \\ -ha_2^{nom} & -ha_1^{nom} & ha_2^{nom} \\ \hline 1 & 0 & 0 \end{array}\right),$$

and where the constraints are defined, for $0 \leq k \leq T$, by $\underline{y}_k = \underline{y}(kh)$, $\overline{y}_k = \overline{y}(kh)$, $\underline{u}_k = \underline{u}(kh)$ and $\overline{u}_k = \overline{u}(kh)$.

Consider the open loop response to the constant input $u = 1$ with initial condition $x_0 = (0, 0)'$. For $h = 0.1$, $a_1^{nom} = 3$, $a_2^{nom} = 9$, a time horizon $T = 50$ steps, Figure 1 shows the output y_k (i.e. the state x of (8)) obtained with the constant input $u = 1$.

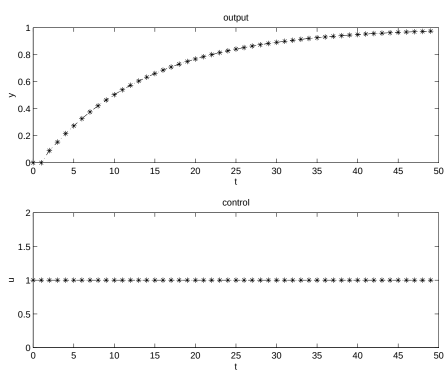

Figure 1: The output of the discrete dynamical system without uncertainty for the input $u = 1$.

We observe that it converges very slowly to the equilibrium $x = 1$. We can look for a new control such that the state converges to the equilibrium in less than 10 iterations of (1). More precisely, let the following problem:

Find a control input u such that, for all $t \geq 0$, $-10 \leq u(t) \leq 10$, and such that the state x in (8) verifies, for all $0 \leq t \leq 4h$, $0 \leq x(t) \leq 2$, for all $4h \leq t \leq 7h$, $0.4 \leq x(t) \leq 1.6$, for all $t \geq 7h$, $0.9 \leq x(t) \leq 1.1$.

The equivalent discrete-time problem is the following:

Find a control sequence $(u_0, \ldots, u_T)$ such that

$$\forall k = 0, \ldots, T, \ -10 \leq u_k \leq \ 10 \quad , \qquad (9)$$

and such that the output of the system (1) satisfies:

$$\begin{aligned} \forall k = 0, \ldots, 3, &\quad 0 \leq y_k \leq 2 \quad , &(10)\\ \forall k = 4, \ldots, 6, &\quad 0.4 \leq y_k \leq 1.6 \quad , &(11)\\ \forall k = 7, \ldots, T, &\quad 0.9 \leq y_k \leq 1.1 \quad . &(12) \end{aligned}$$

We build the affine function $\mathcal{F}$ as shown in the proof of Theorem 2.1 and solve the corresponding SDP problem to obtain the input u and the output y satisfying (9)-(12) for a horizon $T = 9$ steps as shown in Figure 2. Note that the initial value u_0 is large with respect to the others values. Suppose that we have a transition between two equilibria at $t = 0$, more precisely suppose that for the non positive time t we have $u \equiv 0$ and so the equilibrium $x \equiv 0$, and that we want to reach the other equilibrium $x \equiv 1$ for the non negative time. So it may be essential to constraint the control sequence $(u_0, \ldots, u_T)$ to have small variations and the initial value u_0 to be small. This leads to the following new constraint:

$$\begin{aligned} |u_0| &< maxvar \quad (13)\\ \forall k = 0, \cdots, T-1, \ |u_{k+1} - u_k| &< maxvar \ ,(14) \end{aligned}$$

where $maxvar$ is a new parameter which makes the control smoother and the value of $u(0)$ smaller. In Figure 2, the maximal variation ($maxvar$) of u is 6.5. But we can compute a new input u even smoother such that the maximal variation of u is less than $maxvar = 0.8$. We can build (see proof of Theorem 2.1) an affine function $\mathcal{F}$ so that the problem (9)-(12), (13)-(14) with $maxvar = 0.8$ and the SDP problem (5) are equivalent. Figure 3 shows the result of the computation of a possible control. Note that this new constraint makes the output much smoother.

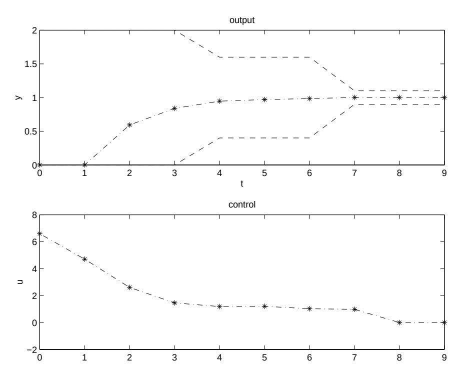

Figure 2: The output and the optimal input of the discrete dynamical system.

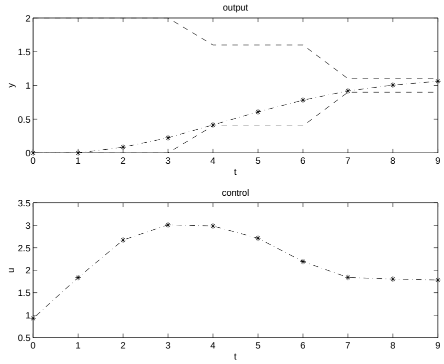

Figure 3: The output and the optimal "smoother" input of the discrete dynamical system.

3 Robust SDP approach

3.1 A discrete-time system subject to structured uncertainty

In this paper, we will consider uncertain systems modelled as, for all $0 \leq k \leq T$,

$$\left\{ \begin{array}{rcl} x_{k+1} & = & \mathbf{A}(\Delta_k)x_k + \mathbf{B}(\Delta_k)u_k \\ y_k & = & \mathbf{C}(\Delta_k)x_k + \mathbf{D}(\Delta_k)u_k \end{array} \right. \quad (15)$$

where Δ_k is a (possible time-varying) uncertain matrix. We want to compute a control sequence $u = (u_0, \ldots, u_T)'$ such that the output of the system (15) satisfies the constraints:

$$\forall k = 0, \ldots, T, \; \underline{y}_k \leq y_k \leq \overline{y}_k \; , \quad (16)$$

and the control satisfies the following constraints:

$$\forall k = 0, \ldots, T, \; \underline{u}_k \leq u_k \leq \overline{u}_k \; , \quad (17)$$

with the objective:

$$\mathtt{minimize} \; c'u \; . \quad (18)$$

where the time horizon T, the initial state x_0, the constraints $\underline{y}_k, \overline{y}_k, \underline{u}_k, \overline{u}_k$ and the objective c are given. We assume that the matrix-valued functions $\mathbf{A}(\Delta)$, $\mathbf{B}(\Delta)$, etc, are given by a *linear-fractional representation* (LFR):

$$\left(\begin{array}{cc} \mathbf{A}(\Delta) & \mathbf{B}(\Delta) \\ \mathbf{C}(\Delta) & \mathbf{D}(\Delta) \end{array} \right) = \left(\begin{array}{cc} A & B \\ C & D \end{array} \right) + L\Delta(I - H\Delta)^{-1}R$$

where A, B, C, D, L, R and H are constant matrices, while $\Delta \in \mathbf{\Delta}$, where $\mathbf{\Delta}$ is a bounded set of matrices. The subspace $\mathbf{\Delta}$ is called the structure set and defines the structure of the perturbation which must be norm-bounded. Together, the matrices A, B, C, D, L, R, H and $\mathbf{\Delta}$, constitute a *linear-fractional representation* (LFR) of our uncertain system. In this paper we will assume that the LFR is well-posed over $\mathbf{\Delta}$ meaning that we have $\forall \Delta \in \mathbf{\Delta}$, $\mathtt{det}(I - H\Delta) \neq 0$. As it turns out, this is easy to cope with in our context, since our system is actually in closed loop with a stabilizing controller that will guarantee all external disturbances are asymptotically rejected. For the uncertainty set $\mathbf{\Delta}$, we can consider a very large class of matrix sets (see examples in [4]) but here we fix the class of sets $\mathbf{\Delta}$ to clarify the exposition. More precisely we consider only *ellipsoidal uncertainty*. This means that the perturbation consists of block vectors, each being subject to an Euclidean-norm bound, more precisely

$$\mathbf{\Delta} = \{\mathtt{diag}(\delta_1 I_{r_1}, \cdots, \delta_l I_{r_l}), \; (\delta_1, \ldots, \delta_l)' \in \mathcal{D}\} \; , \quad (19)$$

where $\mathcal{D}$ is equal to

$$\left\{ \left(\begin{array}{c} \delta_1 \\ \vdots \\ \delta_l \end{array} \right) \in \mathbb{R}^l | \; \delta = \left(\begin{array}{c} \delta^1 \\ \vdots \\ \delta^N \end{array} \right), \; \begin{array}{c} \delta^k \in \mathbb{R}^{n_k}, \\ |\delta^k\|_2 \leq \rho, \\ 1 \leq k \leq N \end{array} \right\} ,$$

where $\rho \geq 0$ is a given parameter that determines the "size" of the uncertainty, the integers n_k denote the lengths of each block vector δ_k (we have of course $n_1 + \cdots + n_N = l$) and $r_1, \ldots, r_l$ are given integers.

3.2 The robust SDP problem

We consider here a robust semidefinite programming problem of the form:

$$\mathtt{minimize} \; c'_{\mathbf{F}}\xi \text{ subject to } \mathbf{F}(\xi, \Delta) \geq 0 \; , \quad (20)$$

where $c_{\mathbf{F}} \in \mathbb{R}^m - \{0\}$, Δ is a "perturbation matrix" that is only known to belong to a set $\mathbf{\Delta}_{\mathbf{F}}$, and $\mathbf{F}$ is a map from $\mathbb{R}^n \times \mathbf{\Delta}_{\mathbf{F}}$ to $\mathcal{S}^n$, the set of the symmetric matrices in $\mathbb{R}^{n \times n}$. We assume that $\mathbf{F}$ is given by a "linear-fractional representation" (LFR):

$$\begin{array}{rcl} \mathbf{F}(\xi, \Delta) & = & F(\xi) + L_{\mathbf{F}}(\xi)\Delta(I - H_{\mathbf{F}}\Delta)^{-1}R_{\mathbf{F}} \\ & & +R'_{\mathbf{F}}(I - \Delta' H'_{\mathbf{F}})^{-1}\Delta' L_{\mathbf{F}}(\xi)' \; , \end{array} \quad (21)$$

where F is an affine function like (5), $L_{\mathbf{F}}(.)$ is an affine mapping taking values in $\mathbb{R}^{n \times p}$, $R_{\mathbf{F}} \in \mathbb{R}^{q \times n}$ and $H_{\mathbf{F}} \in \mathbb{R}^{q \times p}$ are given matrices, while $\Delta \in \mathbf{\Delta}_{\mathbf{F}}$, where $\mathbf{\Delta}_{\mathbf{F}}$ is a bounded set of matrices. We will assume that the LFR is well-posed over $\mathbf{\Delta}$ meaning that we have $\forall \Delta \in \mathbf{\Delta}$, $\mathtt{det}(I - H_{\mathbf{F}}\Delta) \neq 0$. We consider only SDP problem with *ellipsoidal* perturbation sets $\mathbf{\Delta}_{\mathbf{F}}$ equal to

$$\{\mathtt{diag}(\delta_1 I_{r_{1\mathbf{F}}}, \cdots, \delta_{l_{\mathbf{F}}} I_{r_{l_{\mathbf{F}}\mathbf{F}}}), \; (\delta_1, \ldots, \delta_{l_{\mathbf{F}}\mathbf{F}})' \in \mathcal{D}_{\mathbf{F}}\} \; , \quad (22)$$

where $\mathcal{D}_{\mathbf{F}}$ is equal to

$$\left\{ \left(\begin{array}{c} \delta_1 \\ \vdots \\ \delta_{l_{\mathbf{F}}\mathbf{F}} \end{array} \right) \in \mathbb{R}^{l_{\mathbf{F}}} \; | \; \delta = \left(\begin{array}{c} \delta^1 \\ \vdots \\ \delta^{N_{\mathbf{F}}} \end{array} \right), \; \begin{array}{c} \delta^k \in \mathbb{R}^{n_{k\mathbf{F}}}, \\ \|\delta^k\|_2 \leq \rho_{\mathbf{F}}, \\ 1 \leq k \leq N_{\mathbf{F}} \end{array} \right\} ,$$

where $\rho_{\mathbf{F}} \geq 0$ is a given parameter that determines the "size" of the ellipsoidal uncertainty in the function $\mathbf{F}$, the integers $n_{k\mathbf{F}}$ denote the lengths of each block vector δ_k (we have of course $n_{1\mathbf{F}} + \cdots + n_{N_{\mathbf{F}}\mathbf{F}} = l_{\mathbf{F}}$)

and $r_{1\mathbf{F}}, \ldots, r_{l_{\mathbf{F}}\mathbf{F}}$ are given integers. But the following Theorem 3.1 is available in a large class of set $\boldsymbol{\Delta}$ (see [4]). Let J_k be the set of indices $\nu_{k-1}+1, \ldots, \nu_k$ with $\nu_0 = 0$, $\nu_k = \sum_{i=1}^{k} n_{i\mathbf{F}}$. The following theorem can be shown with Lagrangian relaxations techniques see [4]:

Theorem 3.1 *Consider the uncertain semidefinite program with rational perturbation described by the LFR (21), where the perturbation matrix Δ lies in the set $\boldsymbol{\Delta}_{\mathbf{F}}$ defined by (22) and assume the LFR is well-posed over $\boldsymbol{\Delta}_{\mathbf{F}}$. Consider the semidefinite program*

$$\texttt{minimize } c'_{\mathbf{F}}\xi \text{ subject to } S \geq 0,$$
$$\begin{pmatrix} F(\xi) & L_{\mathbf{F}}(\xi) \\ L_{\mathbf{F}}(\xi)' & 0 \end{pmatrix} \geq \begin{pmatrix} R_{\mathbf{F}} & H_{\mathbf{F}} \\ 0 & I \end{pmatrix}' \begin{pmatrix} T & 0 \\ 0 & -S \end{pmatrix} \begin{pmatrix} R_{\mathbf{F}} & H_{\mathbf{F}} \\ 0 & I \end{pmatrix},$$

where $S = \mathtt{diag}(S_1, \ldots, S_{N_{\mathbf{F}}})$, with each S_k of size $\sum_{i\in J_k} r_{i\mathbf{F}}$, and T is the block-diagonal matrix formed with the block-diagonal $r_{i\mathbf{F}} \times r_{i\mathbf{F}}$ blocks of S. Then the above semidefinite program in variables ξ, S is an approximation of the robust counterpart (20), i.e. the projection of the feasible set of (20) on the space of ξ-variables is contained in the set of robust feasible solutions.

3.3 Construction of a SDP-problem solving the problem subject to structured uncertainty.

We have Theorem 2.1 in the following *robust* form:

Theorem 3.2 *We can build a vector $c_{\mathcal{F}}$ and an affine function $\mathcal{F}$ such that the SDP problem (5) and the discrete-time problem (16)-(18) are equivalent.*

Proof: Following the proof of Theorem 2.1 to compute the LFR of the *generalized output matrix* of the equation (7), we obtain an SDP problem which is given by an LFR depending of the *generalized input vector U* defined by $U = (u'_0, \ldots, u'_T)'$. So we can make a Lagrangian relaxation and use Theorem 3.1 to build an SDP problem (5) equivalent to (16)-(18). □

3.4 An example

We consider now a second-order, continuous-time controlled system with uncertainty in the data:

$$\begin{cases} x'' + \mathbf{a}_1(t)x' + \mathbf{a}_2(t)x &= \mathbf{a}_2(t)u \\ y &= x \end{cases} \tag{23}$$

where the uncertain, time-varying parameters $\mathbf{a}_i$, $i = 1, 2$ are subject to bounded variation of given relative amplitude ρ, more precisely $\mathbf{a}_i(t) = a_i^{nom}(1 + \rho\delta_i(t))$, $i = 1, 2$, $t \geq 0$, where $-1 \leq \delta_i(t) \leq 1$ for every t, and a_i^{nom}, $i = 1, 2$, is the nominal value of the parameters. We want to find $u(t)$ such that, for all $t \geq 0$, $\underline{u}(t) \leq u(t) \leq \overline{u}(t)$, and such that the output of the system (23), for any unknown function $\delta_i(t)$, $i = 1, 2$ such that $-1 \leq \delta_i(t) \leq 1$ for every t, satisfy, for all $t \geq 0$, $\underline{y}(t) \leq y(t) \leq \overline{y}(t)$. By discretizing this system using a forward-Euler scheme with discretization period h, we obtain a system of the form (15), with the following LFR:

$$\left(\begin{array}{c|c} \mathbf{A}(\boldsymbol{\Delta}) & \mathbf{B}(\boldsymbol{\Delta}) \\ \hline \mathbf{C}(\boldsymbol{\Delta}) & \mathbf{D}(\boldsymbol{\Delta}) \end{array}\right) = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) + L\Delta R,$$

where $\Delta = \mathtt{diag}(\delta_1, \delta_2)$ and

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \left(\begin{array}{cc|c} 1 & h & 0 \\ -ha_2^{nom} & -ha_1^{nom} & ha_2^{nom} \\ \hline 1 & 0 & 0 \end{array}\right),$$

$$L = -h\rho \left(\begin{array}{cc} 0 & 0 \\ a_1^{nom} & a_2^{nom} \\ \hline 0 & 0 \end{array}\right), R = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 0 & -1 \end{array}\right).$$

With this choice of C and D, we have $y_k = y(kh)$ and we take, for all $0 \leq k \leq T$, $\underline{y}_k = \underline{y}(kh)$, $\overline{y}_k = \overline{y}(kh)$. We look for a control such that the state converges to the equilibrium in less than 10 iterations of (15). More precisely, let define the following problem:

Find a controller u such that, for all $t \geq 0$, $-10 \leq u(t) \leq 10$ and such that the state x in (23) verifies, for all $0 \leq t \leq 4h$, $0 \leq x(t) \leq 2$, for all $4h \leq t \leq 6h$, $0.4 \leq x(t) \leq 1.6$ and, for all $t \geq 6h$, $0.7 \leq x(t) \leq 1.3$.

The discrete-time equivalent problem is the following:

Find a control sequence $(u_0, \ldots, u_T)$ such that the output of the system (23) satisfies:

$$\forall k = 0, \ldots, 3, \quad (0;\ -10)' \leq y_k \leq (2;\ 10)' \ , \tag{24}$$
$$\forall k = 4, 5, \quad (0.4;\ -10)' \leq y_k \leq (1.6;\ 10)' \ , \tag{25}$$
$$\forall k = 6, \ldots, T, \quad (0.7;\ -10)' \leq y_k \leq (1.3;\ 10)' \ . \tag{26}$$

We can build as in the proof of Theorem 3.2 an affine function $\mathcal{F}$ so that the problem (24), (25) and (26) and the SDP problem (5) are equivalent, and then we solve the corresponding SDP problem to obtain the control sequence $(u_0, \ldots, u_T)$ and the output y satisfying (24), (25) and (26) for a time horizon $T = 7$ steps as shown in Figure 4. Note that the initial value u_0 is large with respect to the others values. As explained in the nominal case in section 2.4, it may be essential to add the new constraints (13)-(14) where $maxvar$ is a new parameter which makes the control sequence smoother and the value of $u(0)$ smaller. In Figure 4, the maximal variation ($maxvar$) of u is 4.5. But we can compute a new control sequence $(u_0, \ldots, u_T)$ even smoother such that the maximal variation is less than $maxvar = 1$. We can build as in the proof of Theorem 3.2 an affine function $\mathcal{F}$ so that the problem (24)-(26), (13)-(14) with $maxvar = 1$ and the SDP problem (5) are equivalent. The figure 5 shows the result of the computation of a possible control. Note that this new constraint makes the output much smoother.

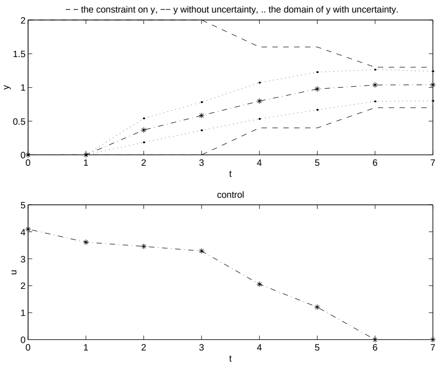

Figure 4: The output and the optimal input of the discrete dynamical system with uncertainty.

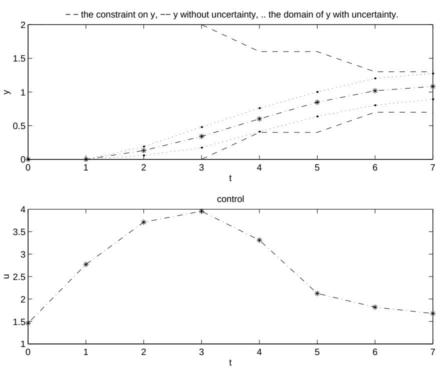

Figure 5: The output and the optimal “smoother” input of the discrete dynamical system with uncertainty.

4 Conclusion

An optimization-based control technique has been developed to defining feasible input trajectories with respect to operating constraints. This constrained problem is cast in an equivalent semidefinite programming (SDP) problem. When the system is subject to structured uncertainty, the nominal SDP problem is extended to account for uncertain parameters evolving in a certain ellipsoidal set. To illustrate the use of this technique, a constrained control problem involving a second order system is solved numerically with and without structured uncertainty. The simulations show that the constraints are satisfied. Furthermore, the results demonstrate that the control can be quite smooth when an additional constraint is used. This could be a crucial aspect from a practical prospective. This technique seems more appropriate to predict trajectories over a long range horizon. In the latter case, the computation is completed off line, but measurement cannot be used and a feedback controller is needed to guarantee stability along the trajectory. An application of such approach using the proposed technique is currently underway [3].

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