On the existence of common Lyapunov triples for ISS and iISS switched systems

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Abstract

In this paper, we present converse Lyapunov theorems for ISS and iISS switched nonlinear systems. Their proofs are based on existing converse Lyapunov theorems for IOSS and iISS nonlinear systems, and on the association of the switched system with a nonlinear system with inputs and disturbances that take values in a compact set.

Key Words: Converse Lyapunov theorems; ISS; IOSS; iISS; Nonlinear; Switched systems.

1 Introduction

Recently, the study of switched systems has received a great deal of attention, being the rapidly developing area of intelligent control an important source of motivation for this study. Informally, a switched system is a family of continuous-time dynamical subsystems and a rule that determines the switching between them. The recent paper [8] is a very interesting survey on the subject, where an updated account of results and open problems may be found.

On the other hand, the notion of input-to-state stability (ISS), introduced a decade ago, resulted in an useful concept upon which much of modern nonlinear feedback analysis and design rest. Recently the concept of integral input-to-state stability (iISS) has been introduced and is expected to play a role at least as prominent as the one that ISS currently has. Both stability concepts have recently been used in relation with switched systems in [9], [2] and [6].

A problem that naturally appears in this context, may be stated as follows: given a switched system whose component subsystems are each ISS (or iISS), find necessary and sufficient conditions for the ISS (or iISS) of the switched system, uniformly with respect to the switching signals.

It can be shown that the sufficient conditions for ISS and iISS established in [13] and [1] respectively carry over with little changes to the switched system. In fact, as was pointed in [9], the existence of a common ISS Lyapunov triple implies that the switched system is ISS for arbitrary switching, and similarly for iISS. The question of the validity of the converses, that arises naturally, originates the problem of the existence of such Lyapunov triples, *i.e.*, of the existence of converse Lyapunov theorems.

In this paper we obtain converse Lyapunov theorems for a certain class of switched nonlinear systems. These theorems are in the spirit of the converse theorem for asymptotic stability of switched nonlinear systems obtained by Mancilla-Aguilar and García ([11]), since we also suppose that the index set is not endowed *a priori* with any topology. Following the ideas there developed, we base our approach on the association of the switched nonlinear system with a perturbed control system whose disturbances take values in a compact set. Once this association is performed, we obtain straightforward proofs of the converse theorems above by extending the results about input-output-to-state stability (IOSS) and iISS obtained in [7] and in [1] respectively.

The outline of the paper is as follows. In section 2. we give the basic definitions including those of switched system, uniform input-to-state stability (UISS), uniform integral-input-to-state stability (UiISS), etc. In section 3 we associate a perturbed control system whose disturbances take values in a compact set, with a class of switched nonlinear systems, and based on this association we prove the converse Lyapunov theorems. Finally, in section 4 we present some conclusions.

2 Switched Systems, UISS and UiISS

Here we introduce some notations and definitions that will be used in the sequel. We denote with $\mathbb{R}^n$ the usual n-dimensional Euclidean space, and with $|\cdot|$ its Euclidean norm. $B_r^n \subset \mathbb{R}^n$ stands for the closed ball of radius r from the origin in $\mathbb{R}^n$. The set of measurable locally essentially bounded functions $\mathbf{u} : [0, +\infty) \to \mathbb{R}^m$

is denoted by $L^m_{\infty,e}$; for each $t \geq 0$ and each $\mathbf{u} \in L^m_{\infty,e}$ we denote with $\mathbf{u}_t$ the truncation of $\mathbf{u}$ at t, *i.e.*, $\mathbf{u}_t(\tau) = \mathbf{u}(\tau)$ if $\tau \leq t$, and $\mathbf{u}_t(\tau) = 0$ if $\tau > t$, and $\|\mathbf{u}_t\| = \text{ess sup}\,\{|\mathbf{u}(\tau)|, 0 \leq \tau \leq t\}$.

Following standard terminology, a continuous function $\alpha : [0, T) \to \mathbb{R}$ is *positive definite* if $\alpha(0) = 0$ and $\alpha(r) > 0$ whenever $r > 0$. α is a $\mathcal{K}$ *class* function if it is positive definite and strictly increasing. A $\mathcal{K}_\infty$ *class* function α is a function of class $\mathcal{K}$, defined in $[0, +\infty)$ that verifies $\lim_{r\to+\infty} \alpha(r) = +\infty$. A continuous function $\beta : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is a $\mathcal{KL}$*-class* function if for any fixed t, $\beta(\cdot, t)$ is of class $\mathcal{K}$, and for a fixed $r > 0$, $\beta(r, \cdot)$ is decreasing and $\lim_{t\to+\infty} \beta(r, t) = 0$.

Finally, we introduce from nonsmooth analysis (see [3]) the notion of proximal subgradient that will be used in the paper.

A vector $\zeta \in \mathbb{R}^n$ is a *proximal subgradient* of the function $\varphi : \mathbb{R}^n \to \mathbb{R}$ at ξ, if there exist positive constants κ and δ such that

$$\varphi(\eta) - \varphi(\xi) \geq \zeta(\eta - \xi) - \kappa|\eta - \xi|^2$$

for all $|\eta - \xi| < \delta$. We denote with $\partial_P\varphi(\xi)$ the (possibly empty) set of proximal subgradients of φ at ξ.

Let the family $\mathcal{P} = \{f_\sigma(x, u),\ f_\sigma : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n,\ \sigma \in \Gamma\}$, where Γ is an index set, $f_\sigma(0, 0) = 0 \forall \sigma \in \Gamma$ and for each $\sigma \in \Gamma$, f_σ is locally Lispchitz.

We will assume in the sequel, with no loss of generality, that there is a one to one correspondence between the elements of Γ and those of $\mathcal{P}$, *i.e.*, given $\sigma \neq \sigma'$ elements of Γ, then $f_\sigma \neq f_{\sigma'}$.

Given the family $\mathcal{P}$, we consider the switched system

$$\dot{x}(t) = f_s(x(t), \mathbf{u}(t)), \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $\mathbf{u} \in L^m_{\infty,e}$ and s is a *switching signal*, *i.e.*, s is a piecewise constant function $s : [0, +\infty) \to \Gamma$; we will denote by $\mathcal{S}$ the family of the switching signals of a given switched system. Associated with each $s \in \mathcal{S}$ there is a sequence of real numbers $0 = t_0 < t_1 < \cdots < t_k < \cdots$ and a sequence of indexes $\sigma_0, \sigma_1, \cdots, \sigma_k, \cdots$ such that $s(t) = \sigma_k$ for all $t_k \leq t < t_{k+1}$.

We recall that a trajectory of (1) corresponding to $s \in \mathcal{S}$, $\mathbf{u} \in L^m_{\infty,e}$ and originating from $\xi \in \mathbb{R}^n$, is a locally absolutely continuous curve $\eta : [0, T) \to \mathbb{R}^n$, such that $\eta(0) = \xi$ and $\dot{\eta}(t) = f_{\sigma_k}(\eta(t), \mathbf{u}(t))$ *a.e.* $t \in [t_k, t_{k+1}) \cap [0, T)$. Then, since the members of $\mathcal{P}$ are locally Lipschitz, for each $s \in \mathcal{S}$, each initial condition $\xi \in \mathbb{R}^n$ and each control $\mathbf{u} \in L^m_{\infty,e}$ there exists a unique maximally defined trajectory corresponding to s, ξ and $\mathbf{u}$. We denote this curve and its maximal interval of definition by $x(t, \xi, \mathbf{u}, s)$ and $[0, T^x_{\xi,\mathbf{u},s})$ respectively.

Definition 2.1 The system (1) is *uniformly (with respect to $s \in \mathcal{S}$) input-to-state stable*, (UISS) if there exist a $\mathcal{KL}$ function β and a $\mathcal{K}$ function γ such that, for each input $\mathbf{u} \in L^m_{\infty,e}$ and each $\xi \in \mathbb{R}^n$, it holds that

$$|x(t, \xi, \mathbf{u}, s)| \leq \beta(|\xi|, t) + \gamma(\|\mathbf{u}_t\|), \tag{2}$$

for each $t \geq 0$ and for all $s \in \mathcal{S}$.

Definition 2.2 The system (1) is *uniformly (with respect to $s \in \mathcal{S}$) integral input-to-state stable*, (UiISS) if there exist a $\mathcal{K}_\infty$ function α, a $\mathcal{KL}$-function β and a $\mathcal{K}$ function γ such that, for each input $\mathbf{u} \in L^m_{\infty,e}$ and each $\xi \in \mathbb{R}^n$, it holds that

$$\alpha(|x(t, \xi, \mathbf{u}, s)|) \leq \beta(|\xi|, t) + \int_0^t \gamma(|\mathbf{u}(\tau)|)d\tau, \tag{3}$$

for each $t \geq 0$ and all $s \in \mathcal{S}$.

It follows from the definitions above that a necessary, but not sufficient, condition for the UISS (UiISS) of the switched system (1) is that each subsystem of the family $\mathcal{P}$ be ISS (iISS).

In order to establish sufficient conditions for the UISS (UiISS) of (1), we introduce the following:

Definition 2.3 A *common ISS-Lyapunov triple* (V, ρ, χ) *for* $\mathcal{P}$ consists of a positive definite radially unbounded $\mathcal{C}^1$ function $V : \mathbb{R}^n \to [0, +\infty)$ and $\mathcal{K}_\infty$ functions ρ and χ that verify:

$$\nabla V(\xi) f_\sigma(\xi, u) \leq -\rho(|\xi|) + \chi(|u|)$$

for all $\xi \in \mathbb{R}^n$, all $u \in \mathbb{R}^m$ and all $\sigma \in \Gamma$.

Definition 2.4 A *common iISS-Lyapunov triple* (V, ρ, χ) *for* $\mathcal{P}$ consists of a positive definite radially unbounded $\mathcal{C}^1$ function $V : \mathbb{R}^n \to [0, +\infty)$ a positive definite function ρ and a $\mathcal{K}$ function χ that verify:

$$\nabla V(\xi) f_\sigma(\xi, u) \leq -\rho(|\xi|) + \chi(|u|)$$

for all $\xi \in \mathbb{R}^n$, all $u \in \mathbb{R}^m$ and all $\sigma \in \Gamma$.

As was pointed in the Introduction, it can be shown, with similar arguments to those used in [13] and [1] to prove sufficient conditions for ISS and iISS respectively, that the existence of a common ISS-Lyapunov triple (V, ρ, χ) for $\mathcal{P}$ assures that the system (1) is UISS, and that a similar result holds in the iISS case, *i.e.*,

Theorem 2.1 Suppose that there exists a common ISS (iISS) Lyapunov triple for $\mathcal{P}$. Then the system (1) is UISS (UiISS).

3 Converse Lyapunov Theorems

The question naturally arises whether the converse to Theorem 2.1 hold.

In order to assure the existence of common ISS-Lyapunov and iISS-Lyapunov triples we assume that $\mathcal{P}$ verifies the following condition:

- **C** : The family $\mathcal{P}$ is uniformly locally Lipschitz, i.e. for each $N \in \mathbb{N}$, there exists $l_N \geq 0$ such that

  $$|f_\sigma(x, u) - f_\sigma(x', u')| \leq l_N(|x - x'| + |u - u'|)$$

  for all $(x, u), (x', u') \in B^n_N \times B^m_N$ and all $\sigma \in \Gamma$.

The main theorem of this paper may be stated as follows:

Theorem 3.1 Suppose that $\mathcal{P}$ satisfies **C**, then:

1. if the system (1) is UISS then there exists a common ISS-Lyapunov triple (V, ρ, χ) for $\mathcal{P}$ with V smooth ($\mathcal{C}^\infty$).
2. If the system (1) is UiISS then there exists a common iISS-Lyapunov triple (V, ρ, χ) for $\mathcal{P}$ with V smooth.

Remark 3.1 When $\mathcal{P}$ is an indexed family of locally Lipschitz vector fields, $\mathcal{P} = \{f_\sigma : \mathbb{R}^n \to \mathbb{R}^n, \sigma \in \Gamma\}$, and (1) is globally asymptotically stable uniformly with respect to $s \in \mathcal{S}$, then it is trivially UISS. In this case Theorem 3.1 asserts the existence of a common Lyapunov function for $\mathcal{P}$, and in consequence Theorem 3.1 generalizes Theorem IV.1 of [4]. In fact, Theorem 3.1 treats with a family of controlled subsystems and in addition its hypotheses are weaker that those of [4] in that we do not assume the local exponential stability of the switched system. (See [11] for details).

The following result, that was stated and proved in [11] will be useful in the proof of Theorem 3.1:

Theorem 3.2 Suppose that $\mathcal{P}$ verifies **C**. Then there exist a compact metric space D, an injective function $\iota : \Gamma \to D$ and a continuous function $F : \mathbb{R}^n \times \mathbb{R}^m \times D \to \mathbb{R}^n$ such that:

1. $\iota(\Gamma)$ is dense in D.
2. $F(x, u, d)$ is locally Lipschitz on (x, u) uniformly on d, *i.e.*, for each compact subset K of $\mathbb{R}^n \times \mathbb{R}^m$ there is some constant c_K so that $|F(x, u, d) - F(x', u', d)| \leq c_K(|x - x'| + |u - u'|)$, for all $(x, u), (x', u') \in K$, and all $d \in D$.
3. $F(x, u, \iota(\sigma)) = f_\sigma(x, u)$ for all $x \in \mathbb{R}^n$, all $u \in \mathbb{R}^m$ and all $\sigma \in \Gamma$.

Remark 3.2 It is clear that when Γ is a finite set the family $\mathcal{P}$ satisfies **C**. In this case we can take $D = \Gamma$ endowed with the discrete metric, ι the identity map and $F(x, u, \sigma) = f_\sigma(x, u)$ for all $x \in \mathbb{R}^n$, all $u \in \mathbb{R}^m$ and all $\sigma \in \Gamma$.

Now we denote with $\mathcal{D} = \{\mathbf{d} : [0, \infty) \to D, \mathbf{d}$ measurable$\}$ and with $\mathcal{D}_\Gamma \subset \mathcal{D}$ those that are piecewise-constant $\iota(\Gamma)$-valued functions, and associate to the switched system (1) the following perturbed control system:

$$\dot{z}(t) = F(z(t), \mathbf{u}(t), \mathbf{d}(t)), \tag{4}$$

where the state z evolves in $\mathbb{R}^n$, the control $\mathbf{u} \in L^m_{\infty,e}$, the disturbance $\mathbf{d} \in \mathcal{D}$, and F is as in Theorem 3.2. Given $\xi \in \mathbb{R}^n$, $\mathbf{u} \in L^m_{\infty,e}$ and $\mathbf{d} \in \mathcal{D}$, if we denote with $z(t, \xi, \mathbf{u}, \mathbf{d})$ the maximally defined trajectory of (4) originating from ξ with control $\mathbf{u}$ and disturbance $\mathbf{d}$, and with $[0, T^z_{\xi,\mathbf{u},\mathbf{d}})$ its maximal interval of definition, then the following hold:

- For each $s \in \mathcal{S}$, if $\mathbf{d}_s = \iota \circ s$, then $\mathbf{d}_s \in \mathcal{D}_\Gamma$, $T^z_{\xi,\mathbf{u},\mathbf{d}_s} = T^x_{\xi,\mathbf{u},s}$ and $x(\cdot, \xi, \mathbf{u}, s) = z(\cdot, \xi, \mathbf{u}, \mathbf{d}_s)$.
- For each $\mathbf{d} \in \mathcal{D}_\Gamma$, there exists a unique switching signal $s_{\mathbf{d}}$ such that $\iota \circ s_{\mathbf{d}} = \mathbf{d}$.

Remark 3.3 We conclude from Theorem 3.2 that a switched system, whose associated family $\mathcal{P}$ verifies **C**, can be considered as a perturbed control system whose disturbances are piecewise-constant functions that take values in a dense subset of a compact metric space D.

In the following lemma we show that the perturbed control system (4) has the same stability properties of the switched system (1).

Lemma 3.1 Suppose that the family $\mathcal{P}$ verifies **C**. Then

1. If the switched system (1) is UISS then the perturbed control system (4) is uniformly (with respect to $\mathbf{d} \in \mathcal{D}$) ISS, *i.e.*, there exist a $\mathcal{KL}$ function β and a $\mathcal{K}$ function γ such that, for each input $\mathbf{u} \in L^m_{\infty,e}$ and each $\xi \in \mathbb{R}^n$, it holds that
$$|z(t, \xi, \mathbf{u}, \mathbf{d})| \leq \beta(|\xi|, t) + \gamma(\|\mathbf{u}_t\|), \tag{5}$$
for each $t \geq 0$ and for all $\mathbf{d} \in \mathcal{D}$.
2. If the switched system (1) is UiISS then the perturbed control system (4) is uniformly (with respect to $\mathbf{d} \in \mathcal{D}$) iISS, *i.e.*, there exist exist a $\mathcal{K}_\infty$ function α, a $\mathcal{KL}$-function β and a $\mathcal{K}$ function γ such that, for each input $\mathbf{u} \in L^m_\infty$ and each $\xi \in \mathbb{R}^n$, it holds that
$$\alpha(|z(t, \xi, \mathbf{u}, \mathbf{d})|) \leq \beta(|\xi|, t) + \int_0^t \gamma(|\mathbf{u}(\tau)|) d\tau, \tag{6}$$
for each $t \geq 0$ and all $\mathbf{d} \in \mathcal{D}$.

Proof: Suppose now that $\xi \in \mathbb{R}^n$, $\mathbf{u} \in L^m_{\infty,e}$ and $\mathbf{d} \in \mathcal{D}$. Then, due to the density of $\iota(\Gamma)$ in D, there exists a sequence $\{s_n : n \in I\!N\} \subset \mathcal{S}$ such that $\mathbf{d}_n = \iota \circ s_n \to d$ *a.e.*
Let t such that $z(\tau, \xi, \mathbf{u}, \mathbf{d})$ is defined for all $\tau \in [0, t]$. Then due to Theorem 1 pp. 57 of [12], for all $\tau \in [0, t]$,

$$\begin{aligned}\lim_{n\to\infty} x(\tau, \xi, \mathbf{u}, s_n) &= \lim_{n\to\infty} z(\tau, \xi, \mathbf{u}, \mathbf{d}_n) \\ &= z(\tau, \xi, \mathbf{u}, \mathbf{d}).\end{aligned}$$

If (2) holds, then $|z(t, \xi, \mathbf{u}, \mathbf{d})| = \lim_{n\to\infty} |x(t, \xi, \mathbf{u}, s_n)| \leq \beta(|\xi|, t) + \gamma(\|\mathbf{u}_t\|)$. In a similar way, if (3) holds, then $|z(t, \xi, \mathbf{u}, \mathbf{d})| = \lim_{n\to\infty} |x(t, \xi, \mathbf{u}, s_n)| \leq \beta(|\xi|, t) + \int_0^t \gamma(|\mathbf{u}(\tau)|) d\tau$. ∎

Remark 3.4 We recall that a positive definite radially unbounded $\mathcal{C}^1$ function $V : \mathbb{R}^n \to \mathbb{R}$ is:

1. an *UISS-Lyapunov function for system* (4) if there exist $\mathcal{K}_\infty$ functions ρ and χ that verify:

   $$\nabla V(\xi) F(\xi, u, d) \leq -\rho(|\xi|) + \chi(|u|)$$

   for all $\xi \in \mathbb{R}^n$, all $u \in \mathbb{R}^m$ and all $d \in D$.

2. An *UiISS-Lyapunov function V for system* (4) if there exist a $\mathcal{K}$ function χ and a positive definite function ρ that verify:

   $$\nabla V(\xi) F(\xi, u, d) \leq -\rho(|\xi|) + \chi(|u|)$$

   for all $\xi \in \mathbb{R}^n$, $d \in D$ and $u \in \mathbb{R}^m$.

It follows that if the family $\mathcal{P}$ verifies **C**, then V is an UISS-Lyapunov function for system (4) with ρ and χ as above if and only if (V, ρ, χ) is a common ISS-Lyapunov triple for $\mathcal{P}$. A similar statement holds in the iISS case.

Due to this last remark, Theorem 3.1 is a corollary of the following:

Theorem 3.3

1. Suppose that system (4) is UISS. Then there exist a smooth UISS Lyapunov function for it.

2. If system (4) is UiISS then there exist a smooth UiISS Lyapunov function for this system.

Sketch of the proof:
Part 1. Since system (4) is UISS and can be seen as a system "with outputs" $y = h(z) = 0 \in \mathbb{R}$, $\forall z \in \mathbb{R}^n$, then it is *uniformly input-output-to-state stable* according to Definition 2.1 of [7]. Then, due to Theorem 1 of the same reference that holds, with minor modifications of its proof if the disturbance set is a compact metric space instead of $[-1, 1]^p$, $p \in I\!N$, there exists a smooth UISS-Lyapunov function for (4).
Part 2: In order to prove it, some steps of the proof of Theorem 1 of [1] will be followed.
Since the perturbed system (4) is uniformly (with respect to $\mathbf{d} \in \mathcal{D}$) iISS, then it is 0-UGAS, *i.e.*, the zero solution of the system

$$\dot{z}(t) = F(z(t), 0, \mathbf{d}(t)) \tag{7}$$

is uniformly (with respect to $\mathbf{d} \in \mathcal{D}$) globally asymptotically stable.
Then, following similar steps as in the proof of Proposition 2.5 of [1], using and extended version of Theorem 2.9 of Lin *et al.*,[10], that assumes that the disturbance value set is a compact metric space (D in this case) instead of a compact subset of a finite-dimensional Euclidean space, and also using the fact that $F(z, u, d)$ is locally Lipschitz on (z, u) uniformly on $d \in D$, a smooth semiproper function W, a $\sigma \in \mathcal{K}$ and a positive definite function $\rho_1 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ may be found such that

$$\nabla W(\xi) F(\xi, u, d) \leq -\rho_1(|\xi|) + \sigma(|u|), \tag{8}$$

for all $\xi \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $d \in D$.
Assume that α and γ are as in (6) and let φ any locally Lipschitz $\mathcal{K}_\infty$ function such that $\gamma(\varphi(\tau)) \leq \alpha(\tau)$ $\forall\, \tau \geq 0$. Now, consider $\tilde{D} := B_1^m \times D$, $\tilde{\mathcal{D}} := \{\tilde{\mathbf{d}} : [0, \infty) \to \tilde{D},\ \tilde{\mathbf{d}} \text{ measurable}\}$ and the following system:

$$\dot{z}(t) = F(z(t), \varphi(|z(t)|)\mathbf{w}(t), \mathbf{d}(t)) := F_\varphi(z(t), \tilde{\mathbf{d}}(t)) \tag{9}$$

with $\tilde{\mathbf{d}} = (\mathbf{w}, \mathbf{d}) \in \tilde{\mathcal{D}}$.

Remark 3.5 Note that $\tilde{D}$ is a compact metric space, and that $F_\varphi(z, \tilde{d})$ is locally Lipschitz in z uniformly with respect to $\tilde{d} = (w, d) \in \tilde{D}$ and jointly continuous in z and $\tilde{d}$.
In the sequel we will consider $\tilde{D}$ as the set where the perturbations of (9) take values and $\tilde{d}$ as the values of these perturbations.

If we denote with $z_\varphi(t, \xi, \tilde{\mathbf{d}})$ the trajectory of (9) from $\xi \in \mathbb{R}^n$ corresponding to $\tilde{\mathbf{d}} \in \tilde{\mathcal{D}}$, the following hold:
1. Due to (6) and since $\gamma \circ \varphi(\tau) \leq \alpha(\tau)\, \forall \tau \geq 0$, the system (9) is forward complete.
2. For each $\eta \in \mathbb{R}^n$ and $\tilde{d} = (w, d) \in \tilde{D}$, let $\mathcal{M}(\eta) := \alpha(|\eta|)$, $\mathcal{L}(\eta, \tilde{d}) := \gamma(|w|\varphi(|\eta|))$, and for each $\xi \in \mathbb{R}^n$, $\tilde{\mathbf{d}} \in \tilde{\mathcal{D}}$ and $t \geq 0$ consider

$$y(t, \xi, \tilde{\mathbf{d}}) = \mathcal{M}(z_\varphi(t, \xi, \tilde{\mathbf{d}})) - \int_0^t \mathcal{L}(z_\varphi(\tau, \xi, \tilde{\mathbf{d}}), \tilde{\mathbf{d}}(\tau)) d\tau.$$

Consequently, from (6), it holds that

$$y(t, \xi, \tilde{\mathbf{d}}) \leq \beta(|\xi|, t), \tag{10}$$

for all $\xi \in \mathbb{R}^n$, all $\tilde{\mathbf{d}} \in \tilde{\mathcal{D}}$ and all $t \geq 0$.
If we define $g : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ by

$$g(\xi) = \sup\{y(t, \xi, \tilde{\mathbf{d}}) \,:\, t \geq 0,\, \tilde{\mathbf{d}} \in \tilde{\mathcal{D}}\}, \tag{11}$$

then it can be proved that g is continuous in $\mathbb{R}^n$ and verifies

$$\alpha(|\xi|) \leq g(\xi) \leq \beta(|\xi|, 0).$$

Consider now $\bar{d} = (w, d) \in \tilde{D}$, $\xi \in \mathbb{R}^n$ and $h \geq 0$; if we define $\bar{\mathbf{d}} \in \tilde{\mathcal{D}}$ as $\bar{\mathbf{d}}(t) = (w, d)$ $\forall t \geq 0$, then due to standard results of optimal control, (see [5]), it follows that

$$g(z_\varphi(h, \xi, \bar{\mathbf{d}})) \leq g(\xi) + \int_0^h \mathcal{L}(z_\varphi(\tau, \xi, \bar{\mathbf{d}}), \bar{d}) d\tau. \tag{12}$$

Let $V_1(\xi) := W(\xi) + g(\xi)\, \forall\, \xi \in \mathbb{R}^n$, where W is given by (8); then there exist $\mathcal{K}_\infty$ functions $\tilde{\alpha}_1$, $\tilde{\alpha}_2$ such that for all $\xi \in \mathbb{R}^n$ it holds that

$$\tilde{\alpha}_1(|\xi|) \leq V_1(\xi) \leq \tilde{\alpha}_2(|\xi|). \tag{13}$$

Consider $\zeta \in \partial_P V_1(\xi)$ a proximal subgradient of V_1 at ξ; hence there exists some positive κ such that for h small enough

$$\begin{aligned}\zeta(z_\varphi(h,\xi,\bar{\mathbf{d}})-\xi) &\leq W(z_\varphi(h,\xi,\bar{\mathbf{d}}))-W(\xi)\\ &+ g(z_\varphi(h,\xi,\bar{\mathbf{d}}))-g(\xi)\\ &+ \kappa|z_\varphi(h,\xi,\bar{\mathbf{d}})-\xi|^2\\ &\leq W(z_\varphi(h,\xi,\bar{\mathbf{d}}))-W(\xi)\\ &+ \int_0^h \mathcal{L}(z_\varphi(\tau,\xi,\bar{\mathbf{d}}),\bar{d})d\tau\\ &+ \kappa|z_\varphi(h,\xi,\bar{\mathbf{d}})-\xi|^2.\end{aligned}$$

It follows that

$$\begin{aligned}\zeta F_\varphi(\xi,\bar{d}) &= \lim_{h\to0^+}\zeta\cdot\frac{z_\varphi(h,\xi,\bar{\mathbf{d}})-\xi}{h}\\ &\leq \lim_{h\to0^+}\frac{W(z_\varphi(h,\xi,\bar{\mathbf{d}}))-W(\xi)}{h}\\ &+ \lim_{h\to0^+}\frac{1}{h}\int_0^h\mathcal{L}(z_\varphi(\tau,\xi,\bar{\mathbf{d}}),\bar{d})d\tau\\ &+ \lim_{h\to0^+}\frac{\kappa}{h}|z_\varphi(h,\xi,\bar{\mathbf{d}})-\xi|^2\\ &= \nabla W(\xi)F_\varphi(\xi,\bar{d})+\mathcal{L}(\xi,\bar{d})\\ &= \nabla W(\xi)F_\varphi(\xi,\bar{d})+\gamma(|w|\varphi(|\xi|)).\end{aligned}$$

Then, from (8), we obtain

$$\begin{aligned}\zeta F(\xi,w\varphi(|\xi|),d) &\leq -\rho_1(|\xi|)+\sigma(|w|\varphi(|\xi|))\\ &+ \gamma(|w|\varphi(|\xi|)) \qquad (14)\end{aligned}$$

for all $\bar{d}=(w,d)\in\tilde{D}$.
Now from Corollary 4.22 of [7], there exists a function $V_2:\mathbb{R}^n\to\mathbb{R}$, smooth on $\mathbb{R}^n-\{0\}$ and such that

$$\begin{aligned}\frac{\tilde{\alpha}_1(|\xi|)}{2} &\leq V_2(\xi)\leq 2\tilde{\alpha}_2(|\xi|)\\ \nabla V_2(\xi)F(\xi,w\varphi(|\xi|),d) &\leq -\frac{\rho_1(|\xi|)}{2}\\ &+ \sigma(|w|\varphi(|\xi|))+\gamma(|w|\varphi(|\xi|))\\ &= -\frac{\rho_1(|\xi|)}{2}+\gamma_1(|w|\varphi(|\xi|)),\end{aligned}$$

for all $\xi\in\mathbb{R}^n-\{0\}$, and $(w,d)\in\tilde{D}$ and with $\gamma_1(\cdot):=\sigma(\cdot)+\gamma(\cdot)$ of class $\mathcal{K}$. By Proposition 4.2 of [10], there exists a smooth function μ of class $\mathcal{K}_\infty$ with $\mu'(r)>0\ \forall r>0$ such that $\mu\circ V_2$ is smooth everywhere. In [1] it is shown that μ can be taken so that $\mu'(r)\leq 1\ \forall\, r>0$ and that if we let $V=\mu\circ V_2$, then the following hold:

$$\begin{aligned}\alpha_1(|\xi|) &\leq V(\xi)\leq\alpha_2(|\xi|)\\ \nabla V(\xi)F(\xi,w\varphi(|\xi|),d) &\leq -\rho(|\xi|)+\gamma_1(|w|\varphi(|\xi|))\end{aligned}$$

for all $\xi\in\mathbb{R}^n$, and all $(w,d)\in\tilde{D}$, where ρ is a positive definite function and α_1 and α_2 are of class $\mathcal{K}_\infty$. It then follows that

$$\nabla V(\xi)F(\xi,u,d)\leq-\rho(|\xi|)+\gamma_1(|u|)$$

for all $\xi\in\mathbb{R}^n$, all $|u|\leq\varphi(|\xi|)$ and all $d\in D$.
Let $\hat{\gamma}(r)=\max\{\nabla V(\xi)F(\xi,u,d)+\rho(|\xi|)\ :\ |\xi|\leq\varphi^{-1}(|u|),\ |u|\leq r,\ d\in D\}$ and $\chi(r)=\max\{\hat{\gamma}(r),\gamma_1(r)\}$. Then $\chi\in\mathcal{K}$ and

$$\nabla V(\xi)F(\xi,u,d)\quad\leq\quad-\rho(|\xi|)+\chi(|u|),$$

for all $\xi\in\mathbb{R}^n$, $u\in\mathbb{R}^n$ and $d\in D$. It follows that V is a smooth UiISS-Lyapunov function for (4) and this fact concludes the proof. ∎

4 Conclusions

In this paper we have presented converse Lyapunov theorems for a class of ISS and iISS switched nonlinear systems. The proofs of the theorems were greatly facilitated by the association of these system with perturbed nonlinear control systems, and by the use of generalizations of results about converse Lyapunov theorems for IOSS and iISS systems with bounded time-varying perturbations.

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