Solution to Brockett's problem on finite-dimensional estimation algebras of maximal rank in nonlinear filtering

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Abstract

The Kalman-Bucy filter is widely used in modern industry. Despite its usefulness, however, the Kalman-Bucy filter is not perfect. One of the weakness is that it needs a Gaussian assumption for the initial data. The other weakness is that it requires the drift term $f(x)$ be a linear function. Brockett [Br], Brockett and Clark [Br-Cl], and Mitter [Mi] proposed independently using a Lie algebraic method to solve Duncan-Mortensen-Zakai equation for nonlinear filtering. This method requires only n sufficient statistics, where n is the state space dimension, and it allows the initial condition be modeled by an arbitrary distribution. The idea was worked out in detail by Tam-Wong-Yau [TWY] and Yau [Ya 1] [Ya 2]. However, in the Lie algebraic method, one has to know explicitly the structure of the estimation algebra. In 1983, Brockett proposed to classify all finite dimensional filters. In this paper, we report the recent results on classification of finite dimensional maximal rank estimation algebras with arbitrary state space dimension.

1 INTRODUCTION

In 1961, Kalman and Bucy published a historically important mathematics paper on filtering. Since then, the Kalman-Bucy filtering has proved useful in many areas such as navigational and guidance systems, radar tracking, solar mapping, and satellite orbit determination. Despite its usefulness, however, the Kalman-Bucy filter is not perfect. One of its weaknesses is that it needs Gaussian assumption for the initial data. The situation is much more complex when the statistics of the initial condition are modeled by an arbitrary distribution. Indeed, the filtering question becomes one of nonlinear filtering, an area in which few results have been obtained.

In the 1960s Ducan [Du], Mortensen [Mo], and Zakai [Za] independently derived the so-called DMZ equation for the nonlinear filtering problem, which the unnormalized conditional probability density of the state $x(t)$, given the observation $\{y(s) : 0 \leq s \leq t\}$, has to satisfy. The Kalman-Bucy filter can be deduced immediately from the DMZ equation . Unfortunately, since the DMZ equation is a stochastic differential equation, there is no easy way to derive a recursive algorithm for solving this equation.

The idea of using estimation algebras to construct finite-dimensional nonlinear filters was first proposed by Brockett and Clark [Br-Cl], Brockett [Br], and Mitter [Mi]. The motivation came from the Wei-Norman approach [We-No] of using Lie algebraic ideas to solve a linear time-varying differential equation. The advantage of this approach is that as long as the estimation algebra is finite dimensional, we will get a finite-dimensional recursive filter; there is no need to make any assumption in the initial data. Moreover, the approach applies well to nonlinear dynamical systems. This approach has been worked out in detail in [T-W-Y] and especially in the so-called Yau filtering system described in [Ch] (cf. [Ya 1] and [Ya 2]). In [Ya 2], it was shown that only n sufficient statistics are needed to solve the DMZ equation explicitly (see section 2 below). For a linear filtering system, it is quite easy to see that the corresponding estimation algebra is finite dimensional. So we can apply the Wei-Norman approach to construct a finite-dimensional recursive filter with arbitrary initial data. However, in the Wei-Norman approach, one has to know explicitly the basis as vector space of the estimation algebra in order to reduce the DMZ equation to a finite system of ordinary differential equations, Kolmogorov equation, and several first-order linear partial differential equation. Classically, one knows the explicit basis for the estimation algebra only in the case that the linear system is controllable and observable.

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In [Ch-Ya] Chiou and Yau introduced the concept of an estimation algebra of maximal rank. They were able to classify all finite dimensional estimation algebras of maximal rank with state space dimension $n = 2$. In [C-Y-L 1] and [C-Y-L 2], Chen, Leung and Yau classified all finite-dimensional estimation algebras of maximal rank with state space dimension $n = 3, 4$ respectively. Recently Hu and Yau [Hu-Ya] have succeeded in classifying all finite dimensional estimation algebras of maximal rank with state space dimension $n \leq 5$. The novelty of their theorem is that there is no a priori assumption on the drift term of the nonlinear filtering system. On the other hand, if the drift term has a potential function (i.e. the drift term is a gradient vector field), the corresponding estimation algebra is called exact. In [T-W-Y] and [D-T-W-Y], Dong, Tam, Wong and Yau classified all finite dimensional exact estimation algebras of maximal rank with arbitrary state space dimension. In this paper, we would like to report the result of the classification of finite dimensional estimation algebra of maximal rank with arbitrary state space dimension without any assumption on the drift term.

2 THE LIE ALGEBRAIC METHOD IN DMZ EQUATION

The filtering problem considered here is based on the following signal observation model:

$$\begin{cases} dx(t) = f\big(x(t)\big)dt + g\big(x(t)\big)dv(t) & x(0) = x_0 \\ dy(t) = h\big(x(t)\big)dt + dw(t) & y(0) = 0 \end{cases} \quad (1)$$

in which x, v, y, and w are, respectively, R^n, R^p, R^m, and R^m valued processes, and v and w have components that are independent, standard Brownian processes. We further assume that $n = p$, f, h and C^∞ smooth and that g is an orthogonal matrix. We will refer to $x(t)$ as the state of the system at time t and $y(t)$ as the observation at time t.

Let $\rho(t,x)$ denote the condiional probability density of the state given the observation $\{y(s) : 0 \leq s \leq t\}$. It is well known that $\rho(t,x)$ is given by normalizing a function $\sigma(t,x)$ that satisfies the following Duncan-Mortensen-Zakai equation:

$$\begin{cases} d\sigma(t,x) & = L_o\sigma(t,x)dt + \sum_{i=1}^{m} L_i\sigma(t,x)dy_i(t) \\ \sigma(0,x) & = \sigma_0 \end{cases} \quad (2)$$

where

$$L_0 = \frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2} - \sum_{i=1}^{n} f_i\frac{\partial}{\partial x_i} - \sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i} - \frac{1}{2}\sum_{i=1}^{m} h_i^2$$

and for $i = 1, \ldots, m$, L_i is the zero-degree differential operator of multiplication by h_i, the i-th component of h. σ_0 is the probability density of the initial point, x_0. Equation (2) is a stochastic partial differential equation. In real applications, we are interested in constructing robust state estimators from observed sample paths with some property of robustness. Using the transformation,

$$\xi(t,x) = e^{-\sum_{i=1}^{m} h_i(x)y_i(t)}\sigma(t,x),$$

we reduce (2) to the folloowing time-varying partial differential equation, which is called the robust DMZ equation:

$$\begin{cases} \frac{\partial\xi}{\partial t}(t,x) & = L_0\xi(t,x) + \sum_{i=1}^{m} y_i(t)[L_0, L_i]\xi(t,x) \\ & \quad + \frac{1}{2}\sum_{i,j=1}^{m} y_i(t)y_j(t)\big[[L_0, L_i], L_j\big]\xi(t,x) \\ \xi(0,x) & = \sigma_0 \end{cases} \quad (3)$$

where $[\cdot,\cdot]$ is the Lie bracket as described by the following definition.

Definition: If X and Y are differential operators, the Lie bracket of X and Y, $[X,Y]$, is defined by

$$[X,Y]\phi = X(Y\phi) - Y(X\phi)$$

for any C^∞ function ϕ.

The objective of constructing a robust finite-dimensinal filter to (1) is equivalent to finding a smooth manifold M and computing C^∞ vector fields μ_i on M and C^∞ functions ν on $M \times R \times R^n$ and ω_i on R^m, such that $\xi(t,x)$ can be represented in the form

$$\begin{cases} \frac{dz}{dt}(t) & = \sum_{i=1}^{k}\mu_i\big(z(t)\big)\omega_i\big(y(t)\big), \ z(0) \in M \\ \xi(t,x) & = \nu\big(z(t), t, x\big). \end{cases} \quad (4)$$

Following [Ch-Mi], we say that system (1) has a robust universal finite-dimensional filter if for each initial probability density σ_0 there exists a z_0 such that (4) holds if $z(0) = z_0$, and μ_i, ω_i are independent of σ_0.

The Wei-Norman approach [We-No] of using Lie algebraic ideas to solve time-varying linear differential equations is roughly as follows. Consider the equation

$$\begin{cases} \frac{d}{dt}X(t) & = A(t)X(t) \equiv \sum_{i=1}^{m} a_i(t)A_iX(t) \\ X(0) & = X_0 \end{cases} \quad (5)$$

where X and A_i's are $n \times n$ matrices and a_i's are scalar-valued functions. Let $B_1, \ldots, D_\ell$ be a basis of the Lie algebra generated by $A_1, \ldots, A_m$. Then the Wei-Norman theorem states that locally in t, $X(t)$ has a representation of the form

$$X(t) = e^{b_1(t)B_1} \ldots e^{b_1(t)B_1}X_0, \quad (6)$$

where b_i's satisfy an ordinary differential equation of the form

$$\frac{db_i}{dt} = c_i(b_1, \ldots, b_\ell),\ b_i(0) = 0$$

for all i. The functions c_i in the above equation are determined by the structure constants of the Lie algebra (generated by the A_i's) relative to the basis $\{B_1, \ldots, B_\ell\}$.

The extension of the Wei-Norman approach to the nonlinear filtering problem is much more complicated. Instead of an ordinary differential equation, we have to solve the robust DMZ equation, which is a time-varying partial differential equation. For this purpose, we introduce the concept of the estimation algebra of (1) and examine its algebraic structure.

Definition: The estimation algebra E of a filtering problem (1) is defined to be the Lie algebra generated by $\{L_0, L_1, \ldots, L_m\}$. E is an estimation algebra of maximal rank if for any i, there is a constant c_i such that $x_i + c_i$ is in E.

Remark 2.1: If $h(x)$ in (1) is of the form $Hx + K$, where H is an $m \times n$ matrix of maximal rank, $m \geq n$, and K is an $m \times 1$ matrix, then the estimation algebra E is of maximal rank.

The following theorem was announced in [Ya 1] and proved in detail in [Ya 2], which includes Kalman-Bucy filtering system as a special case.

Theorem 2.2: [Ya 2]. *Let E be an estimation algebra of (1) satisfying $\frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j} = c_{ij}$, where c_{ij}'s are constants for all $1 \leq i,\ j \leq n$. Suppose that E is a finite-dimensional estimation algebra of maximal rank. Then E has a basis of the form 1, $x_1, \ldots, x_n$, $D_1, \ldots, D_n$, and L_0, where$D_i = \frac{\partial}{\partial x_i} - f_i$, and $\eta := \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^{n} f_i^2 + \sum_{i=1}^{m} h_i^2$ is a degree two polynomial $\sum_{i,j=1}^{n} a_{ij}x_ix_j + \sum_{i=1}^{n} b_ix_i + d$. The robust DMZ equation (3) has a solution for all $t \geq 0$ of the form*

$$\xi(t,x) = e^{T(t)}e^{r_n(t)x_n} \ldots e^{r_1(t)x_1}e^{s_n(t)D_n} \ldots e^{s_1(t)D_1}e^{tL_0}\sigma_0, \tag{7}$$

where $T(t)$, $r_1(t), \ldots, r_n(t)$, $s_1(t), \ldots, s_n(t)$ satisfies the following ordinary differential equations: For $1 \leq i \leq n$,

$$\frac{dr_i(t)}{dt} = \frac{1}{2}\sum_{j=1}^{n} s_j(t)(a_{ij} + a_{ji}) \tag{8}$$

$$\frac{ds_i}{dt}(t) = r_i(t) + \sum_{j=1}^{n} s_j(t)c_{ji} + \sum_{k=1}^{m} h_{ki}y_k(t) \tag{9}$$

where $h_k(x) = \sum_{j=1}^{n} h_{kj}x_j + e_k$, for $1 \leq k \leq m$, h_{kj} and e_k are constant; and

$$\begin{aligned}\frac{dT}{dt}(t) \;=\; & -\frac{1}{2}\sum_{i=1}^{n} r_i^2(t) - \frac{1}{2}\sum_{i=1}^{n} s_i^2(t)\Big(\sum_{j=1}^{n} c_{ij}^2 - a_{ii}\Big) \\ & + \sum_{i=1}^{n} r_i(t) - \sum_{j=2}^{n}\sum_{i=1}^{j} s_j(t)c_{ij} - \sum_{i,j=1}^{n} s_i(t)r_j(t)c_{ij} \\ & + \sum_{i\leq i<k\leq n} s_i(t)s_k(t)\Big(\sum_{j=1}^{n} c_{ij}c_{jk} + \frac{1}{2}(a_{ik} + a_{ki}\Big) \\ & + \frac{1}{2}\sum_{i=1}^{n} s_i(t)b_i + \frac{1}{2}\sum_{i,j=1}^{m} y_i(t)y_j(t)\Big(\sum_{k=1}^{n} h_{ik}h_{jk}\Big)\end{aligned} \tag{10}$$

Remark 2.3: In view of (10), $T(t)$ can be expressed explicitly in terms of $r_i(t)$ and $s_i(t)$, $1 \leq i \leq n$. thus, the characterization given in (7) requires $2n$ sufficient statistics. In fact, by inspecting (8) and (9), one easily sees that the characterization in (7) really requires n sufficient statistics, the same complexity as the classical Kalman-Bucy filter with Gaussian initial condition.

3 QUADRATIC STRUCTURE OF EXTIMTION ALGEBRA AND CONSTANT STRUCTURE OF WONG MATRIX

Chen and Yau [Ch-Ya 1] made important progress in the program of classification of finite-dimensional estimation algebras of maximal rank. They study the quadratic forms in E and show that the Ω-matrix is linear in the sense that all ω_{ij} are degree one polynomials.

Definition: Let Q be the space of quadratic forms in n variables, namely, realy vector space spanned by x_ix_j, $1 \leq i \leq j \leq n$. Let $X = (x_1, \ldots, x_n)^T$. For any quadratic form $p \in Q$, there exists a symmetric matrix A such that $p(x) = X^TAX$. The rank of the quadratic form p is denoted by $r(p)$ and is defined to be the rank of the matrix A. A fundamental quadratic form of the estimation algebra E is an element $p_0 \in E \cap Q$ with the greatest positive rank, that is, $r(p_0) \geq r(p)$ for any $p \in E \cap Q$. The maximal rank of quadratic forms in the estimation algebra E is defined to be $k = r(p_0)$ and is called the quadratic rank of E.

After an orthogonal transformation, p_0 can be written as

$$p_0(x) = c_1x_i^2 + c_2x_2^2 + \cdots + c_kx_k^2, \qquad c_i \neq 0, \qquad 0 \leq k \leq n.$$

From $p_0(x)$, we can construct a sequence of quadratic forms in $E \cap Q$ as follows:

$$q_0(x) \;=\; p_0(x)$$

$$q_j(x) \quad = \quad [[L_0, q_{j-1}], q_0] = \sum_{i=1}^{k} 4^j c_i^{j+1} x_i^2.$$

In view of the inertibility of the Vandermonde matrix, we can assume that

$$p_0(x) = x_1^2 + x_2^2 + \cdots + x_k^2 \in E. \qquad (11)$$

Lemma 3.1: (Chen and Yau) [Ch-Ya 1]. *If p is a quadratic form in the estimation algebra E of (1), then p is independent of x_j for $j \geq k$, where $k = r(p_0)$ is the quadratic rank of E. In other words, $\frac{\partial p}{\partial x_j} = 0$ for $k+1 \leq j \leq n$.*

Let $p_1 \in E \cap Q$ be an element with least positive rank, that is, $0 < r(p_1) \leq r(q)$ for any nonzero $q \in E \cap Q$. After an orthogonal transform that fixes $x_{k+1}, \ldots, x_n$ variables (i.e. an orthogonal transform on $x_1, x_2, \ldots, x_k$), and the Vandermonde matrix procedure as above, we can assume

$$p_1 = \sum_{i=1}^{k_1} x_i^2 \in E, \qquad 1 \leq k_1 \leq k. \qquad (12)$$

Notice that the orthogonal transform on $x_1, \ldots, x_k$ leaves p_0 invariant. In summary, we deduce that $p_0 = \sum_{i=1}^{k} x_i^2$ has the greatest positive rank and $p_1 = \sum_{i=1}^{k_1} x_i^2$ has the least positive rank. Define

$$S_1 = \{1, 2, \ldots, k_1\} \subseteq S = \{1, 2, \ldots, k\} \qquad (13)$$

and $Q_1 =$ real vector space spanned by $\{x_i x_j : k_1+1 \leq i \leq j \leq k\} \subseteq Q$. If $k_1 < k$, then $Q_1 \cap E$ and is a nontrivial space, since $p - p_0 \in E \cap Q$. In a similar procedure as above, there exist $k_2 > k_1$ and

$$p_2 = \sum_{i=k_1+1}^{k_2} x_i^2 \in E \cap Q \qquad (14)$$

with the least positive rank in $E \cap Q$. By induction, we can construct a series of S_i, Q_i and p_i such that

$$S_i = \{k_{i-1}+1, \ldots, k_i\}, \qquad k_0 = 0 < k_1 < \cdots < k_i < \cdots \leq k \qquad (15)$$

$$Q_i = \text{real vector space spanned by} \quad x_i x_j : k_i+1 \leq \ell \leq j \leq k\} \qquad (16)$$

$$p_i = \sum_{j=k_{i-1}+1}^{k_i} x_j^2 = \sum_{j \in S_i} x_j^2, \qquad i > 0 \qquad (17)$$

and p_i has the least positive rank in $E \cap Q_{i-1}$, for $i > 0$.

Lemma 3.2: (Chen and Yau) [Ch-Ya 1]. *If $p \in E \cap Q$, then there exists a constant λ such that*

$$p(0, \ldots, 0, x_{k_{i-1}+1}, \ldots, x_{k_i}, 0, \ldots, 0) = \lambda p_i, \qquad \text{for } i > 0$$

Lemma 3.3: (Chen and Yau) [Ch-Ya 1]. *If $p \in E \cap Q$, then*

$$p(x_1, \ldots, x_{k_{i-1}+1}, 0, \ldots, 0, x_{k_{i+1}}, \ldots, x_n) \in E \qquad \text{for } i > 0.$$

Lemma 3.4: (Chen and Yau) [Ch-Ya 1]. *Let $p = \sum_{i \in s_{\ell_1}} \sum_{j \in S_{\ell_2}} 2_{a_{ij}} x_i x_j \in E$, where $a_{ij} \in \boldsymbol{R}$ and $\ell_1 < \ell_2$. Then $|S_{\ell_1}| = |S_{\ell_2}|$ and $A = (a_{ij}) = bT$ where b is a constant and T is an orthogonal matrix.*

The following theorem is the main result of Chen and Yau in [Ch-Ya 1].

Theorem 3.5: [Chen-Yau]. *If E is a finite-dimensional estimation algebra of maximal rank, than all the entries $\omega_{ij} = \frac{\partial f_i}{\partial x_i} - \frac{\partial f_i}{\partial x_j}$ of Ω are degree one polynomials. Let k be the quadratic rank of E. Then there exists an orthogonal change of coordinates such that ω_{ij} are constants for $1 \leq i, j \leq k$, ω_{ij} be degree one polynomials in $x_1, \ldots, x_k$ for $1 \leq i \leq k$ or $1 \leq j \leq k$; and ω_{ij} are degree one polynomials in $x_{k+1}, \ldots, x_n$ for $k+1 \leq i, j \leq n$.*

Let $\Omega = (\omega_{ij})$, where $\omega_{ij} = \frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j}$, be the matrix introduced by Wong [Wo]. For finite dimensional estimation algebra of maximal rank, it is easy to see that ω_{ij} is in E. In view of Ocone's theorem, ω_{ij} is a polynomial of degree 2. Let $\omega_{ij}^{(2)}, \omega_{ij}^{(1)}$ and $\omega_{ij}^{(0)}$ be the homogeneous part of degree 2, 1 and 0 of ω_{ij} respectively.

Lemma 3.6: ([Ch-Ya 1]). *Suppose that E is a finite dimensional estimation algebra of maximal rank. Then*

(i) $\omega_{ij}^{(2)}$ *depends only on $x_1, \ldots, x_k$ for $i \leq k$ or $j \leq k$*

(ii) $\omega_{ij}^{(2)} = 0, \; \forall \; k+1 \leq i, \; j \leq n$

(iii) $\frac{\partial \omega_{ij}^{(2)}}{\partial x_\ell} + \frac{\partial \omega_{j\ell}^{(2)}}{\partial x_i} + \frac{\partial \omega_{\ell i}^{(2)}}{\partial x_j} = 0 \qquad \forall \; 1 \leq i, j, \ell \leq n$

(iv) $\frac{\partial \omega_{ij}^{(1)}}{\partial x_\ell} + \frac{\partial \omega_{j\ell}^{(1)}}{\partial x_i} + \frac{\partial \omega_{\ell i}^{(1)}}{\partial x_j} = 0 \qquad \forall \; 1 \leq i, j, \ell \leq n.$

The following lemma was observed in [Ya 2].

Lemma 3.7: *Let E be a finite-dimensional estimation algebra with maximal rank. Then $< 1, x_1, \ldots, x_n, D_1, \ldots, D_n, \; L_0 > \subseteq E$.*

In [WYH], a weak Hessian non-decomposition theorem was proved.

Theorem 3.8: [WYH]. *Let $\eta_4(x_1,\dots,x_n)$ be a homogeneous polynomial of degree 4 in $x_1,\dots,x_n$. Let $H(\eta_4) = \left(\frac{\partial^2\eta_4}{\partial x_i\partial x_j}\right)_{1\le i,j\le n}$ be the Hessian matrix of η_4. Then $H(\eta_4)$ cannot be decomposed as $\Delta\Delta^T$ where $\Delta = (\beta_{ij})_{1\le i,j\le n}$ is an anti-symmetric matrix with entries in homogeneous polynomial of degree one satisfying the cyclic relation*

$$\frac{\partial\beta_{ij}}{\partial x_\ell} + \frac{\partial\beta_{\ell i}}{\partial x_j} + \frac{\partial\beta_{j\ell}}{\partial x_i} = 0 \text{ for all } \ 1\le i,j,\ell\le n$$

unless η_4 and Δ are trivial, i.e. $H(\eta_4)=\Delta\Delta^T$ implies $\Delta = 0$.

As a consequence of Theorem 3.8 and Theorem 3.5, we have the following theorem.

Theorem 3.9: *Let E be a finite-dimensional estimation algebra of maximal rank, k be the maximal rank of quadratic forms in E and n be the dimension of the state space. Then ω_{ij} are constants for $1\le i,j\le k$, or $k+1\le i,j\le n$; ω_{ij} are degree one polynomials in $x_1,\dots,x_k$ for $1\le i\le k$, $k+1\le j\le n$ or $k+1\le i\le n$, $1\le j\le k$.*

4 CLASSIFICATION THEOREM

In view of the theory developed in section 3 above, Yau and Hu [Ya-Hu] have proved the following propositions.

Proposition 4.1: *If $x^2_{k_{p-1}+1}+\cdots+x^2_{k_p}$ is a basic quadratic form in E* (cf. (17)) *and $\frac{\partial\omega_{j\ell}}{\partial x_i}=0$ for all $k+1\le\ell\le n$, $k_{p-1}+1\le i,j\le k_p$ and $i\ne j$, the $\frac{\partial\omega_{i\ell}}{\partial x_i}=0$ for all $k_{p-1}+1\le i\le k_p$.*

Proposition 4.2: *Let $x^2_{k_{p-1}+1}+\cdots+x^2_{k_r}$ and $x^2_{k_{s-1}+1}+\cdots+x^2_{k_s}$ be the basic forms in E* (cf. (17)) *where $k_{r-1}<k_r\le k_{s-1}<k_s$. Let $\xi_{ij}=\sum_{\ell=k+1}^{n}\frac{\partial\omega_{j\ell}}{\partial x_i}D_\ell$. Suppose $\sum_{j=k_{s-1}+1}^{k_s}\xi_{pj}\xi_{qj}=0$ for all $k_{r-1}+1\le p,q\le k_r$, $p\ne q$. Then $\frac{\partial\omega_{j\ell}}{\partial x_i}=0$ for all $k+1\le\ell\le n$, $k_{r-1}+1\le i\le k_r$, and $k_{s-1}+1\le j\le k_s$.*

Proposition 4.3: *$x^2_{k_{r-1}+1}+\cdots+x^2_{k_r}$ and $x^2_{k_{s-1}+1}+\cdots+x^2_{k_s}$ be the basic quadratic forms in E* (cf. (17)), *where $k_{r-1}<k_r\le k_{s-1}<k_s$. Let $\xi_{ij}=\sum_{\ell=k+1}^{n}\frac{\partial\omega_{j\ell}}{\partial x_i}D_\ell$. Then $\sum_{j=k_{s-1}+1}^{k_s}\xi_{pj}\xi_{qj}=0$ for all $k_{r-1}+1\le p,q\le k_r$, $p\ne q$ if and only if $\sum_{j=k_{s-1}+1}^{k_s}a^p_{j\ell_1}a^q_{j\ell_2}=0$ for all $k+1\le\ell_1,\ell_2\le n$, $k_{r-1}+1\le p,q\le k_r$, $p\ne q$, where $a^p_{j\ell_1}=\frac{\partial\omega_{j\ell_1}}{\partial x_p}$*

Proposition 4.4: *Let $x^2_{k_{r-1}+1}+\cdots+x^2_{k_r}$ and $x^2_{k_{s-1}+1}+\cdots+x^2_{k_s}$ be the basic quadratic forms in E* (cf. (17)) *where $k_{r-1}<k_r\le k_{s-1}<k_s$. Assume that $Q_\ell=\sum_{i=k_{r-1}+1}^{k_r}\sum_{j=k_{s-1}+1}^{k_s}a^i_{j\ell}x_ix_j\in E$ for all $k+1\le\ell\le n$, where $a^i_{j\ell}=\frac{\partial\omega_{j\ell}}{\partial x_i}$. Then $\sum_{j=k_{s-1}+1}^{k_s}a^p_{j\ell_1}a^q_{j\ell_2}=0$ for all $k+1\le\ell_1,\ell_2\le n$, $k_{r-1}+1\le p,q\le k_r$.*

Proposition 4.5: *Let $x^2_{k_{r-1}+1}+\cdots+x^2_{k_r}$ and $x^2_{k_{s-1}+1}+\cdots+x^2_{k_s}$ be the basic quadratic forms in E* (cf. (17)) *where $k_{r-1}<k_r\le k_{s-1}<k_s$. Then $\frac{\partial\omega_{j\ell}}{\partial x_i}=0$ for all $k+1\le\ell\le n$, $k_{r-1}+1\le i\le k_r$ and $k_{s-1}+1\le j\le k_s$.*

Proposition 4.6: *Let $x^2_{k_{r-1}+1}+\cdots+x^2_{k_r}$ be a basic quadratic form in E* (cf. (17)). *Then $\frac{\partial\omega_{j\ell}}{\partial x_i}=0$ for all $k+1\le\ell\le n$, $k_{r-1}+1\le i,j\le k_r$ and $i\ne j$.*

Theorem 3.9 together with Proposition 4.1 to Proposition 4.6 imply the following theorem.

Theorem 4.7: *Suppose that E is a finite-dimensional estimation algebra of maximal rank. Then $\Omega=(\omega_{ij})$ is a matrix with constant coefficients.*

The following is the classification theorem of finite dimensional estimation algebra of maximal rnak, which is a solution to the Brockett problem mentioned above. It follows from Theorem 2.2 and Theorem 4.7.

Theorem 4.8: *Suppose that the state space of the filtering system (2.1) is of dimension n. If E is the finite-dimensional estimation algebra with maximal rank, then E is a real vector space of dimension $2n+2$ with basis given by $1, x_1,\dots,x_n, D_1,\dots,D_n$ and L_0.*

References

[Br] R.W. Brockett, Nonlinear systems and nonlinear estimation theory, in The Mathematics of Filtering and Identification and Applications, M. Hazewinkel and J.C. Willems, eds., Reidel, Dordrecht, 1981.

[Br-Cl] R.W. Brockett and J.M.C. Clark, The geometry of the conditional density functions, in Analysis and Optimization of Stochastic Systems, O.L.R. Jacobs, et al., eds., Academic Press, New York, 1980, pp. 399-309.

[Ch] J. Chen, On ubiquity of Yau filters, Proceedings of the American Control Conference, Baltimore, Maryland, June 1994, 252-254.

[Ch-Mi] M. Chaleyat-Maurel and D. Michel, Des resultats de nonexistence de filtre de dimension finite, Stochastics 13 (1984), 83-102.

[Ch-Ya] W.L. Chiou and S.S.-T. Yau, Finite dimensional filters with nonlinear drift II: Brockett's problem on classification of finite dimensiona estimation algebras, SIAM J. Control and Optimization 32, no. 1 (1994), 297-310.

[Ch-Ya 1] J. Chen and S.S.-T. Yau, Finite dimensional filters with nonlinear drift VI: Linear structure of Ω matrix, Mathematics of Control, Signal and Systems 9 (1996), 370-385.

[Ch-Ya 2] J. Chen and S.S.-T. Yau, Finite dimensional filters with nonlinear drift VII: Mitter conjecture and structure of η, SIAM J. Control and Optimization, vol. 35, no. 4 (1997), 1116-1131.

[C-L-Y 1] J. Chen, C.-W. Leung and S.S.-T. Yau, Finite dimensional filters with nonlinear drift IV: Classification of finite dimensional estimation algebras of maximal rank with state space dimension 3, SIAM J. Control and Optimization 34 (1996), 179-198.

[C-L-Y 2] J. Chen, C.-W. Leung and S.S.-T. Yau, Finite dimensional filters with nonlinear drift VIII: Classification of finite dimensional estimation algebras of maximal rank with state space dimension 4, SIAM J. Control and Optimization, vol. 35, no. 4 (1997), 1132-1141.

[D-T-W-Y] R.T. Doug, L.-F. Tam, W.-S. Wong, and S.S.-T. Yau, Structure and classification theorems of finite-dimensional exact estimation algebras, SIAM J. Control and Optimization 29, no. 4 (1991), 866-877.

[Da] M.H.A. Davis, On a multiplicative functional transformation arising in nonlinear filtering theory, Z. Wahrsch. Verw. Gebiete, 54 (1980), 125-139.

[Du] T.E. Duncan, Probability densities for diffusion processes with applications to nonlinear filtering theory, Ph.D. thesis, Stanford, 1967.

[Hu-Ya] G.Q. Hu and S.S.-T. Yau, Finite-dimensional filters with nonlinear drift XIII: classification of finite-dimensional estimation algebras of maximal rank with state space dimension five, (to appear) Asian Journal of Mathematics.

[Mi] S.K. Mitter, On the analogy between mathematical problems of non-linear filtering and quantum physics, Ricerche Automat 10 (1979), 163-216.

[Mo] R.E. Mortensen, Optimal control of continuous time stochastic systems, Ph.d. thesis, Berkeley, 1966.

[T-W-Y] L.-F. Tam, W.S. Wong, and S.S.-T. Yau, On a necessary and sufficient condition for finite dimensionality of estimation algebras, SIAM J. Control and Optimization 28, no. 1 (1990), 173-185.

[We-No] J. Wei and E. Norman, On the global representation of the solutions of linear differential equations as a product of exponentials, Proc. Amer. Math. Sci. 15 (1964), 327-334.

[Wo] W.S. Wong, On a new class of finite-dimensional estimation algebras, Systems Control Letters 9 (1987), 79-83.

[WYH] X. Wu, S.S.-T. Yau and G.Q. Hu, Finite dimensional filters with nonlinear drift XII: Linear and constant structure of Ω, submitted for publication.

[Ya 1] S.S.-T. Yau, Recent results on nonlinear filtering: New class of finite dimensional filters, Proceedings of the 29th Conference on Decision and Control at Honolulu, Hawaii, Dec. 1990, 231-233.

[Ya 2] S.S.-T. Yau, Finite dimensional filters with nonlinear drift I: A class of filters including both Kalman-Bucy filters and Benés filters, J. Math. Systems, Estimation and Control 4, no. 2 (1994), 181-203.

[Ya-Hu] S.S.-T. Yau and G.Q. Hu, Finite-dimensional filters with nonlinear drift XIV: Classification of finite-dimensional estimation algebras of maximal rank with arbitrary state space dimension and Mitter conjecture, submitted for publication.

[YWW] S.S.-T. Yau, X. Wu and W.S. Wong, Hessian matrix non-decomposition theorem, Mathematical Research Letter 6, (1999), 1-11.

[Ya-Ya] S.S.-T. Yau and S.-T. Yau, New direct method for Kalman-Bucy filtering system with arbitrary initial condition, Proceedings of the 33rd Conference on Decision and Control ar Lake Buena Vista, Florida, Dec. 14-16, (1994), 1221-1225.

[Za] M. Zakai, On the optimal filtering of diffusion processes, Z. Wahrsch, Verw. Geb. 11 (1969), 230-243.