UGAS of nonlinear time-varying systems: a δ-persistency of excitation approach*

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Abstract

We study the problem of stability analysis for certain nonlinear systems. Our contributions are new tools to guarantee uniform global asymptotic stability (UGAS) of nonlinear time-varying (NLTV) systems. Firstly, we provide new definitions of *persistency of excitation* (PE). In particular, we give here a new definition of uniform δ-PE (uδ-PE) which, though conceptually equivalent to the original one introduced [7], is mathematically less conservative. We also provide with some properties of δ-PE pairs and contribute with a result which establishes UGAS of NLTV systems under uδ-PE.

Notations. We denote the open ball $B_r := \{x \in \mathbb{R}^n : \|x\| < r\}$. A continuous function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ ($\alpha \in \mathcal{K}$), if $\alpha(s)$ is strictly increasing and $\alpha(0) = 0$; $\alpha \in \mathcal{K}_\infty$ if in addition, $\alpha(s) \to \infty$ as $s \to \infty$. A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if $\beta(\cdot, t) \in \mathcal{K}$ for each fixed $t \geq 0$ and $\beta(s,t) \to 0$ as $t \to \infty$ for each $s \geq 0$. $\dot{V}_{(\#)}(t,x)$ is the time derivative of the Lyapunov function $V(t,x)$ along the solutions $x(t; t_\circ, x_\circ)$ of the differential equation (#). By $\|x\|_p$ we mean $(\int_{t_\circ}^{\infty} \|x(t)\|^p \, dt)^{1/p}$ for $p \in [1, \infty)$ and $\|x\|_\infty := \sup_{t \geq t_\circ} \|x(t)\|$.

1 Introduction

Time-varying nonlinear systems of the form

$$\dot{x} = f(t,x) \qquad x(t_\circ) = x_\circ \tag{1}$$

arise in many problems including the study of trajectory tracking control for nonlinear systems and stabilization of nonholonomic systems by continuous (time-varying) feedback. For such problems, a common objective is to establish uniform (global) asymptotic stability of the origin. The simplest way to guarantee uniform asymptotic stability is to find a uniformly positive definite, decrescent (Lyapunov) function having a uniformly negative definite derivative along trajectories. Unfortunately, the latter condition is often hard to meet. Over the years, many authors have provided sufficient conditions for UAS when the derivative of the Lyapunov function is only negative *semidefinite* (see e.g. [15, 4, 10, 3] and references therein).

This paper is another contribution in that direction. Specifically, we focus on conditions related to the notion of *persistency of excitation*, first introduced in [1] and known in the literature of identification and adaptive control for many years now. For our purposes, an illustrative application is the stability analysis of the linear time-varying system

$$\dot{x} = \begin{bmatrix} A & B\phi(t)^\top \\ -\phi(t)C^\top & 0 \end{bmatrix} x \, . \tag{2}$$

This and other related problems were studied in the concurrent papers [2, 8, 5, 18] where the concept of PE was employed to establish various stability results. The conclusion for (2), which is nicely summarized in many textbooks is that the origin is uniformly (globally) exponentially stable if the triple (A,B,C) is strictly positive real, i.e., satisfies the Kalman-Yakubovich-Popov (KYP) lemma (see e.g. [4]) and: (c1) $\phi(t)$ is bounded (and absolutely continuous); (c2) $\dot{\phi}(t)$ is bounded almost everywhere, and (c3) $\phi(t)$ is PE, i.e., there exist $\mu, T > 0$ such that

$$\mu I \leq \int_t^{t+T} \phi(\tau)\phi(\tau)^\top d\tau \, , \qquad \forall\, t \geq 0 \, . \tag{3}$$

In the context of model reference adaptive control, the following nonlinear version of the system (2) is encountered:

$$\dot{x} = \begin{bmatrix} A & B\phi(t,x)^\top \\ -\phi(t,x)C^\top & 0 \end{bmatrix} x \tag{4}$$

i.e., the regressor depends also on the state and hence it is not possible to apply directly the classical result mentioned above. Some authors (see [13] for a detailed literature review) use a PE condition on the *time* function

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$\tilde{\phi} := \phi(t, x(t, t_\circ, x_\circ))$ where $x(t, t_\circ, x_\circ)$ is a trajectory of the system. This technique has two main disadvantages: 1) no uniformity can be guaranteed since $\tilde{\phi}$ depends on the initial conditions $t_\circ, x_\circ$; 2) if the regressor $\phi(t,x)$ is such that $\phi(t,0) \equiv 0$ then the PE property is lost near the origin. In [7] we provided a UGAS result for the system (4) and, to overcome these two difficulties we introduced the introduced the concept of (uniform) δ-PE.

In this paper we study the class of systems considered in [20, 19, 11], namely given by

$$\dot{x}_1 = H(t,x) + B(t,x)^\top x_2 \tag{5a}$$

$$\dot{x}_2 = D(t,x) \ , \tag{5b}$$

with state $x = \mathrm{col}(x_1, x_2)$, where $x_1 \in R^n$, $x_2 \in \mathbb{R}^m$, all the functions are at least locally Lipschitz continuous and the system is UGS. In [20, 19] only exponential convergence for each trajectory is established. In [11], uniform global asymptotic (rather than exponential) stability is asserted; however, the proof is via Barbalat's lemma which does not give uniform convergence directly.

With respect to [7], we will present a mathematically less conservative (though conceptually equivalent) definition of *uniform* δ-persistency of excitation for state-dependent functions $\phi(t,x)$ and use it to establish UGAS for (5). Also, we will formulate properties for uδ-PE pairs, similar to those for time-dependent functions $\phi(t)$ which can be found in adaptive control books (see e.g. [16, 10, 3]).

Remark 1 It is interesting to remark that, in [9] the authors *pointed out* that, for a nonlinear system $\dot{x} = f(t,x)$ where the time variations are due to an external input $u(t)$, the persistency of excitation condition (3) is neither sufficient nor necessary to assure uniform asymptotic stability. As a possible alternative, in [9, p. 157], the authors *comment on* the idea of defining a new persistency of excitation condition for the vector $u(\cdot)$, with respect to the differential equation $\dot{x} = f(t,x)$.

2 (Uniform) δ-persistency of excitation

2.1 Definitions

Consider the system

$$\dot{x} = f(t,x), \qquad x := \mathrm{col}[x_1, x_2], \tag{6}$$

where $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ is such that (6) is forward complete. Let $\phi : \mathbb{R}_{\geq 0} \times \mathbb{R}^{n+m} \to \mathbb{R}^{p\times q}$ be such that $\phi(t, x(t, t_\circ, x_\circ))$ is locally integrable for each $(t_\circ, x_\circ)$.

For the pair (ϕ, f) we introduce the following definitions and properties.

Definition 1 (δ-Persistency of excitation) *The pair $(\phi,\ f)$ is said to be δ-PE with respect to x_1 if, for each $(t_\circ, x_\circ) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^{n+m}$ there exists a constant $T(t_\circ, x_\circ) > 0$ and for each $\delta > 0$, there exist a constant $\mu(t_\circ, x_\circ, \delta)$ such that for all $t \geq t_\circ$,*

$$\left\{ \min_{s\in[t,\,t+T]} \|x_1(s)\| \geq \delta \right\} \Rightarrow$$
$$\left\{ \int_t^{t+T} \phi(\tau, x(\tau, t_\circ, x_\circ))\phi(\tau, x(\tau, t_\circ, x_\circ))^\top d\tau \geq \mu I \right\} . \tag{7}$$

Definition 2 (uδ-Persistency of excitation) *The pair $(\phi,\ f)$ is called uniformly δ-PE (uδ-PE) with respect to x_1 if, for each $r > 0$ there exists $T(r) > 0$ and for each $\delta > 0$, there exists a constant $\mu(r, \delta) > 0$ such that (7) holds for all $(t_\circ, x_\circ) \in \mathbb{R}_{\geq 0} \times B_r$ and all $t \geq t_\circ$.*

With an abuse of notation we may call the function $\phi(t,x)$, δ-PE (resp. uδ-PE) with respect to x_1 if the pair $(\phi,\ f)$ is δ-PE (resp. uδ-PE) with respect to x_1.

In words, we say that a pair (ϕ, f) is δ-PE if the time signal $\phi(t, x(t))$ is PE in the usual sense whenever the trajectories $x(t)$, generated by the ODE $\dot{x} = f(t,x)$, are away from a δ-neighborhood of the origin on a time window of length T. If this property holds with the same parameters μ and T for all initial conditions in a compact set, then we say that the pair is *uniformly* δ-PE.

Example 1 Let $v : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be locally integrable and consider the system (6), where $x = col(x_1, x_2) \in \mathbb{R}^2$,

$$f(t,x) = \begin{bmatrix} -x_1 + v(t)({x_1}^2 + {x_2}^2)x_2 \\ -v(t)({x_1}^2 + {x_2}^2)x_1 \end{bmatrix}$$

where $v(t)$ is PE in the usual sense. Define

$$\phi(t,x) = v(t)({x_1}^2 + {x_2}^2), \tag{8}$$

then the pair (ϕ, f) is uδ-PE with respect to x_1. Also, it is uδ-PE with respect to x_2 and with respect to x.

2.2 Properties of uδ-PE pairs

Below we present some properties of uδ-PE pairs, which are analogous to properties of time signals which are PE in the usual sense. The proofs of these properties are provided in [13].

Lemma 1 (Properties of uδ-PE pairs) *The following properties hold true for uδ-PE pairs (ϕ, f).*

P1. Let $e : \mathbb{R}_{\geq 0} \times \mathbb{R}^{n+m} \to \mathbb{R}^{p\times q}$ be such that, for each $r > 0$ there exist $M_e > 0$ and $\gamma \in \mathcal{K}$ such that, for all $(t_\circ, x_\circ) \in \mathbb{R}_{\geq 0} \times B_r$, *either* of the following holds:

$$(i) \qquad \int_{t_\circ}^{\infty} \|e(\tau, x(\tau, t_\circ, x_\circ))\|^2 \, d\tau \leq \gamma(r)\,, \qquad (9)$$

(ii) for all $t \geq t_\circ$, $\|e(t, x(t, t_\circ, x_\circ))\| \leq M_e$ and

$$\int_{t_\circ}^{\infty} \|e(\tau, x(\tau, t_\circ, x_\circ))\| \, d\tau \leq \gamma(r)\,. \qquad (10)$$

Then, the pair (ϕ_f, f) where $\phi_f(t,x) = \phi(t,x) + e(t,x)$, is u$\delta$-PE.

P2. Let $e(t,x)$ be uniformly bounded and uniformly convergent, that is, assume that for each $r, \varepsilon > 0$ there exist M_e and $T_e \geq 0$, such that, for all $(t_\circ, x_\circ) \in \mathbb{R}_{\geq 0} \times B_r$ and all $t \geq t_\circ$ we have $\|e(t, x(t, t_\circ, x_\circ))\|^2 \leq M_e$ and

$$\|e(t, x(t, t_\circ, x_\circ))\| \leq \varepsilon \qquad \forall\, t \geq t_\circ + T_e\,. \qquad (11)$$

Then the pair (ϕ_f, f) where $\phi_f(t,x) = \phi(t,x) + e(t,x)$, is u$\delta$-PE.

P3. Let $\phi_i(t,x) \in \mathbb{R}^p$ be the i-th column of $\phi(t,x)$, $i = 1, \ldots, q$ and consider the system

$$\dot{x}_\phi =: \begin{bmatrix} \dot{x} \\ \dot{\phi}_{f1} \\ \vdots \\ \dot{\phi}_{fq} \end{bmatrix} = \begin{bmatrix} f(t,x) \\ -a\phi_{f1} + \phi_1(t,x) \\ \vdots \\ -a\phi_{fq} + \phi_q(t,x) \end{bmatrix} =: F(t, x_\phi)$$

with state $x_\phi := \operatorname{col}[x, \phi_{f1}, \ldots, \phi_{fq}]$, and a constant $a > 0$. If $\phi(\cdot,\cdot)$ is an absolutely continuous function of its arguments and if for every $r > 0$ there exists a constant $M_\phi(r) > 0$ such that for all initial conditions $(t_\circ, x_\circ) \in \mathbb{R}_{\geq 0} \times B_r$, we have

$$\max\left\{ \|\phi(t, x(t, t_\circ, x_\circ))\|\,, \left\| \frac{d\phi(t, x(t, t_\circ, x_\circ))}{dt} \right\| \right\} \leq M_\phi$$

for almost all $t \geq t_\circ$, then the pair (ϕ_f, F) where $\phi_f := [\phi_{f1}, \ldots, \phi_{fq}]$, is u$\delta$-PE with respect to x.

Remark 2 Property **P3** above is analogous to the well known property of time functions which are PE in the usual sense, which – roughly speaking – states that the output of a strictly proper and strictly stable filter, driven by a PE input signal, is also PE.

2.3 Verifying the uδ-PE property

It may be clear from Example 1, that in order for a pair (ϕ, f) to satisfy Definition 2, it is not necessary in general, to know a priori the solutions of the system. The following proposition states the uδ-PE property for a fairly general class of pairs (ϕ, f), based on conditions imposed on $\phi(t,x)$ and stability conditions on $\dot{x} = f(t,x)$. The proof makes use of **P1** and is given in [13].

Proposition 1 *(Class of uδ-PE pairs) Let $\phi(t,x)$ be locally Lipschitz in both arguments. Consider the system (6) and assume that there exists:*

(i). a number $\gamma > 0$ such that

$$\max\{\|x\|_\infty\,, \|x_1\|_2\} \leq \gamma \|x_\circ\|\,,$$

(ii). a nondecreasing function $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that

$$\max\left\{ \left\| \frac{\partial \phi(t,x)}{\partial x_j} \right\|, \left\| \frac{\partial \phi(t,x)}{\partial t} \right\| \right\} \leq \rho(\|x\|), \text{ a.e.,} \qquad (12)$$

(iii). a function $\bar{\phi} : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n+m}$ and constants μ_b, M_b and $T_b > 0$, such that

$$\int_t^{t+T_b} \bar{\phi}(\tau)\bar{\phi}(\tau)^\top \geq \mu_b I \qquad \forall t \geq 0 \quad (13)$$

$$\max_{t \geq 0}\left\{ \|\bar{\phi}(t)\|\,, \|\dot{\bar{\phi}}(t)\| \right\} \leq M_b \quad \text{a.e.,} \qquad (14)$$

(iv). a nondecreasing function $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, and two constants c_1, $c_2 \geq 0$ such that $c_1 + c_2 > 0$ and such that for any unitary vector $\xi \in \mathbb{R}^m$,

$$\|\phi_0(t,x)^\top \xi\| \geq [c_1 + c_2 \psi(\|x_2\|) \|x_2\|] \|\bar{\phi}(t)^\top \xi\|$$

where $\phi_0(t,x) := \phi(t,x)|_{x_1 \equiv 0}$.

Then, the pair (ϕ, f) is uδ-PE with respect to x.

3 UGAS of NLTV systems

In this section we present our main stability result for a particular class of nonlinear time-varying systems using our uδ-PE property. We will formulate our results for systems of the form

$$\dot{x} = F(t,x), \qquad x_\circ = x(t_\circ),\ t_\circ \geq 0, \qquad (15)$$

with state $x = \operatorname{col}(x_1, x_2)$, where $x_1 \in R^n$, $x_2 \in \mathbb{R}^m$, and

$$F(t,x) := \begin{bmatrix} H(t,x) + B(t,x)^\top x_2 \\ D(t,x) \end{bmatrix} \qquad (16)$$

where $H : \mathbb{R}_{\geq 0} \times \mathbb{R}^{n+m} \to \mathbb{R}^n$, $D : \mathbb{R}_{\geq 0} \times \mathbb{R}^m$ and $B : \mathbb{R}_{\geq 0} \times \mathbb{R}^{n+m} \to \mathbb{R}^{m \times n}$ are locally Lipschitz continuous in both arguments. We assume further that the system (15) satisfies the following hypotheses.

A1. There exist nondecreasing functions ρ_1, ρ_2, ρ_3 : $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, such that

$$\max\{\|H(t,x)\|\,, \|D(t,x)\|\} \leq \rho_1(\|x\|)\,\|x_1\| \qquad (17)$$

$$\|B(t,x)\| \leq \rho_2(\|x\|) \qquad (18)$$

$$\max\left\{ \left\| \frac{\partial B(t,x)}{\partial x_j} \right\|, \left\| \frac{\partial B(t,x)}{\partial t} \right\| \right\} \leq \rho_3(\|x\|), \text{ a.e.,} (19)$$

A2. There exist a function $\gamma_1 \in \mathcal{K}_\infty$, a locally bounded function $\gamma_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and a positive definite continuous function $\gamma_3 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that

$$\|x\|_\infty \leq \gamma_1(\|x_\circ\|) \qquad (20)$$
$$\|\gamma_3(\|x_1\|)\|_1 \leq \gamma_2(\|x_\circ\|) \qquad (21)$$

Remark 3 In particular, assumption **A2** is satisfied if there exist a continuously differentiable function $V(t,x)$ and two functions $\bar{\alpha}(s), \underline{\alpha}(s) \in \mathcal{K}_\infty$ such that

$$\underline{\alpha}(\|x\|) \leq V(t,x) \leq \bar{\alpha}(\|x\|),$$
$$\dot{V}_{(16)}(t,x) \leq -\gamma_3(\|x_1\|) \,. \qquad (22)$$

Systems of the form above include applications in adaptive control of linear time varying and nonlinear systems (see e.g. [4, 3, 10, 16]). Indeed, notice that if there exists $V(t,x)$ satisfying (22) with $\gamma_3(s) = s^2$ the system $\dot{x} = F(t,x) + v$, with external input $v := \mathrm{col}[v_1; 0]$, $v_1 \in \mathcal{L}_2^n$, and output $y = x_1$, is output strictly passive. Systems belonging to this class often include Euler-Lagrange systems (see e.g. [6, 12].) ■

In addition to the system (15) let us consider the following system

$$\dot{x} = F_1(t,x) \;:=\; \begin{bmatrix} -x_1 + B(t,x)^\top x_2 \\ -B(t,x)x_1 \end{bmatrix} . \qquad (23)$$

The reason for introducing this 'simplified' system is the following. It is easy to see that system (15) can be considered as a system (23) with "output" injection $K(t,x) = F(t,x) - F_1(t,x)$ which, under the assumption **A1** satisfies the bound

$$\|F(t,x) - F_1(t,x)\| \leq \rho_4(\|x\|)\,\|x_1\| \,. \qquad (24)$$

where $\rho_4 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a nondecreasing function. Then, if the system (23) is UGAS, assumption **A2** and (24) imply that all conditions of Lemma 3 presented in the Appendix, are satisfied. Hence UGAS of (15) follows from UGAS of (23).

Theorem 1 (UGAS under uδ-PE) *If* **A1**, **A2** *hold and the pair (B, F_1) is uδ-PE with respect to x_2 the system (15), (16) is UGAS.*

Proof. As mentioned above the proof of UGAS for (15) follows invoking Lemma 3. We start with proving that (23) is UGAS, for which we will use Lemma 2. For this, let us define $V_\circ(t,x) := \frac{1}{2}\|x\|^2$, taking the time derivative along the solutions of (23) and integrating from $t_\circ$ to ∞ we obtain that

$$\|x\|_\infty \leq \|x_\circ\| \qquad (25)$$
$$\|x_1\|_2^2 \leq \frac{1}{2}\|x_\circ\|^2 \,. \qquad (26)$$

In the next part of the proof we will show that (38) holds for $x_2(t)$, which together with (25) and (26) allows to conclude UGAS of (23). Towards this end, define $V_1(t,x) := -x_1^\top B^\top(t,x)x_2$ and for all t and x, where $B(t,x)$ is differentiable, define

$$M(t,x) := B(t,x) - \left(-x_1 + B^\top(t,x)x_2\right)^\top \frac{\partial B^\top(t,x)}{\partial x_1} + x_1^\top B^\top(t,x)\frac{\partial B^\top(t,x)}{\partial x_2} + \frac{\partial B^\top(t,x)}{\partial t} \,.$$

From Assumption **A1** it follows that there exists a non-decreasing function ρ_4 such that almost everywhere

$$\|M(t,x)\| \leq \rho_5(\|x\|) \,. \qquad (27)$$

Since the pair (B, F_1) is uδ-PE for each $r > 0$ there exists $T(r) > 0$ and for each $\delta > 0$ there exists $\mu(r,\delta)$ such that (7) holds. Fix r and δ arbitrarily. Taking the time derivative of $V_1(t,x)$ (wherever it exists) along the solutions of (23), we obtain with an abuse of notation,

$$\dot{V}_1(t,x) \leq -\|B^\top(t,x)x_2\|^2 + \frac{\mu}{8T}\|x_2\|^2 + \rho_6(r)\|x_1\|^2 \,,$$

for almost all $t \geq t_\circ$, where $\rho_6(s) := \frac{2T}{\mu}\rho_5^2(s) + \rho_2(s)$. Integrating the inequality above from t to $t+T$ (for any $t \geq t_\circ$) we get[1]

$$V_1(t+T, x(t+T)) - V_1(t, x(t)) \;\leq\; \rho_6(r)\int_t^{t+T} \|x_1(\tau)\|^2 d\tau - \int_t^{t+T} \|B^\top(\tau, x(\tau))x_2(\tau)\|^2 d\tau + \frac{\mu}{8T}\int_t^{t+T} \|x_2(\tau)\|^2 d\tau. \qquad (28)$$

Using the following representation of the solutions:

$$x_2(\tau, t_\circ, x_\circ) = x_2(t, t_\circ, x_\circ) + \textstyle\int_t^\tau B(s, x(s, t_\circ, x_\circ))x_1(s, t_\circ, x_\circ)ds \,,$$

completing some squares and using (18), it is straightforward to find that

$$\|x_2(\tau)\|^2 \leq 2\|x_2(t)\|^2 + 2\rho_2^2(r)\int_t^\tau \|x_1(s)\|^2 ds \qquad (29)$$

while

$$\|B^\top(\tau, x(\tau))x_2(\tau)\|^2 \geq \tfrac{1}{2}\|B^\top(\tau, x(\tau))x_2(t)\|^2 - \rho_2^4(r)\textstyle\int_t^\tau \|x_1(s)\|^2 ds \,. \qquad (30)$$

Substitute (30) and (29) in (28) to obtain

$$V_1(t+T, x(t+T)) - V_1(t, x(t)) \;\leq\; \left[\rho_2(r)^4 + \frac{\mu}{4T}\rho_2(r)^2\right]\int_t^{t+T}\int_t^\tau \|x_1(s)\|^2\, ds d\tau - \frac{1}{2}\int_t^{t+T} \left\|B(\tau, x(\tau))^\top x_2(t)\right\|^2 d\tau + \frac{\mu}{4T}\int_t^{t+T} \|x_2(t)\|^2\, d\tau + \rho_6(r)\int_t^{t+T} \|x_1(\tau)\|^2\, d\tau$$

[1] noting that $V(t, x(t))$ is an absolutely continuous function of time.

Changing the order of integration for the first term on the right hand side of the inequality above, we obtain

$$V_1(t+T, x(t+T)) - V_1(t, x(t)) \le \rho_7 \int_t^{t+T} \|x_1(\tau)\|^2 d\tau$$
$$-\frac{1}{2}\int_t^{t+T} \left\|B(\tau, x(\tau))^\top x_2(t)\right\|^2 d\tau + \frac{\mu}{4}\|x_2(t)\|^2$$

where we defined $\rho_7 := T\rho_2^4 + 0.25\mu\rho_2^2 + \rho_6$. Hence,

$$\tfrac{1}{2}\left(\int_t^{t+T} \left\|B(\tau, x(\tau))^\top x_2(t)\right\|^2 d\tau - \tfrac{\mu}{2}\|x_2(t)\|^2\right) \le$$
$$\rho_7 \int_t^{t+T} \|x_1(\tau)\|^2 d\tau - V_1(t+T, x(t+T)) + V_1(t, x(t)) . \tag{31}$$

Define $T^* \ge t_\circ$ as follows: $T^* := \min_{\tau \ge t_\circ}\left\{\|x_2(\tau)\|^2 = \delta\right\}$, $T^* = \infty$ if $\|x_2(\tau)\|^2 > \delta$ for all $\tau \ge t_\circ$, and $T^* = t_\circ$ if $\|x_2(t_\circ)\|^2 < \delta$. Let us consider two cases:

Case 1: If $T^* \ge t_\circ + T$,

$$\int_{t_\circ}^{t}\left[\|x_2(\tau)\|^2 - \delta\right] d\tau =$$
$$\left(\int_{t_\circ}^{T^*-T} + \int_{T^*-T}^{T^*} + \int_{T^*}^{t}\right)\left[\|x_2(\tau)\|^2 - \delta\right] d\tau ,$$

and it follows from (25), that $\|x(\tau)\|^2 \le \|x(T^*)\|^2 \le \delta$ for all $\tau \ge T^*$, therefore the last integral on the right hand side of the inequality above is non-positive, hence

$$\int_{t_\circ}^{t}\left[\|x_2(\tau)\|^2 - \delta\right] d\tau \quad \le \quad \int_{t_\circ}^{T^*-T}\left[\|x_2(\tau)\|^2 - \delta\right] d\tau + \int_{T^*-T}^{T^*} \|x_2(\tau)\|^2 d\tau . \tag{32}$$

We show next that there exists $\beta_{r\delta} \in \mathcal{K}$ such that the first term on the right hand side above is smaller or equal to $\beta_{r\delta}(\|x_\circ\|)$. Using (7) and since (B, F_1) is uδ-PE with respect to x_2, we have that, for all $t \in [t_\circ, T^* - T]$,

$$\int_t^{t+T} \left\|B(\tau, x(\tau))^\top x_2(t)\right\|^2 d\tau \ge \mu\|x_2(t)\|^2$$

hence using this bound in (31), we obtain that

$$\frac{\mu}{4}\|x_2(t)\|^2 \quad \le \quad \rho_7 \int_t^{t+T} \|x_1(\tau)\|^2 d\tau$$
$$-V_1(t+T, x(t+T)) + V_1(t, x(t)) .$$

Integrating from $t_\circ$ to $T^* - T$, we obtain that

$$\int_{t_\circ}^{T^*-T} \|x_2(t)\|^2 dt \le \frac{4T}{\mu}[0.5\rho_7 + \rho_2]\|x_\circ\|^2 =: \beta_1(\|x_\circ\|)$$

while for the second term on the right hand side of (32), using (25) we have that

$$\int_{T^*-T}^{T^*} \|x_2(\tau)\|^2 d\tau \le \int_{T^*-T}^{T^*} \|x_\circ\|^2 d\tau \le T\|x_\circ\|^2 , \tag{33}$$

thus for any $\delta > 0$

$$\int_{t_\circ}^{t}\left[\|x_2(\tau)\|^2 - \delta\right] d\tau \le \beta_1(\|x_\circ\|) + T\|x_\circ\|^2 . \tag{34}$$

Case 2: If $T^* < t_\circ + T$, we can use the partition

$$\int_{t_\circ}^{t}\left[\|x_2(\tau)\|^2 - \delta\right] d\tau =$$
$$\left(\int_{t_\circ}^{T^*} + \int_{T^*}^{t}\right)\left[\|x_2(\tau)\|^2 - \delta\right] d\tau , \tag{35}$$

and the observation that

$$\int_{t_\circ}^{T^*} \|x_2(\tau)\|^2 d\tau \le T\|x_\circ\|^2$$

to obtain

$$\int_{t_\circ}^{t}\left[\|x_2(\tau)\|^2 - \delta\right] d\tau \le T\|x_\circ\|^2 . \tag{36}$$

UGAS for the system (23) follows from Lemma 2 with $\gamma(s) = s^2$, $\nu = \delta$ and $\beta_{r\nu}(\|x_\circ\|) := \beta_1(\|x_\circ\|) + (0.5 + T)\|x_\circ\|^2$.

We finish the proof of UGAS for (15) invoking Lemma 3. Let $y = x_1$, hence (42) holds with $k_2(s) = s$. Inequality (41) holds due to (24) with $k(s) = s$ and $k_1 = \rho_4$. Inequality (40) holds with $\gamma = \gamma_3$ and $\beta = \gamma_2$. The UGS condition is equivalent to (20). ■

We close the paper with the following Corollary which follows directly from Proposition 1 and Theorem 1 and illustrates that, for large classes of systems we do not need to know the solutions a priori.

Corollary 1 *The system (15) is UGAS if for this system, items (iii) and (iv) of Proposition 1 and Assumptions* **A1** *–* **A2** *hold.*

4 Conclusions

The sufficient conditions for UGAS of NLTV systems that one can find in classical texts on Lyapunov theory, are often too restrictive since they rely on finding a Lyapunov function with a negative definite derivative. In this report we have made a step forward in the direction of stability analysis for a determined (but wide) class of NLTV systems. Our contributions consist on new definitions of persistency of excitation, which apply in a formal way, to nonlinear functions depending on time and the state. Based on these tools we have established sufficient conditions for UGAS.

References

[1] K. J. Åström and. Numerical identification of linear dynamic systems from normal operating records. In P. H. Hammond, editor, Proc. of the 2nd IFAC Symp. on Theory of Self-adaptive Control Systems, pages 96–111, Nat. Phys. Lab., Teddington, England, 1965.

[2] B. O. Anderson. Exponential stability of linear equations arising in adaptive identification. *IEEE Trans. on Automat. Contr.*, 22(1):83–88, 1977.

[3] P. Ioannou and J. Sun. *Robust adaptive control.* Prentice Hall, New Jersey, USA, 1996.

[4] H. Khalil. *Nonlinear systems.* Macmillan Publishing Co., 2nd ed., New York, 1996.

[5] G. Kreisselmeier. Adaptive observers with exponential rate of convergence. *IEEE Trans. on Automat. Contr.*, 22(1):2–8, 1977.

[6] A. Loría, T I. Fossen, and A. Teel. UGAS and ULES of nonautonomous systems: Applications to integral control of ships and manipulators. In *Proc. 5th. European Contr. Conf.*, Karlsruhe, Germany, 1998. Paper no. 551.

[7] A. Loría, E. Panteley, and A. Teel. A new persistency-of-excitation condition for UGAS of NLTV systems: Application to stabilization of nonholonomic systems. In *Proc. 5th. European Contr. Conf.*, 1999. paper no. 500.

[8] A. P. Morgan and K. S. Narendra. On the stability of nonautonomous differential equations $\dot{x} = [A + B(t)]x$ with skew-symmetric matrix $B(t)$. *SIAM J. on Contr. and Opt.*, 15(1):163–176, 1977.

[9] K. Narendra and A. Annaswamy. Persistent excitation in adaptive systems. *Int. J. of Contr.*, 45(1):127–160, 1987.

[10] K. S. Narendra and A. M. Anaswamy. *Stable adaptive systems.* Prentice-Hall, Inc., New Jersey, 1989.

[11] R. Ortega and A. L. Fradkov. Asymptotic stability of a class of adaptive systems. *Int. J. Adapt. Contr. Sign. Process.*, 7:255–260, 1993.

[12] R. Ortega, A. Loría, P. J. Nicklasson, and H. Sira-Ramírez. *Passivity-based Control of Euler-Lagrange Systems: Mechanical, Electrical and Electromechanical Applications.* Comunications and Control Engineering. Springer Verlag, London, 1998. ISBN 1-85233-016-3.

[13] E. Panteley, A. Loría, and A. Teel. UGAS of NLTV systems: Applications to adaptive control. Technical Report 99-160, Lab. d'Automatique de Grenoble, UMR 5528, CNRS, France, 1999.

[14] E. Panteley, A. Loría, and A. Teel. Integral characterizations of asymptotic stability. In *Proc. Int. Symp. on Math. Theory of Networks and Systems*, Perpignan, France, 2000. paper no. SI28_4.

[15] N. Rouche and J. Mawhin. *Ordinary differential equations II: Stability and periodical solutions.* Pitman publishing Ltd., London, 1980.

[16] S. Sastry and M. Bodson. *Adaptive control: Stability, convergence and robustness.* Prentice Hall Intl., 1989.

[17] A. Teel, E. Panteley, and A. Loría. Integral characterizations of uniform asymptotic and exponential stability with applications. Technical Report AP00-015, LAG UMR 5528, CNRS, January 2000. Submitted to *Math. of Cont. Sign. and Syst.*

[18] J. S-C Yuan and W. M. Wonham. Probing signals for model reference identification. *IEEE Trans. on Automat. Contr.*, 22(4):530–538, 1977.

[19] Y. Zhang, P. Ioannou, and C. Chien. Parameter convergence of a class of adaptive controllers. In *Proc. American Control Conference*, volume 2, pages 1248–1252, 1994.

[20] Y. Zhang, P. Ioannou, and C. Chien. Parameter convergence of a new class of adaptive controllers. *IEEE Trans. on Automat. Contr.*, 41(10):1489–1493, 1996.

A Integral lemmas for UGAS

The lemmas below are extracted from [13]. See also [14, 17] for extensions of these to stability of sets and the case of differential inclusions.

Lemma 2 *The system*

$$\dot{x} = f(t, x) \tag{37}$$

where $x \in R^n$, the function $f(t,x)$ satisfies the Caratheodory conditions[2]*, is UGAS if the system is UGS and there exists continuous positive definite function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and for each r, $\nu > 0$ there exists a locally bounded function $\beta_{r\nu} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, such that for all $(t_\circ, x_\circ) \in \mathbb{R}_{\geq 0} \times B_r$, all solutions $x(\cdot, t_\circ, x_\circ)$ and all $t \geq t_circ$*

$$\int_{t_\circ}^{t} \left[\gamma(\|x(\tau, t_\circ, x_\circ)\|) - \nu\right] d\tau \leq \beta_{r\nu}(\|x_\circ\|)\,. \tag{38}$$

Lemma 3 *Assume that the system (37) is UGAS and function $f(t,x)$ is locally Lipschitz in x uniformly in t. Then the system*

$$\dot{x} = f(t, x) + K(t, x) \tag{39}$$

where $K(t,x)$ is continuous in x, measurable in t and locally Lipschitz in x, is UGAS if the system (39) is UGS and there exist an output $y = h(t,x)$, nondecreasing functions β, k_1, $k_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, continuous positive definite function γ and class $\mathcal{K}_\infty$ function k, such that for all $(t_\circ, x_\circ) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$

$$\|\gamma(\|y\|)\|_1 \leq \beta(\|x_\circ\|) \tag{40}$$

$$\|K(t, x)\| \leq k_1(\|x\|)k(\|y\|) \tag{41}$$

$$\|h(t, x)\| \leq k_2(\|x\|)\,. \tag{42}$$

[2] That is, 1) $f(t,x)$ is continuous in x for each fixed t and measurable in t for each fixed x; 2) For each compact set $\mathcal{U} := [a,b] \times \mathcal{C} \subset \mathbb{R} \times \mathbb{R}^n$, there exists a function $m_{\mathcal{U}} : [a,b] \to \mathbb{R}_{\geq 0}$, Lebesgue integrable on $[a,b]$, such that $\|f(t,x)\| \leq m_{\mathcal{U}}(t)$ for all $(t,x) \in \mathcal{U}$.