

# Optimal control with HARA utility functions

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## Abstract

In this paper, a control methodology based on the HARA utility function is presented as an alternative to the exponential-of-an-integral approach to finding robust controllers. This work is inspired by the intuition that HARA controllers, while being robust, may give better performance than exponential controllers in normal situations. The HARA problem is shown to be equivalent to a certain differential game and the asymptotic properties of the HARA problem and this differential game are studied. As an example, a linear-quadratic HARA problem is studied, where the problem of finding a robust HARA controller is proved to be equivalent to solving a standard linear-quadratic problem for a system with a higher noise intensity. This reveals an interesting relationship between robustness and uncertainty.

*Key words:* Risk-sensitive control, differential games, HARA utility function, upper/lower Isaacs equations, viscosity solutions.

## 1 Introduction

In this paper, we propose an approach based on the HARA utility function ( $U(x) = \frac{1}{\gamma}x^\gamma$ , for  $x \geq 0$  and  $\gamma > 0$ ) as an alternative to the exponential utility ( $U(x) = \gamma \exp(\gamma x)$ , for  $\gamma \geq 0$ ) approach to finding robust controllers. This work is inspired by the intuition that HARA controllers, while being robust, may perform better than exponential controllers when applied to a system that is operating under normal conditions.

Suppose that the evolution of a given system is determined by the following stochastic differential equation

(SDE):

$$\begin{cases} dx(t) = b(t, x(t), u(t)) dt \\ \quad + \sigma(t, x(t), u(t)) dB(t), \quad t \in [s, T], \\ x(s) = x, \end{cases} \quad (1)$$

where  $(s, x)$  represents the initial time and state. Typically, the objective is to find a control input  $u(\cdot)$  such that the system (1) with input  $u(\cdot)$  satisfies a given set of performance specifications; for example, certain specifications on the minimum rise time, constraints on the maximum overshoot, etc., may need to be satisfied. In recent years, many control methodologies (e.g., PID, LQR,  $H_\infty$ , just to name a few) have been proposed as alternative techniques for finding controllers, each having its own advantages, disadvantages and characteristics. However, we emphasize that irrespective of the methodology used to come up with a given controller, the controller is designed as the input for the system (1) and for this reason, the *same* set of performance specifications are used to evaluate whether or not it is suitable. For example, suppose we have two controllers  $u_1(\cdot)$  and  $u_2(\cdot)$  such that  $u_1(\cdot)$  minimizes an  $H_\infty$  norm while  $u_2(\cdot)$  minimizes an  $H_2$  norm. When evaluating  $u_1(\cdot)$  and  $u_2(\cdot)$  and comparing their performance, the crucial issue is the behavior of the system (1) under each input; that is, both controllers are tested on the same system (1) and evaluated according to how well the *original* performance criteria are met (e.g., ‘Does it satisfy the rise time specifications?’ and ‘Does it meet the constraints on maximum overshoot?’, etc.), and not the cost functions (e.g.,  $H_\infty$  or  $H_2$ ) that may have been used to determine it.

If the system being controlled is not too complicated, then finding a controller which satisfies all of the specifications is reasonably straightforward. However, things may not be so easy when dealing with more complex systems (e.g., nonlinear, time-varying, stochastic). In this case, the following simplifications are made. Rather than introducing a *set* of performance specifications, one assumes that *all* performance specifications are summarized by a *single* performance measure

$$J(s, x; u(\cdot)) = E\left\{ \int_s^T f(t, x(t), u(t)) dt + g(x(T)) \right\}, \quad (2)$$

and the following convention is adopted: a controller

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which makes (2) small satisfies the performance specifications better than controllers that makes (2) large. That is, the set of performance specifications is replaced by a single performance measure and, as in the classical case, all controllers, irrespective of how they are determined, are evaluated using the same performance measure (2).

The risk-sensitive methodology using the exponential utility function is one approach to finding robust controllers for the system (1) (with performance measure (2)). Such controllers are obtained by minimizing criteria of the form

$$J(s, x; u(\cdot)) = E \exp \frac{1}{\epsilon} \left( \int_s^T f(t, x(t), u(t)) dt + g(x(T)) \right). \quad (3)$$

Again, we emphasize that the *performance* of the optimal exponential controller  $\bar{u}(\cdot)$  (which minimizes (3)) is evaluated using the cost functional (2): it is ‘good’ if (2) is small, and ‘bad’ if (2) is large. One limitation of the exponential approach is that optimal risk-sensitive controllers, while being robust, may result in poor performance under normal conditions. The simple explanation for this is that the exponential utility emphasizes the large values of the (random) exponent

$$\int_s^T f(t, x(t), u(t)) dt + g(x(T)), \quad (4)$$

thus greatly amplifying the contribution of the large values of (4) in the cost (3). That is, optimal exponential controllers are conservative in that they are designed so that the largest (or ‘worst case’) values of (4), should they occur, are kept small. For this reason, however, the performance of the optimal exponential controller in a ‘less than worst case’ environment (which often corresponds to ‘normal conditions’) may not be satisfactory. Alternatively, one can look at the close relationship between exponential risk-sensitive control and the so-called  $H_\infty$  approach to robust control; see [9].  $H_\infty$  controllers are designed to perform well in a ‘worst case disturbances’ environment. Consequently, optimal  $H_\infty$  controllers (and therefore, optimal risk-sensitive controllers) may perform poorly, according to (2), in normal conditions.

In this paper, we study certain issues related to the performance and robustness of controllers obtained by minimizing the HARA utility of the cost. In particular, for the system (1) with performance measure (2), the HARA cost functional is given by

$$J(s, x; u(\cdot)) = E \left\{ \left( \int_s^T f(t, x(t), u(t)) dt + g(x(T)) \right)^{\frac{1}{\epsilon}} \right\}, \quad (5)$$

where  $\epsilon > 0$  is a parameter. Since the HARA utility function is of polynomial rather than exponential order, the resulting controllers should be still robust but less risk-averse than optimal exponential controllers. For this reason, HARA controllers may perform better in normal conditions than optimal exponential controllers which are obtained by minimizing (3). While this remains to be verified conclusively, certain weaker, though related, comparisons between the value functions of the HARA and exponential problems can be obtained. (For more details, we the interested reader should consult the paper [12]). In particular, this comparison may be viewed, in some sense, as a comparison between the *guaranteed performance* of HARA controllers and exponential controllers. In addition, we show that the HARA problem is equivalent to a class of stochastic differential games which are characterized by a cost function that contains a logarithmic term that acts like a weight on different components of the cost. The logarithmic term in the cost may be viewed as making the disturbance player less aggressive; that is, the (HARA) controller is designed in a ‘less than worst case’ environment. This is consistent with the intuition that HARA controllers may perform better than exponential controllers when operating in normal conditions. In addition, the asymptotic properties (small noise limits) of the HARA problem and the associated differential game are also studied.

Our results on the relationship between the HARA problem and differential games as well as the asymptotic properties of the HARA problem are in line with those presented in [9] for the exponential-of-an-integral problem. However, there are certain differences which we wish to point out. In this paper, we consider a broader class of nonlinear systems. In particular, our results apply to systems in which the diffusion term may be control dependent and/or degenerate, and the drift term may depend nonlinearly on the control. (In fact, our analysis can be modified to obtain parallel results for the exponential case for the same class of nonlinear systems that we study in this paper). These results are obtained using results from nonsmooth analysis and viscosity solutions. The inclusion of systems with control dependent diffusions in our analysis has particular relevance to finance applications; see [6, 7, 10, 15].

On the other hand, the robust control literature has centered, by and large, around the exponential utility function. This is due to its relationship to differential games and  $H_\infty$  control; see [2, 9]. However, the HARA utility function and its associated differential game are alternatives that should be kept in mind, especially when exponential and  $H_\infty$  controllers are found to be too conservative.

In this section, we introduce the HARA control problem. Suppose that  $s \in [0, T)$  is fixed,  $(\Omega, \mathcal{F}, \{\mathcal{F}_t^s\}_{t \geq s}, P)$  is a filtered probability space, and  $B(\cdot)$  an  $\mathbf{R}^k$ -valued standard Brownian motion defined on this space. Suppose that  $x \in \mathbf{R}^n$  and  $\epsilon > 0$  are given. We consider systems with dynamics given by the following stochastic differential equation (SDE):

$$\begin{cases} dx(t) = b(t, x(t), u(t)) dt \\ + \sqrt{\epsilon} \sigma(t, x(t), u(t)) dB(t), \quad t \in [s, T], \\ x(s) = x. \end{cases} \quad (6)$$

The cost functional is given by:

$$J(s, x; u(\cdot)) = E \left\{ \left( \int_s^T f(t, x(t), u(t)) dt + g(x(T)) \right)^{\frac{1}{\epsilon}} \right\} \quad (7)$$

where  $x(\cdot)$  is the solution of the SDE (6) corresponding to  $(s, x)$  and  $u(\cdot)$ . Since we shall be using dynamic programming, the weak formulation of admissible controls will be used with  $u(\cdot)$  taking values in a set  $U$ ; see [14, Chapter 2]. In addition, it is assumed that (6) has a unique weak solution and (7) is well defined for every admissible control; see [12] for relevant assumptions. In order to study (6)-(7) via dynamic programming, we introduce the following equivalent problem: Let  $\bar{y} \in (0, C)$ , where  $0 < C < \infty$  is a certain constant, required for technical reasons; see [12]. For every  $(s, x, y) \in [0, T) \times \mathbf{R}^n \times (-\bar{y}, \infty)$ , consider the following SDE:

$$\begin{cases} dx(t) = b(t, x(t), u(t)) dt \\ + \sqrt{\epsilon} \sigma(t, x(t), u(t)) dB(t), \\ dy(t) = f(t, x(t), u(t)) dt, \quad t \in [s, T], \\ x(s) = x, y(s) = y \end{cases} \quad (8)$$

and cost functional:

$$J(s, x, y; u(\cdot)) = E \left\{ \left( y(T) + g(x(T)) \right)^{\frac{1}{\epsilon}} \right\}. \quad (9)$$

The Hamilton-Jacobi-Bellman (HJB) equation associated with (8)-(9) is:

$$\begin{cases} v_t + \inf_{u \in U} \left\{ \frac{\epsilon}{2} \text{tr} \left[ v_{xx} \sigma \sigma' \right] \right. \\ \left. + v_x' b + v_y f \right\} = 0, \\ (t, x, y) \in [0, T) \times \mathbf{R}^n \times (-\bar{y}, \infty) \\ v(T, x, y) = (y + g(x))^{\frac{1}{\epsilon}}, \\ (x, y) \in \mathbf{R}^n \times (-\bar{y}, \infty). \end{cases} \quad (10)$$

The following result proven in [12].

**Theorem 2.1** *The value function  $v$  associated with (8)-(9) is the unique viscosity solution of (10).*

Let  $(\Omega, \mathcal{F}, \{\mathcal{F}_t^s\}_{t \geq s}, P)$  be a filtered probability space and  $B(\cdot)$  a standard  $\mathbf{R}^k$ -valued Brownian motion. Let  $x \in \mathbf{R}^n$  and  $\epsilon > 0$  be given and fixed. Consider the following SDE:

$$\begin{cases} dx(t) = \\ [b(t, x(t), u(t)) + \sigma(t, x(t), u(t)) w(t)] dt \\ + \sqrt{\epsilon} \sigma(t, x(t), u(t)) dB(t), \quad t \in [s, T], \\ x(s) = x. \end{cases} \quad (11)$$

We consider cost functionals of the following form:

$$J(s, x; u(\cdot), w(\cdot)) = E \left\{ \ln \left( \int_s^T f(t, x(t), u(t)) dt + g(x(T)) \right) - \frac{1}{2} \int_s^T |w(t)|^2 dt \right\}. \quad (12)$$

Intuitively, the *Upper Differential Game* associated with (11)-(12) can be understood as follows. Player 1 goes first, and chooses an admissible input process  $u(\cdot)$  from some admissible set  $\mathcal{U}[s, T]$ . On the other hand, the input process  $w(\cdot)$  of player 2 may depend on the choice that player 1 makes; that is,  $w(\cdot) = \Gamma(u(\cdot))$ , where  $\Gamma(\cdot)$  denotes the *strategy* of player 2 which is constrained to belonging to a set of *admissible strategies*,  $\Gamma[s, T]$ . We refer the reader to [5, 8, 9, 12] for the definitions of  $\mathcal{U}[s, T]$  and  $\Gamma[s, T]$  as well as further details of the Elliott-Kalton formulation of differential games. In the deterministic case (i.e.  $\epsilon = 0$ ), we denote the admissible sets by  $\mathcal{U}_d[s, T]$  and  $\Gamma_d[s, T]$ . Finally, it is assumed throughout that  $u(\cdot)$  and  $w(\cdot)$  take values in subsets  $U$  and  $W$ , respectively.

In order to study (11)-(12) using dynamic programming, we follow the same procedure that we used for the HARA problem (Section 2) by considering the following related differential game. Let  $\bar{y} \in (0, C)$  be fixed, where  $C > 0$  is the constant associated with (8)-(9), and  $y \in (-\bar{y}, \infty)$ . Consider the SDE:

$$\begin{cases} dx(t) = \\ [b(t, x(t), u(t)) + \sigma(t, x(t), u(t)) w(t)] dt \\ + \sqrt{\epsilon} \sigma(t, x(t), u(t)) dB(t) \\ dy(t) = f(t, x(t), u(t)) dt, \\ t \in [s, T], \\ x(s) = x, y(s) = y \end{cases} \quad (13)$$

with cost

$$J(s, x, y; u(\cdot), w(\cdot)) = E \left\{ \ln (y(T) + g(x(T))) - \frac{1}{2} \int_s^T |w(t)|^2 dt \right\} \quad (14)$$

The *Upper Isaacs Equation* associated with (13)-(14) is:

$$\left\{ \begin{array}{l} \psi_t + \inf_{u \in U} \sup_{w \in W} \left\{ \frac{\epsilon}{2} \text{tr} \left[ \psi_{xx} \sigma \sigma' \right] \right. \\ \left. + \psi_x' (b + \sigma w) + \psi_y' f - \frac{1}{2} |w|^2 \right\} = 0, \\ (t, x, y) \in [0, T] \times \mathbf{R}^n \times (-\bar{y}, \infty), \\ \psi(T, x, y) = \ln(y + g(x)), \\ (x, y) \in \mathbf{R}^n \times (-\bar{y}, \infty). \end{array} \right. \quad (15)$$

The *deterministic differential game* associated with (13)-(14) corresponds to setting  $\epsilon = 0$ . The corresponding upper Isaacs equation is (15) with  $\epsilon = 0$ . Under certain assumptions, we have the following result (see [12] for proofs).

**Theorem 3.1** *Let  $\epsilon \geq 0$  be given and fixed. Then:*

$$\psi(s, x, y) = \inf_{u(\cdot) \in \mathcal{U}[s, T]} \sup_{\alpha[\cdot] \in \Gamma[s, T]} J(s, x, y; u(\cdot), \alpha[u(\cdot)]) \quad (16)$$

*is the unique viscosity solution of (15).*

#### 4 HARA problems and differential games

In this section, we show that (under certain conditions) the HARA problem (8)-(9) is equivalent to the stochastic differential game (13)-(14).

Consider the following transformation:

$$\begin{aligned} \phi(t, x, y) &:= \epsilon \ln v(t, x, y), \\ \forall (t, x, y) &\in [0, T] \times \mathbf{R}^n \times (-\bar{y}, \infty). \end{aligned} \quad (17)$$

Since  $v$  is the unique viscosity solution of (10) (Theorem 2.1), it follows (from substituting (17) into (10)) that  $\phi$  is the unique viscosity solution of the following PDE:

$$\left\{ \begin{array}{l} \phi_t + \inf_{u \in U} \sup_{w \in \mathbf{R}^k} \left\{ \frac{\epsilon}{2} \text{tr} \left[ \phi_{xx} \sigma \sigma' \right] \right. \\ \left. + \phi_x' (b + \sigma w) + \phi_y' f - \frac{1}{2} |w|^2 \right\} = 0, \\ (t, x, y) \in [0, T] \times \mathbf{R}^n \times (-\bar{y}, \infty) \\ \phi(T, x, y) = \ln(y + g(x)), \\ (x, y) \in \mathbf{R}^n \times (-\bar{y}, \infty). \end{array} \right. \quad (18)$$

The following result establishes the equivalence between the HARA problem and the stochastic differential game.

**Theorem 4.1** *Let  $\epsilon > 0$  be fixed, and  $v$  be the unique viscosity solution of (10). Then  $\phi := \epsilon \ln v$  is the*

*unique viscosity solution of (18). Moreover, there exists a compact subset  $W \subseteq \mathbf{R}^k$  which is independent of  $\epsilon > 0$  such that  $\phi$  has the representation*

$$\phi(t, x, y) = \inf_{u(\cdot) \in \mathcal{U}[s, T]} \sup_{\alpha[\cdot] \in \Gamma[s, T]} J(s, x, y; u(\cdot), \alpha[u(\cdot)]) \quad (19)$$

where

$$\begin{aligned} J(s, x, y; u(\cdot), w(\cdot)) &= \\ E \left\{ \ln [y(T) + g(x(T))] - \frac{1}{2} \int_s^T |w(t)|^2 dt \right\} \end{aligned}$$

with  $(x(\cdot), y(\cdot))$  being the solution of (13) associated with  $(u(\cdot), w(\cdot))$ .

#### 5 Asymptotic analysis

For the remainder of this paper, for any given  $\epsilon > 0$ , we shall denote the value function of the HARA problem (8)-(9) by  $v^{(\epsilon)}$ , and  $\phi^{(\epsilon)} := \epsilon \ln v^{(\epsilon)}$ . In the deterministic case of (13)-(14) with  $\epsilon = 0$ , we continue to use  $\psi$ .

We have shown that with  $\epsilon > 0$ , the HARA problem (8)-(9) is equivalent to a stochastic differential game of the form (13)-(14); see Theorem 4.1. In this section, we show that  $\phi^{(\epsilon)} \rightarrow \psi$  when  $\epsilon \rightarrow 0$ . This reveals a relationship between the HARA problem and the deterministic differential game. The convergence proof can be found in [12] and follows the general methods of Barles and Perthame [1].

**Theorem 5.1** *For every  $\epsilon > 0$ , let  $v^{(\epsilon)}$  be the value function of the HARA problem (8)-(9),  $\phi^{(\epsilon)} := \epsilon \ln v^{(\epsilon)}$  and  $W \subseteq \mathbf{R}^k$  be the compact set from Theorem 4.1. Suppose that  $\psi$  is the upper value of the deterministic differential game (13)-(14) associated with  $W$ . Then*

$$\begin{aligned} \lim_{\epsilon \downarrow 0} \phi^{(\epsilon)}(t, x, y) &= \psi(t, x, y), \\ \forall (t, x, y) &\in [0, T] \times \mathbf{R}^n \times (-\bar{y}, \infty), \end{aligned}$$

*uniformly on compact subsets.*

In [9], there is a similar asymptotic result (though proven under stronger assumptions than in [12]) for the exponential risk-sensitive cost. In particular, non-degeneracy of  $\sigma$  is assumed, which guarantees the existence of classical solutions to the HJB equation associated with the  $\epsilon$ -parametrized risk-sensitive problems, whereas Theorem 5.1 is proven under assumptions which can only guarantee viscosity solutions.

In this section, we study a linear-quadratic-regulator (LQR) problem with HARA utility. This problem is interesting because it reveals an interesting relationship between robustness and uncertainty.

### HARA problem:

Consider the following linear-quadratic problem with HARA utility. (We assume throughout that  $H > 0$ ) :

$$\begin{cases} \min_{u(\cdot) \in \mathcal{U}[s, T]} J(s, x; u(\cdot)) = E \left\{ \frac{1}{2} H x(T)^2 \right\}^{\frac{1}{\epsilon}}, \\ dx(t) = [\bar{A}(t) x(t) + \bar{B}(t) u(t)] dt \\ \quad + \sqrt{\frac{2}{\epsilon} - 1} \bar{D}(t) u(t) dB(t), \quad t \in [s, T], \\ x(s) = x. \end{cases} \quad (20)$$

Throughout this section, we shall assume that  $x(\cdot)$  is scalar valued and  $u(\cdot)$  is  $\mathbf{R}^m$ -valued, for  $m \geq 1$ . For simplicity, we assume that  $B(\cdot)$  is a scalar-valued Brownian motion. (However, the analysis carries through to the case when  $B(\cdot)$  is higher-dimensional). Also, we shall assume that  $\bar{D}(t)' \bar{D}(t) > 0$  for all  $t \in [s, T]$ , and that  $\epsilon \in (0, 2)$  is given and fixed. In particular, it should be noted that the model (20) (with scalar state, but a multivariable control) is one that arises quite frequently in financial applications; see [6, 7, 15] for an example of this.

The solution of the HJB equation associated with (20) is:

$$v(t, x) = \left( \frac{1}{2} P(t) x^2 \right)^{\frac{1}{\epsilon}}, \quad (21)$$

where  $P(\cdot)$  is the unique (positive) solution of the *stochastic Riccati equation*<sup>1</sup>:

$$\begin{cases} \dot{P} + P\bar{A} + \bar{A}P \\ -P\bar{B} \left[ \left( \frac{2}{\epsilon} - 1 \right) \bar{D}P\bar{D} \right]^{-1} \bar{B}P = 0, \\ P(T) = H. \end{cases} \quad (22)$$

Note that (22) has an explicit solution:

$$P(t) = H \exp \left\{ \left[ 2\bar{A}(t) - \bar{B}(t) \left( \left( \frac{2}{\epsilon} - 1 \right) \bar{D}(t)' \bar{D}(t) \right)^{-1} \bar{B}(t)' \right] (T - t) \right\}.$$

The optimal feedback control for (20) is:

$$u(t) = - \left( \left( \frac{2}{\epsilon} - 1 \right) \bar{D}(t)' \bar{D}(t) \right)^{-1} \bar{B}(t)' x(t). \quad (23)$$

<sup>1</sup>Although the stochastic Riccati equation (22) is clearly deterministic, it is actually a special case of a Riccati-type backward stochastic differential equation that is introduced in [3], and is deterministic only under the assumptions of this paper.

### Stochastic LQR problem:

Consider now the following stochastic LQR problem:

$$\begin{cases} \min_{u(\cdot) \in \mathcal{U}[s, T]} J(s, x; u(\cdot)) = E \left\{ \frac{1}{2} H x(T)^2 \right\}, \\ dx(t) = [\bar{A}(t) x(t) + \bar{B}(t) u(t)] dt \\ \quad + \sqrt{\frac{2}{\epsilon} - 1} \bar{D}(t) u(t) dB(t), \quad t \in [s, T], \\ x(s) = x. \end{cases} \quad (24)$$

It is easy to show (see [3]) that

$$V(t, x) = \frac{1}{2} P(t) x^2 \quad (25)$$

is the value function associated with (24), where  $P(\cdot)$  is determined by (22), and (23) is the optimal control. This shows an equivalence between the HARA problem (20) and the stochastic LQR problem (24). In particular, the  $\epsilon$ -parameter in the cost (20) has been transferred to the state equation (24).

### Discussion:

The equivalence between (20) and (24) shows an interesting relationship between uncertainty and robustness. Suppose that under 'normal' conditions, the dynamics are given by the system:

$$\begin{cases} dx(t) = [\bar{A}(t) x(t) + \bar{B}(t) u(t)] dt \\ \quad + \bar{D}(t) u(t) dB(t), \quad t \in [s, T], \\ x(s) = x. \end{cases} \quad (26)$$

In addition, suppose that the *performance* of any given controller  $u(\cdot)$  is measured by the cost functional:

$$J(s, x; u(\cdot)) = E \left\{ \frac{1}{2} H x(T)^2 \right\}, \quad H > 0. \quad (27)$$

Note that the optimal controller associated with (26)-(27) is given by

$$u(t) = -(\bar{D}(t)' \bar{D}(t))^{-1} \bar{B}(t)' x(t). \quad (28)$$

Suppose that we wish to find a controller  $u(\cdot)$  for the system (26) that is more robust (or risk-averse) than the optimal controller (28). One method for finding such a controller is to solve the HARA problem (20) corresponding to some  $\epsilon \in (0, 1)$ . (That is, to use the optimal HARA controller (23), with  $\epsilon \in (0, 1)$ , in the system (26)). On the other hand, the equivalence between (20) and (24) shows that the (robust) HARA controller (23) corresponding to  $\epsilon \in (0, 1)$  is also the optimal controller for (24) with the same value of  $\epsilon$ . Moreover, since  $\sqrt{(2/\epsilon) - 1} > 1$  when  $\epsilon \in (0, 1)$ , finding this robust HARA controller corresponds to solving a standard LQR problem (since (24) is of the same form as (26)-(27)) but with a higher noise intensity.

It should also be noted that greater robustness corresponds to a controller of smaller magnitude (i.e., decreasing  $\epsilon \in (0, 1)$  corresponds to the magnitude of  $u(t)$  in (23) decreasing).

When  $\epsilon \in (1, 2)$ , the optimal controller (23) for the HARA problem is risk-seeking. The relationship between the HARA problem (20) and the LQR problem (24) shows that a risk-seeking controller for the system (26) under the criterion (27) is obtained by solving an LQR problem (24) with a smaller noise intensity (since  $\sqrt{(2/\epsilon) - 1} < 1$  when  $\epsilon \in (1, 2)$ ). Note also that risk-seeking controllers have a larger magnitude than the optimal controller (28).

## 7 Conclusion

In this paper, we have studied some of the properties of optimal HARA controllers. Our study of the HARA problem was motivated by the belief that in addition to being robust, optimal HARA controllers are less conservative than optimal exponential controllers. We have shown that the HARA problem is equivalent to a certain stochastic differential game, different from the one commonly encountered in the robust control literature, and have studied the asymptotic properties of both the HARA problem and the associated game problem. One feature of this differential game is that it involves a logarithmic term which acts like a weight for the different components of the cost. This weighting has the effect of making the opposing (disturbance) player less aggressive; that is, the controller in the logarithmic-weighted game (i.e. the HARA controller) is designed in a ‘less than worst case’ environment. This is consistent with the intuition that HARA controllers, while being robust, may perform better than exponential controllers (which are designed in a ‘worst case environment’) when applied to a system operating under normal situations. While conclusive theoretical justification of this intuition remains an important open question, certain related, though weaker, comparisons between the value functions of the HARA exponential problems can be obtained; see [12]. As an example, we examined a particular linear-quadratic case of the HARA problem. For this problem, we showed that finding a robust controller for a certain class of linear systems with a quadratic terminal cost is equivalent to solving a linear-quadratic problem of the *same* form, *but with a larger noise intensity*. This shows an interesting relationship between robustness and uncertainty.

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