More about non square implicit descriptions for modelling and control

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1 Introduction

Rosenbrock [27] was the first to introduce the Implicit Descriptions,

$$(1) \qquad E\dot{x}(t) = Ax(t) + Bu(t) \ \ , \ \ y(t) = Cx(t) \ \ ,$$

where $E : \mathcal{X} \to \underline{\mathcal{X}}$, $A : \mathcal{X} \to \underline{\mathcal{X}}$, $B : \mathcal{U} \to \underline{\mathcal{X}}$ and $C : \mathcal{X} \to \mathcal{Y}$ are linear operators of appropriate dimensions, as a generalization of the standard State Space case ($E = I$). Since this introduction many people have payed attention to this broader class of systems with different points of view: *(i)* The Geometric Theory (e.g. [30], [12], [2], [26], [25], [21]), *(ii)* The Time Domain Approach (e.g. [31]), *(iii)* The Laplace Transform (e.g. [29], [4], [19]), *(iv)* The Kronecker Theory (e.g. [23],[24]), *(v)* The Polynomial Approach (e.g. [20]), *(vi)* The Differential Algebra (e.g. [13]), and *(vii)* The Diffential Inclusion Techniques (e.g. [14]).

Not only many interesting research problems have been solved but also some practical aspects of System Theory have been raised though this vision. In fact, it is now well recognized that Implicit Descriptions are able to describe a wide range of interesting external behaviours in System Theory (e.g. [29],[11], [12], [1], [22]), namely, Proper Systems (E=I), Non Proper Systems with Derivative Actions, Restricted Input Systems, and Systems with Algebraic State Constraints (see [15]).

In [5], it was shown that when $\dim \underline{\mathcal{X}} \leq \dim \mathcal{X}$, it is also possible to describe linear systems with an internal Variable Structure. Indeed, when $\dim \underline{\mathcal{X}} < \dim \mathcal{X}$ and if the system is solvable (i.e. possesses at least one solution), solutions are generally non unique. In some sense there is a degree of freedom in (1), which can be used for instance to take into account a possible *structure variation* in an implicit way. Other kinds of structure variations have taken into account in [17, 3].

In the general case (matrices E and A not necessarily square) Frankowska [14] characterized the controllable subspace using differential inclusion techniques. But in the case of non square (flat) E, A matrices one may be faced to controllable systems even in the absence of any input; this is possible because of the existence of the free descriptor variables (degree of freedom) acting as internal controllers. In order to avoid such pathologies has been introduced in [7] the concept of *output dynamics assignment*, which guarantees controllability by means of the control input, namely, *there exists a P.D. feedback, $u = F_d\dot{x} + F_px$, such that the spectrum of $\lambda(E - BF_d) - (A + BF_p)$ is arbitrarily chosen if and only if the two following geometric conditions hold:*

$$(2) \qquad \mathcal{R}^*_{\mathcal{X}} = \mathcal{X} \ ,$$

$$(3) \qquad \begin{array}{l} \dim\left(\mathcal{V}^*_{\mathcal{X}} \cap \mathrm{Ker}\ E\right) - \dim\left(\frac{\mathrm{Im}\ B}{\mathrm{Im}\ B \cap \mathrm{Im}\ E}\right) \\ \qquad \leq \dim\left(\mathcal{V}^*_{\mathrm{Ker}\ C} \cap E^{-1}\mathrm{Im}\ B\right) \ , \end{array}$$

where $\mathcal{R}^*_{\mathcal{X}}$ and $\mathcal{V}^*_{\mathcal{K}}$ (with $\mathcal{K}$ equal to $\mathcal{X}$ or to Ker C) are respectively the limits of ($i \geq 1$):

$$\mathcal{R}^0_{\mathcal{X}} = \mathcal{V}^*_{\mathcal{X}} \cap \mathrm{Ker}\ E \ ; \ \mathcal{R}^i_{\mathcal{X}} = \mathcal{V}^*_{\mathcal{X}} \cap E^{-1}\left(\mathrm{Im}\ B + A\mathcal{R}^{i-1}_{\mathcal{X}}\right) \ ,$$

$$\mathcal{V}^0_{\mathcal{K}} = \mathcal{K} \ ; \ \mathcal{V}^i_{\mathcal{K}} = \mathcal{K} \cap A^{-1}\left(\mathrm{Im}\ B + E\mathcal{V}^{i-1}_{\mathcal{K}}\right) \ .$$

Now, since up to now people do not know how to synthesize pure derivative actions, we have proposed in [8] a procedure for approximating the non proper control laws, as proposed in [7], by proper controllers guaranteeing internal stability (see also [9] for a general study on this topic). Since the control law needs the knowledge of the descriptor variable (both the non proper controller and the proper one) it is necessary to estimate it. As the synthesis of a *descriptor variable* deeply depends on the knowledge of which *internal structure* is active, it has been proposed in [10] a *structure detector* based on a normalized gradient adaptation algorithm, projected along a given hyper-sphere, which aim is to identify in finite time which *internal structure* is present. In order to try to complete this control scheme, in this paper we propose a procedure to find a flat implicit description for a linear system switching between a finite number of internal models. This procedure is more general that the one proposed in [6]. In Section 2 we introduce the *ladder systems*, taking into account three kind of factors: irreducible factors of order one and two and lead/lag compensation networks. In Section 3 we give an illustrative example which shows how to approximate the so-called logistic function, one of the three most important crop growing curves, into a *ladder system*. Finally in Section 4 we conclude.

2 Ladder Systems

Let us consider a system which external behavior, (u, y), can be described by one of the following 2^{n-1} linear systems:

$$\begin{cases} (\mathrm{p}a_1+1)y(t)=u(t) \\ (\mathrm{p}a_i+1)(\mathrm{p}a_1+1)y(t)=u(t)\ ,\ \text{with: } i\in\{2,\dots,n\} \\ (\mathrm{p}a_j+1)(\mathrm{p}a_i+1)(\mathrm{p}a_1+1)y(t)=u(t)\ , \\ \qquad\qquad \text{with: } i,j\in\{2,\dots,n\}\&\ i\neq j \\ (\mathrm{p}a_k+1)(\mathrm{p}a_j+1)(\mathrm{p}a_i+1)(\mathrm{p}a_1+1)y(t)=u(t)\ , \\ \qquad\qquad \text{with: } i,j,k\in\{2,\dots,n\}\&\ i\neq j\neq k\neq i \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ (\mathrm{p}a_n+1)\dots\dots\dots\dots(\mathrm{p}a_1+1)y(t)=u(t) \end{cases}$$

where the a_i (i = 1, ..., n) are real numbers. All these 2^{n-1} linear systems can be described by the following global implicit system:

$$\begin{cases} \dot{x}_2=-a_1(x_1+x_2)+(x_3+x_4) \\ \dot{x}_4=(a_n-1)x_1-a_n(x_3+x_4)+(x_5+x_6) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \dot{x}_{2n-4}=(a_4-1)x_{2n-7}-a_4(x_{2n-5}+x_{2n-4}) \\ \qquad\qquad +(x_{2n-3}+x_{2n-2}) \\ \dot{x}_{2n-2}=(a_3-1)x_{2n-5}-a_3(x_{2n-3}+x_{2n-2})+(x_{2n-1}) \\ \dot{x}_{2n-1}=(a_2-1)x_{2n-3}-a_2x_{2n-1}+u \end{cases} \tag{4}$$

$$\begin{cases} 0=x_1(t)-\theta_0(x_3(t)+x_4(t)) \\ 0=x_3(t)-\theta_1(x_5(t)+x_6(t)) \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ 0=x_{2n-5}(t)-\theta_{n-3}(x_{2n-3}(t)+x_{2n-2}(t)) \\ 0=x_{2n-3}(t)-\theta_{n-2}(x_{2n-1}(t)) \end{cases} \tag{5}$$

$$y(t)=(x_1(t)+x_2(t)) \tag{6}$$

where the parameters θ_i, $i=0,1,\dots,n-2$, switch between "1" and "0". Afterwards, we consider the following three cases (p is the derivative operator d/dt and s is the complex variable of Laplace):

1. Irreducible factors of order 1, $(\mathrm{p}+a_i)$,
2. Irreducible factors of order 2, $(\mathrm{p}^2+2\rho\omega_o\mathrm{p}+\omega_o^2)$,
3. Lead/Lag compensation network, $\alpha\ (\gamma\alpha\mathrm{s}+1)/(\mathrm{s}+\alpha)$.

The importance of the first two cases is evident. The third case is often useful to add a final touch towards a given desired behaviour. Namely, the step response of the network $\alpha(\alpha s+1)/(s+\alpha)$ is $1+(\alpha^2-1)\mathrm{e}^{-\alpha t}$, and thus the parameter α can be used to fit the step response; this is a well known technique to compensate oscilloscope probes.

2.1 Irreducible Factors of order 1 over $\Re$

Let us first consider a system which input-output behaviour can be described by one of the following eight differential equations ($\bar{\theta}_i=1-\theta_i$):

$$\left(\Pi_{i=0}^{n-2}\left(\bar{\theta}_i(\mathrm{p}+a_{n-i})+\theta_i\right)\right)(\mathrm{p}+a_1)y(t)= \\ =\left(\Pi_{i=0}^{n-2}\left(\bar{\theta}_i a_{n-i}+\theta_i\right)\right)a_1u(t)$$

with $n=4$, and where $\theta_i\in\{0\ ,\ 1\}$, for $i=0,\ 1,\ 2$. From (4)–(6) we get the implicit description (7)-(8):

$$\begin{bmatrix} 0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&1 \end{bmatrix}\dot{x}= \\ \begin{bmatrix} -\bar{a}_1&-a_1&k_4&1&0&0&0 \\ a_4-1&0&-\bar{a}_4&-a_4&k_3&1&0 \\ 0&0&a_3-1&0&-\bar{a}_3&-a_3&1 \\ 0&0&0&0&a_2-1&0&-a_2 \end{bmatrix}x \\ +\begin{bmatrix} 0&0&0&k_2 \end{bmatrix}^T u \\ y=\begin{bmatrix} k_1&1&0&0&0&0&0 \end{bmatrix}x \tag{7}$$

$$0=\begin{bmatrix} 1&0&-\theta_0k_4&-\theta_0&0&0&0 \\ 0&0&1&0&-\theta_1k_3&-\theta_1&0 \\ 0&0&0&0&1&0&-\theta_2 \end{bmatrix}x \tag{8}$$

The parameters $\bar{a}_i$ and the gains k_i are:

$$\begin{cases} \bar{a}_i=k_i(a_i-1)+1 \qquad ; \quad i=1,3,4 \\ k_2=\hat{a}_4\hat{a}_3\hat{a}_2\hat{a}_1\ \ ;\ \ (k_1,\ k_3,\ k_4)=\left(\frac{1}{\hat{a}_4},\ \frac{1}{\hat{a}_2},\ \frac{1}{\hat{a}_3}\right) \\ \text{if } a_i\neq 0 \text{ then } \hat{a}_i=|a_i| \text{ else } \hat{a}_i=1 \end{cases}$$

In Table 1 the involved internal behaviours are shown.

2.2 Irreducible Factors of order 2 over $\Re$

Let us next take into account a system which input-output behaviour can be described by one of the following four differential equations ($\bar{\theta}_i=1-\theta_i$):

$$\left(\bar{\theta}_1(\mathrm{p}+a_2)+\theta_1\right)\ \left(\bar{\theta}_0(\mathrm{p}^2+2\rho\omega_o\mathrm{p}+\omega_o^2)+\theta_0\right)\cdot \\ \cdot(\mathrm{p}+a_1)y(t)=\left(\bar{\theta}_1a_2+\theta_1\right)\left(\bar{\theta}_0\omega_o^2+\theta_0\right)a_1u(t),$$

$\theta_1,\ \theta_0\in\{0\ ,\ 1\}$. For this case (7)-(8) is modified as:

$$\begin{bmatrix} 0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&1 \end{bmatrix}\dot{x}= \\ \begin{bmatrix} -\bar{a}_1&-a_1&k_4&1&0&0&0 \\ a_4-1&0&-\bar{a}_4&-a_4&k_3&1&0 \\ \omega_o^2&0&-\hat{a}_3&-\omega_o^2&-\bar{a}_3&-a_3&1 \\ 0&0&0&0&a_2-1&0&-a_2 \end{bmatrix}x \\ +\begin{bmatrix} 0&0&0&k_2 \end{bmatrix}^T u \\ y=\begin{bmatrix} k_1&1&0&0&0&0&0 \end{bmatrix}x \tag{9}$$

$$0=\begin{bmatrix} 1&0&-\theta_0k_4&-\theta_0&0&0&0 \\ 0&0&1&0&-\theta_1k_3&-\theta_1&0 \\ 0&0&0&0&1&0&-\theta_1 \end{bmatrix}x \tag{10}$$

The parameters a_3, a_4, $\bar{a}_i$ ($i=1,3,4$), $\tilde{a}_3$ and the gains k_i have the following values:

$$\begin{cases} a_3=\omega_o^2\ ,\quad a_4=2\rho\omega_o,\quad \tilde{a}_3=k_4\omega_o^2+1,\quad \bar{a}_3=1-k_3 \\ \bar{a}_1=k_1(a_1-1)+1,\quad \bar{a}_4=k_4(2\rho\omega_o-1)+1 \\ k_2=\hat{a}_3\hat{a}_2\hat{a}_1,\quad (k_1,\ k_3,\ k_4)=\left(\frac{1}{\sqrt{\hat{a}_3}},\ \frac{1}{\hat{a}_2},\ \frac{1}{\sqrt{\hat{a}_3}}\right) \\ \text{if } a_i\neq 0 \text{ then } \hat{a}_i=|a_i| \text{ else } \hat{a}_i=1,\quad (i=1,2,3) \end{cases}$$

In Table 2 the involved internal behaviours are shown.

2.3 Lead/Lag Compensation Network

Let us finally take into account a system which input-output behaviour can be described by one of the following four differential equations ($\bar{\theta}_i=1-\theta_i$):

$$\left(\bar{\theta}_1(\mathrm{p}+a_2)+\theta_1\right)\left(\bar{\theta}_0(\mathrm{p}+\alpha)+\theta_0\right)(\mathrm{p}+a_1)y(t) = \left(\bar{\theta}_0(\gamma\alpha\mathrm{p}+1)+\theta_0\right)\left(\bar{\theta}_1 a_2+\theta_1\right)\left(\bar{\theta}_0\alpha+\theta_0\right)a_1 u(t),$$

$\theta_1,\ \theta_0 \in \{0\ ,\ 1\}$. For this case (7)-(8) is modified as:

$$(11)\quad \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \dot{x} = \begin{bmatrix} -\bar{a}_1 & -a_1 & k_3-\gamma\alpha & 1 & \gamma\alpha \\ a_2-1 & 0 & -\bar{a}_2 & -a_2 & 1-a_2\gamma\alpha \\ 0 & 0 & \alpha-1 & 0 & -\alpha \end{bmatrix} x + \begin{bmatrix} 0 & 0 & k_2 \end{bmatrix}^T u$$

$$y = \begin{bmatrix} k_1 & 1 & 0 & 0 & 0 \end{bmatrix} x$$

$$(12)\, 0 = \begin{bmatrix} 1 & 0 & -\theta_0(k_3-\gamma\alpha) & -\theta_0 & -\theta_0\gamma\alpha \\ 0 & 0 & 1 & 0 & -\theta_1 \end{bmatrix} x$$

The parameters $\bar{a}_1$, $\bar{a}_2$, and the gains k_i $(i = 1,2,3)$ have the following values:

$$\begin{cases} \bar{a}_1 = k_1(a_1-1)+1, \quad \bar{a}_2 = k_3(a_2-1)+(1-a_2\gamma\alpha) \\ a_3 = \alpha, \quad k_2 = \hat{a}_3\hat{a}_2\hat{a}_1, \quad (k_1,\ k_3) = (1/\hat{a}_2,\ 1/\hat{a}_3) \\ \text{if } a_i \neq 0 \text{ then } \hat{a}_i = |a_i| \text{ else } \hat{a}_i = 1, \quad (i=1,2,3) \end{cases}$$

In Table 3 the involved internal behaviours are shown.

2.4 Output Dynamics Assignment

Let us do the following observations over the matrices E, A, B, and C of the previous cases:

1. The matrices E of systems (7), (9) and (11) are full row rank, which implies: Im $E = \underline{\mathcal{X}}$, $\mathcal{V}^*_{\mathcal{X}} = \mathcal{X}$, and $\dim(\text{Im } B/\text{Im } B \cap \text{Im } E) = 0$,

2. From the matrices E and C of systems (7), (9) and (11), we get EKer $C = \underline{\mathcal{X}}$, which implies: $\mathcal{V}^*_{\text{Ker } C} =$ Ker C,

3. From the above observations, condition (3) is equivalent to: $\dim \text{Ker } E \leq \dim\left(\text{Ker } C \cap E^{-1}\text{Im } B\right)$. This inequality is satisfied (with the equality) by systems (7), (9) and (11),

4. With respect to condition (2), i.e. the reachability subspace, let us note that systems (7), (9) and (11) satisfy (for system (11), note that $\det\begin{vmatrix} 1 & \gamma\alpha \\ -a_2 & 1-a_2\gamma\alpha \end{vmatrix} = 1$): Im $A + ImB = \underline{\mathcal{X}}$, which implies that $\mathcal{R}^*_{\mathcal{X}} = \mathcal{X}$.

And then all the above Implicit Flat Descriptions have the Output Dynamics Assignment Property, and then, they can be controlled by an external control input.

3 Illustrative Example

Let us take into account the logistic function:

$$(13)\qquad \bar{y}(t) = \bar{y}^* \frac{\mathrm{e}^{\kappa\bar{y}^*(t-t^*)}}{1+\mathrm{e}^{\kappa\bar{y}^*(t-t^*)}}$$

which is solution of the differential equation:

$$\dot{\bar{y}}(t)/\bar{y}(t) = \kappa(\bar{y}^* - \bar{y}(t))$$

This function was introduced by Robertson in 1923 and it is one of the three most important functions used to

$(\theta_0\theta_1\theta_2)$	*Internal Behaviour*
(111)	$(\mathrm{p}+a_1)y = a_1u$ $(\mathrm{p}+a_1)x_2 = k_1(1-a_1)(x_4+k_4(x_6+k_3x_7))$ $(\mathrm{p}+1)x_4 = 0\ ;\ (\mathrm{p}+1)x_6 = 0$ $(\mathrm{p}+1)x_7 = k_2u$
(011)	$(\mathrm{p}+a_2)(\mathrm{p}+a_1)y = a_2a_1u$ $(\mathrm{p}+a_1)x_2 = k_1(1-a_1)(x_4+k_4x_6)$ $(\mathrm{p}+1)x_4 = 0\ ;\ (\mathrm{p}+1)x_6 = x_7$ $(\mathrm{p}+a_2)x_7 = k_2u$
(101)	$(\mathrm{p}+a_3)(\mathrm{p}+a_1)y = a_3a_1u$ $(\mathrm{p}+a_1)x_2 = k_1(1-a_1)x_4$ $(\mathrm{p}+1)x_4 = (x_6+k_3x_7)$ $(\mathrm{p}+a_3)x_6 = k_3(1-a_3)x_7\ ;\ (\mathrm{p}+1)x_7 = k_2u$
(110)	$(\mathrm{p}+a_4)(\mathrm{p}+a_1)y = a_4a_1u$ $(\mathrm{p}+a_1)x_2 = (x_4+k_4(x_6+k_3x_7))$ $(\mathrm{p}+a_4)x_4 = k_4(1-a_4)(x_6+k_3x_7)$ $(\mathrm{p}+1)x_6 = 0\ ;\ (\mathrm{p}+1)x_7 = k_2u$
(001)	$(\mathrm{p}+a_3)(\mathrm{p}+a_2)(\mathrm{p}+a_1)y = a_3a_2a_1u$ $(\mathrm{p}+a_1)x_2 = k_1(1-a_1)x_4\ ;\ (\mathrm{p}+1)x_4 = x_6$ $(\mathrm{p}+a_3)x_6 = x_7\ ;\ (\mathrm{p}+a_2)x_7 = k_2u$
(010)	$(\mathrm{p}+a_4)(\mathrm{p}+a_2)(\mathrm{p}+a_1)y = a_4a_2a_1u$ $(\mathrm{p}+a_1)x_2 = (x_4+k_4x_6)\ ;\ (\mathrm{p}+1)x_6 = x_7$ $(\mathrm{p}+a_4)x_4 = k_4(1-a_4)x_6\ ;\ (\mathrm{p}+a_2)x_7 = k_2u$
(100)	$(\mathrm{p}+a_4)(\mathrm{p}+a_3)(\mathrm{p}+a_1)y = a_4a_3a_1u$ $(\mathrm{p}+a_1)x_2 = x_4\ ;\ (\mathrm{p}+a_4)x_4 = (x_6+k_3x_7)$ $(\mathrm{p}+a_3)x_6 = k_3(1-a_3)x_7\ ;\ (\mathrm{p}+1)x_7 = k_2u$
(000)	$\left(\Pi_{i=1}^4(\mathrm{p}+a_i)\right)y = \left(\Pi_{i=1}^4 a_i\right)u$ $(\mathrm{p}+a_1)x_2 = x_4\ ;\ (\mathrm{p}+a_4)x_4 = x_6$ $(\mathrm{p}+a_3)x_6 = x_7\ ;\ (\mathrm{p}+a_2)x_7 = k_2u$

Table 1: Possible descriptions of system (7)-(8).

describe the growing curve of crops development, the other two are: Gompertz and Richards functions (see [18, 28, 16] for example). Usually the growing curve (13) is classified in three different regions, namely, the exponential region, the linear region, and the logarithmic region. The exponential part of the curve, below the inflection point, characterizes an accelerated growth; the linear part, around the inflection point, characterizes a stationary growing rate; and the logarithmic part, above the inflection point, characterizes a deceleration growth.

Let us choose in this illustrative example: $t^* = 1/2$, $\bar{y}^* = 1$, and $\kappa = 4$; with these values we have the inflection point $\bar{y}(1/2) = 1/2$, the stationary value $\bar{y}(\infty) = 1$, and the initial condition $\bar{y}(0) = 0.12$ $(= \bar{y}_0)$. With these selected values the growing curve (13) can be approximated by the step responses of the three linear differential equations shown in (14) in the time horizon $[0\ ,\ 1.5]$ with an error less than 0.7 %.

$$(14)\begin{cases} (\mathrm{p}+\alpha)(\mathrm{p}+a)(\mathrm{p}+s_1)y_1(t) = (\alpha\mathrm{p}+1)\alpha a_o s_1, & t \in [0\ ,\ t') \\ \mathrm{p}(\mathrm{p}+s_1)y_2(t) = s_1u(t), & t \in [t'\ ,\ t"] \\ (\mathrm{p}+s_2)(\mathrm{p}+s_1)y_3(t) = \omega_o^2u(t), & t \in (t"\ ,\ T] \end{cases}$$

with $t' = 0.4$, $t" = 0.6$, $T = 1.5$ and where the parameters α, a_1, s_1, and s_2 are equal to:

$(\theta_0\theta_1)$	*Internal Behaviour*
(11)	$\boxed{(\mathrm{p}+a_1)y = a_1 u}$ $(\mathrm{p}+a_1)x_2 = k_1(1-a_1)\left(x_4 + k_4\left(x_6 + k_3 x_7\right)\right)$ $(\mathrm{p}+1)x_4 = 0$; $(\mathrm{p}+1)x_6 = 0$; $(\mathrm{p}+1)x_7 = k_2 u$
(10)	$\boxed{(\mathrm{p}+a_2)(\mathrm{p}+a_1)y = a_2 a_1 u}$ $(\mathrm{p}+a_1)x_2 = k_1(1-a_1)\left(x_4 + k_4 x_6\right)$ $(\mathrm{p}+1)x_4 = 0$; $(\mathrm{p}+1)x_6 = x_7$ $(\mathrm{p}+a_2)x_7 = k_2 u$
(01)	$\boxed{(\mathrm{p}^2 + 2\rho\omega_o \mathrm{p} + \omega_o^2)(\mathrm{p}+a_1)y = \omega_o^2 a_1 u}$ $(\mathrm{p}+a_1)x_2 = k_1(1-a_1)x_4$; $\mathrm{p}x_6 = k_3 x_7 - \omega_o^2$ $(\mathrm{p}+2\rho\omega_o)x_4 = x_6 + k_3 x_7$; $(\mathrm{p}+1)x_7 = k_2 u$
(00)	$\boxed{(\mathrm{p}^2 + 2\rho\omega_o \mathrm{p} + \omega_o^2)\left(\Pi_{i=1}^2(\mathrm{p}+a_i)\right) y = \omega_o^2 a_2 a_1 u}$ $(\mathrm{p}+a_1)x_2 = x_4$; $(\mathrm{p}+2\rho\omega_o)x_4 = x_6$ $\mathrm{p}x_6 = x_7 - \omega_o^2 x_4$; $(\mathrm{p}+a_2)x_7 = k_2 u$

Table 2: Possible descriptions of system (9)-(10).

$(\theta_0\theta_1)$	*Internal Behaviour*
(11)	$\boxed{(\mathrm{p}+a_1)y = a_1 u}$ $(\mathrm{p}+a_1)x_2 = k_1(1-a_1)\left(x_4 + k_3 x_5\right)$ $(\mathrm{p}+1)x_4 = 0$; $(\mathrm{p}+1)x_5 = k_2 u$
(10)	$\boxed{(\mathrm{p}+\alpha)(\mathrm{p}+a_1)y = (\gamma\alpha\mathrm{p}+1)\alpha a_1 u}$ $(\mathrm{p}+a_1)x_2 = k_1(1-a_1)\left(x_4 + \gamma\alpha x_5\right)$ $(\mathrm{p}+1)x_4 = (1-\gamma\alpha)x_5$; $(\mathrm{p}+\alpha)x_5 = k_2 u$
(01)	$\boxed{(\mathrm{p}+a_2)(\mathrm{p}+a_1)y = a_2 a_1 u}$ $(\mathrm{p}+a_2)x_4 = k_3(1-a_2)x_5$ $(\mathrm{p}+a_1)x_2 = x_4 + k_3 x_5$; $(\mathrm{p}+1)x_5 = k_2 u$
(00)	$\boxed{(\mathrm{p}+\alpha)(\mathrm{p}+a_2)(\mathrm{p}+a_1)y = (\gamma\alpha\mathrm{p}+1)\alpha a_2 a_1 u}$ $(\mathrm{p}+a_2)x_4 = (1-a_2\gamma\alpha)x_5$ $(\mathrm{p}+a_1)x_2 = x_4 + \gamma\alpha x_5$; $(\mathrm{p}+\alpha)x_5 = k_2 u$

Table 3: Possible descriptions of system (11)-(12).

$$\left\{ \begin{array}{l} s_1 = \omega_o\rho + \omega_o\sqrt{\rho^2-1} \;\; ; \;\; s_2 = \omega_o\rho - \omega_o\sqrt{\rho^2-1} \\ \alpha = 0.4 \;\; ; \;\; a = -1.09 \;\; ; \;\; \omega_o = 4.7 \;\; ; \;\; \rho = 1.003 \end{array} \right.$$

and the initial conditions are:

$$y_1(0) = \bar{y}_0 \; ; \; \dot{y}_1(0) = \bar{y}_0(1-s_1) + \alpha\bar{k}u(0)$$
$$\ddot{y}_1(0) = \bar{y}_0(s_1^2 - a - s_1) + (1 + (1 - as_1)\alpha - \alpha^2)\bar{k}u(0)$$
$$y_2(t') = y_1(t'_-) \; ; \; \dot{y}_2(t') = \dot{y}_1(t'_-)$$
$$y_3(t") = y_2(t"_-) \; ; \; \dot{y}_3(t') = (\omega_o/(2\rho))\left(1 - y_2(t"_-)\right)$$

where: $\bar{k} = |\alpha|\,|a|\,|s_1|$ and $u(0) = 1$. From the previous Section the associated implicit description of (14) is:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \dot{x} = \begin{bmatrix} -\bar{s}_1 & -s_1 & \frac{1}{|\alpha|}-\alpha & 1 & \alpha & \alpha & 0 & 0 & 0 \\ a-1 & 0 & -\bar{a} & -a & 1-a\alpha & 1-a\alpha & 0 & 0 & 0 \\ 0 & 0 & \alpha-1 & 0 & -\alpha & -\alpha & \frac{1}{|s_2|} & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & \frac{1}{|s_2|}-1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & s_2-1 & -s_2 & 0 \end{bmatrix} x$$
$$+ \begin{bmatrix} 0 & 0 & 0 & 0 & (\omega_o^2 \cdot |\alpha| \cdot |a|) \end{bmatrix}^T u$$
$$y = \begin{bmatrix} |a|^{-1} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} x$$

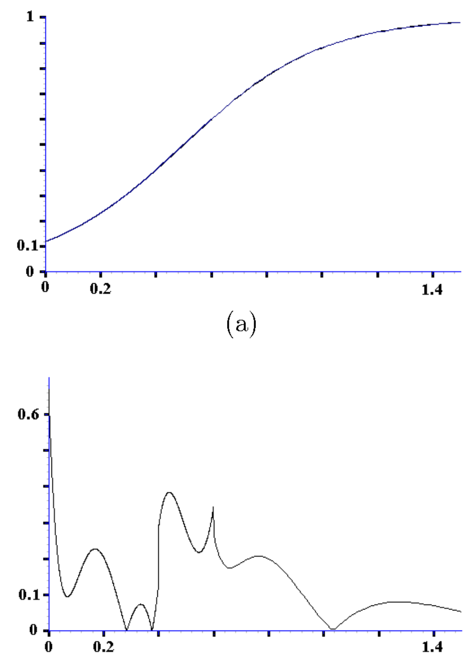
(15)

where: $\bar{s}_1 = |a|^{-1}(s_1 - 1) + 1$ and $\bar{a} = |\alpha|^{-1}(a-1) + (1 - a\alpha)$. Its associated algebraic restriction is:

$$0 = \begin{bmatrix} 1 & 0 & -\theta_0\bar{\alpha} & -\theta_0 & \theta_0\alpha & \theta_0\alpha & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -\theta_0 & -\theta_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -\frac{\theta_1}{|s_2|} & \theta_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -\theta_2 \end{bmatrix} x$$

(16)

where: $\bar{\alpha} = 1/|\alpha| - \alpha$. We show in Figure 1 some Simnon simulation results of $\bar{y}(t)$ and the step response of (15)-(16), with $(\theta_0\theta_1\theta_2) = (011)$ if $t < t'$, $(\theta_0\theta_1\theta_2) = (101)$ if $t' \leq t \leq t"$, and $(\theta_0\theta_1\theta_2) = (110)$ if $t > t"$; we also show the relative error $100|y(t) - \bar{y}(t)|/\bar{y}(t)$.[1] We can verify that system (15) satisfy (since $\det\begin{vmatrix} 1 & \alpha \\ a & (1-\alpha) \end{vmatrix} = 1$): $\mathrm{Im}\, E = \underline{\mathcal{X}}$, $E\mathrm{Ker}\, C = \underline{\mathcal{X}}$, $\dim \mathrm{Ker}\, E = \dim(\mathrm{Ker}\, C \cap E^{-1}\mathrm{Im}\, B)$, and $\mathrm{Im}\, A + \mathrm{Im}\, B = \underline{\mathcal{X}}$. Other words saying, system (15) has the output dynamics assignment property.



(a)

(b)

Figure 1: Simnon simulation results. (a) $\bar{y}(t)$, y(t); (b) relative error $100|y(t) - \bar{y}(t)|/\bar{y}(t)$.

4 Concluding Remarks

In this paper we have introduced the ladder systems as a tool for synthesizing Implicit Flat Systems, with the output dynamics assignment property, which are able to describe linear systems with a variable internal structure.

With this paper we complement our previous results [5]–[8], [10]. We are specially interested in applying these kinds of systems to agronomical systems.

[1] Special care have to paid in the simulation with respect to i.c. $t = 0$, $t = t'$, and $t = t"$ in order to avoid undesired jumpings.

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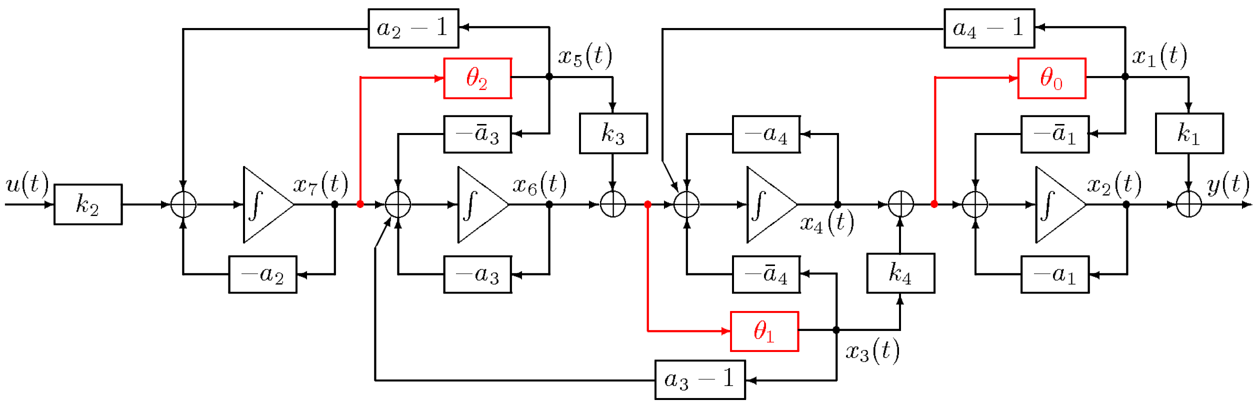
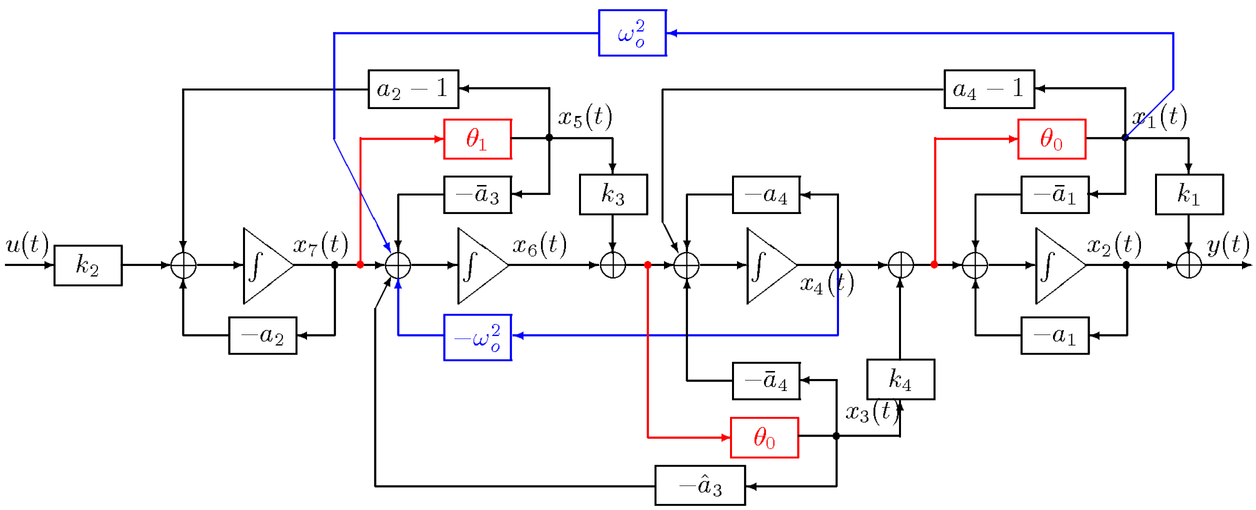
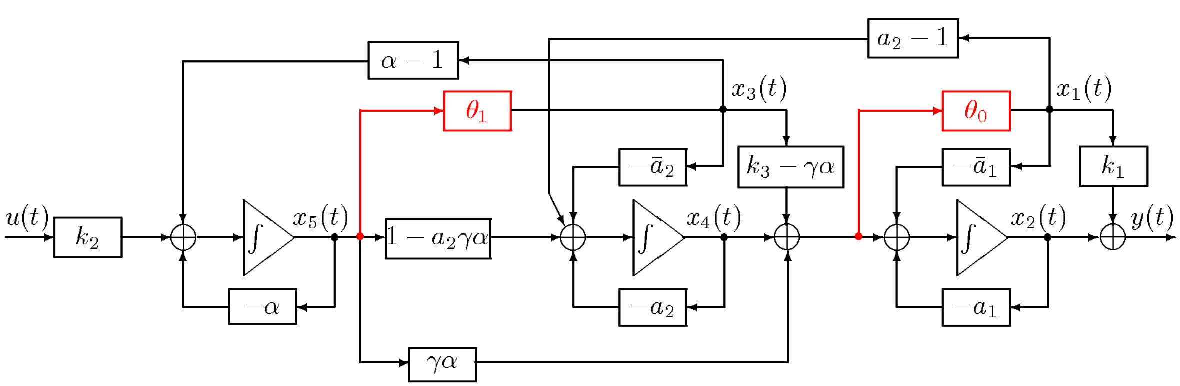


(a) Block Diagram of system (7)-(8).



(b) Block Diagram of system (9)-(10)



(c) Block Diagram of system (11)-(12).

Figure 2: Block diagrams of the ladder systems shown in subsection 2.1.