Stabilization of Linear Systems with Input Constraints[1]

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Abstract

We consider the control of an unstable LTI system with a constraint on the control input. We show that for every compact subset of the null-controllable region of the system, we can constructively design a nonlinear state feedback controller which ensures the internal exponential stability of the closed loop system, i.e., the state and control signals go to zero exponentially, for every initial condition in this subset. Two controllers are explicitly constructed: one has the property of being continuous and homogeneous, the other one can be considered as a one step ahead model predictive control (MPC) scheme with a special cost function.

1 Introduction

Control of systems with input and state constraints has a strong practical motivation and is currently an active research area, see recent survey paper [3], collection of articles [1, 20] and monographs [11] and [18]. In this paper, we consider the state feedback stabilization of a general unstable LTI system with a constraint on the control input. We aim to design a controller which makes the closed loop domain of attraction as large as possible. If possible, this controller should have such desirable properties as simplicity and continuity.

In the case of semi-stable LTI systems, i.e., continuous time systems with all poles in the closed left half of the complex plane and discrete time systems with all poles in the closed unit disk, it was shown that nonlinear controllers could be designed to make the closed loop system globally stable, e.g. see [19]; it was shown that saturated linear controllers could be designed to make any prescribed bounded set attractive e.g. see [12, 13, 14, 17]. The input constrained stabilization of general unstable LTI systems has also been studied before. For such a system, the closed loop domain of attraction has to be a subset of the null-controllable region, i.e., the set of states that can be steered to the origin by a constrained open loop control. One group of such works [7, 2] aimed at finding a simple saturated linear controller so that the domain of attraction is reasonably large and the performance is reasonably good, but in general the achievable domain of attraction using linear saturated feedback might have quite a distance from the domain of attraction (except in the case of planar systems as shown in [10]). It was recognized in [6, 4] that by using nonlinear control it is possible to make any prescribed compact subset of the null-controllable region attractive. However, the controllers developed in these works and other related works are based on online linear programing and they have the tendency that the dimension of the linear programming problems involved grow exponentially as the domain of attraction required approaches the null controllable region.

In this paper, two controllers are obtained. The first controller can be considered as an infinite horizon model predictive control and it has the properties of being homogeneous and continuous. The second controller can be considered as a one step ahead model predictive control scheme with a special cost function. Neither controller has the tendency to lead to online mathematical programming of high dimension when the domain of attraction approaches the null controllable region.

The notation used in this paper is standard. The Hölder p-norm in $\mathbb{R}^n$ is denoted by $|\cdot|_p$. ℓ_∞^m denotes the standard $\mathbb{R}^m$-valued sequence space with norm $\|\cdot\|_\infty$. ℓ_0^m denotes the space of all $\mathbb{R}^m$-valued sequences with only finite many nonzero terms. $\ell_1^{p\times m}$ denotes the $\mathbb{R}^{p\times m}$-valued sequence space with norm $\|\cdot\|_1$. Let $\mathcal{S}$ be a subset of a normed linear space. Then its closure is denoted by $\overline{\mathcal{S}}$. If $\mathcal{S}$ is bounded, absorbing and convex, its Minkowski functional is denoted by $\rho_{\mathcal{S}}$.

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2 Preliminaries

Consider the plant

$$x(k+1) = Ax(k) + Bu(k) \tag{1}$$

where $x(k) \in \mathbb{R}^n$ and $u(k) \in \mathbb{R}^m$. A control signal u is said to be admissible if $\|u\|_\infty \le 1$.

Definition 1 A state $x_0 \in \mathbb{R}^n$ of system (1) is said to be *null-controllable* if there exists an admissible control so that the time response x of (1) with initial condition $x(0) = x_0$ satisfies $x(K) = 0$ for some $K > 0$. The set of all null-controllable states of the system (1) is denoted by $\mathcal{C}(A, B)$ and is called the null-controllable region of the system (1) or the pair (A, B).

By carrying out a similarity transformation if necessary, we may as well assume that A and B are of the form

$$\begin{aligned} A &= \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \in \mathbb{R}^{(n_1+n_2)\times(n_1+n_2)}, \\ B &= \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \in \mathbb{R}^{(n_1+n_2)\times m} \end{aligned} \tag{2}$$

with A_1 antistable, i.e., having all eigenvalues outside of the unit circle, and A_2 semistable, i.e., having all eigenvalues on or inside the unit circle.

The following is well-known.

Lemma 1 ([8]) *Assume that (A, B) is controllable in the usual sense. Then $\mathcal{C}(A_1, B_1)$ is an open, bounded, absorbing, convex, and balanced subset of $\mathbb{R}^{n_1}$; $\mathcal{C}(A_2, B_2) = \mathbb{R}^{n_2}$; and $\mathcal{C}(A, B) = \mathcal{C}(A_1, B_1) \oplus \mathcal{C}(A_2, B_2)$.*

In the rest of this section we assume that A is antistable.

Lemma 2 *Assume that A is antistable. Then*

$$\mathcal{C}(A,B) = \left\{ -\sum_{l=0}^{\infty} A^{-l-1}Bv(l) : v \in \ell_0^m, \|v\|_\infty \le 1 \right\} \tag{3}$$

$$\overline{\mathcal{C}(A,B)} = \left\{ -\sum_{l=0}^{\infty} A^{-l-1}Bv(l) : v \in \ell_\infty^m, \|v\|_\infty \le 1 \right\} \tag{4}$$

Proof: Let $x_0 \in \mathcal{C}(A, B)$. Then

$$0 = A^K x_0 + \sum_{l=0}^{K-1} A^{K-l-1}Bv(l) \tag{5}$$

for some $\|v\|_\infty \le 1$ and $K > 0$. Equation (5) is equivalent to

$$x_0 = -\sum_{l=0}^{K-1} A^{-l-1}Bv(l).$$

Hence

$$\mathcal{C}(A,B) = \left\{ -\sum_{l=0}^{\infty} A^{-l-1}Bv(l) : v \in \ell_0^m, \|v\|_\infty \le 1 \right\}.$$

Let

$$x_0 \in \left\{ -\sum_{l=0}^{\infty} A^{-l-1}Bv(l) : v \in \ell_\infty^m, \|v\|_\infty \le 1 \right\}.$$

Then

$$x_0 = -\sum_{l=0}^{\infty} A^{-l-1}Bv(l)$$

for some $v \in \ell_\infty^m$ with $\|v\|_\infty \le 1$. Let

$$x_K = -\sum_{l=0}^{K-1} A^{-l-1}Bv(l).$$

Then $x_K \in \mathcal{C}(A, B)$ and $\lim_{K\to\infty} x_K = x_0$. This shows that $x_0 \in \overline{\mathcal{C}(A,B)}$. It remains to show that

$$\left\{ -\sum_{l=0}^{\infty} A^{-l-1}Bv(l) : v \in \ell_\infty^m, \|v\|_\infty \le 1 \right\}$$

is closed. This is equivalent to the fact that its intersection with any one-dimensional subspace is a closed interval. This intersection is given by

$$\left\{ -\xi' \sum_{l=0}^{\infty} A^{-l-1}Bv(l) : v \in \ell_\infty^m, \|v\|_\infty \le 1 \right\}.$$

It is a standard result that this is a closed interval given by

$$\left[-\sum_{l=0}^{\infty} \|\xi' A^{-l-1}B\|_1, \sum_{l=0}^{\infty} \|\xi' A^{-l-1}B\|_1 \right].$$

□

Lemma 2 implies that it is better to work with $\overline{\mathcal{C}(A,B)}$ instead of $\mathcal{C}(A, B)$. $\overline{\mathcal{C}(A,B)}$ is a projection of the unit ball of ℓ_∞^m, so it can be considered as a generalized zonotope [21]. Even if we approximate it by taking a finite sum in (4), we get a zonotope whose numbers of vertices and facets depend combinatorially on the number of terms taken, except in the case when $n \le 2$. Hence it is not a numerically feasible problem to describe, even approximately, $\overline{\mathcal{C}(A,B)}$ in terms of its vertices or facets when $n > 2$. A more feasible problem is as follows: given $x_0 \in \mathbb{R}^n$, find out computationally if $x_0 \in \mathcal{C}(A, B)$, or more generally compute $\rho_{\mathcal{C}(A,B)}(x_0)$. In principle, this can be formulated straightforwardly as a linear programming problem, see [4] for example. However, the following lemma suggests that it may have numerical advantages to work on the dual problem.

Lemma 3 ([9])

$$\begin{aligned} &\rho_{\mathcal{C}(A,B)}(x_0) \\ &\quad = \max_{\xi\in\mathbb{R}^n} \{\xi' x_0 : \|\{-\xi' A^{-k-1}B\}_{k=0}^{\infty}\|_1 \le 1\} && (6) \\ &\quad = \left[\min_{\xi \in x_0^\perp} \left\| \left\{ -\left(\frac{x_0}{x_0' x_0} + \xi \right)' A^{-k-1}B \right\}_{k=0}^{\infty} \right\|_1 \right]^{-1} && (7) \end{aligned}$$

The computation of $\rho_{\mathcal{C}(A,B)}(x_0)$ using (6) requires an n-dimensional constrained convex programming whereas that using (7) requires an $n-1$ dimensional unconstrained convex programming. Intuition suggests and numerical experience shows that (7) is more convenient than (6). A result somewhat related to Lemma 3 was reported in [15].

We can see from Lemma 2 that for each $x_0 \in \mathbb{R}^n$, there exists a sequence $v \in \ell_\infty^m$ with $\|v\|_\infty = \rho_{\mathcal{C}(A,B)}(x_0)$ such that

$$x_0 = -\sum_{l=0}^{\infty} A^{-l-1} B v(l).$$

Lemma 4 *Each such v has the property that $|v(k)|_\infty = \rho_{\mathcal{C}(A,B)}(x_0)$ for infinitely many $k \geq 0$.*

Proof: Following (6), let $\rho_{\mathcal{C}(A,B)}(x_0) = \xi' x_0$ for some ξ with

$$\|\{-\xi' A^{-k-1} B\}_{k=0}^{\infty}\|_1 = 1.$$

Then

$$\begin{aligned} \rho_{\mathcal{C}(A,B)}(x_0) &= \xi' x_0 = -\sum_{l=0}^{\infty} \xi' A^{-l-1} B v(l) \\ &\leq \|\{-\xi' A^{-k-1} B\}_{k=0}^{\infty}\|_1 \|v\|_\infty \\ &= \rho_{\mathcal{C}(A,B)}(x_0). \end{aligned}$$

This shows that $\{-\xi' A^{-k-1} B\}_{k=0}^{\infty}$ and v are aligned. This is possible only when $|v(k)|_\infty = \rho_{\mathcal{C}(A,B)}(x_0)$ for all k with $-\xi' A^{-k-1} B \neq 0$. Since (A,B) is controllable, it is an elementary exercise to show that $-\xi' A^{-k-1} B$ cannot be zero for n consecutive integer values of k. □

3 Development

Our problem is as follows: for every compact subset $\mathcal{S}$ of $\mathcal{C}(A,B)$, design a state feedback controller which, for every initial condition in $\mathcal{S}$, results in the state and control going exponentially to zero.

We will abbreviate $\mathcal{C}(A,B)$ by $\mathcal{C}$ in this section. We first assume that A has all eigenvalues outside or on the unit circle; the extension to the general case is easy and will be carried out in the end of this section.

For $\lambda \in (0,1)$, define $\mathcal{C}_\lambda = \mathcal{C}(\lambda^{-1}A, \lambda^{-1}B)$. Since $\lambda^{-1}A$ is antistable, it follows from Lemma 1 that $\mathcal{C}_\lambda$ is open, bounded, absorbing, convex and balanced. So $\rho_{\mathcal{C}_\lambda}$ is a norm. Furthermore $\mathcal{C}_\lambda$ is monotonically increasing as $\lambda \to 1$ and

$$\bigcup_{\lambda \in (0,1)} \mathcal{C}_\lambda = \mathcal{C}.$$

This implies that for each compact subset $\mathcal{S}$ of $\mathcal{C}$, there exists $\lambda \in (0,1)$ such that $\mathcal{S} \subset \mathcal{C}_\lambda$. Since

$$\overline{\mathcal{C}_\lambda} := \left\{ -\sum_{l=1}^{\infty} \lambda^l A^{-l-1} B v(l) : \|v\|_\infty \leq 1 \right\},$$

for $\zeta \in \overline{\mathcal{C}_\lambda}$, we have $\rho_{\mathcal{C}_\lambda}(\zeta) \leq 1$ and there exists a sequence $v \in \ell_\infty^m$ with $\|v\|_\infty = \rho_{\mathcal{C}_\lambda}(\zeta)$ such that

$$\zeta = -\sum_{l=0}^{\infty} \lambda^l A^{-l-1} B v(l).$$

Define the feedback control map h: $\overline{\mathcal{C}_\lambda} \to \mathbb{R}^m$ by $h(\zeta) = v(0)$. Since v given above is not uniquely determined, the function h for now is not well defined. Such nonuniqueness can be exploited and removed by further restricting $v(0)$ to meet other requirements, for example by choosing $h(\zeta)$ to be such a $v(0)$ with the smallest 2-norm. We can even leave the choice of $h(\zeta)$ to some agent which is not part of the controller. In this case the controller is unconventional and is said to be nondeterministic. At this stage let us assume that $h(\zeta)$ is arbitrarily chosen among all possible $v(0)$.

Theorem 1 *For every initial condition $x_0 \in \overline{\mathcal{C}_\lambda}$, the solution of (1) under the nonlinear state feedback controller $u(k) = h[x(k)]$ satisfies $x(k) \in \lambda^k \rho_{\mathcal{C}_\lambda}(x_0)\overline{\mathcal{C}_\lambda}$ or equivalently $\rho_{\mathcal{C}_\lambda}[x(k)] \leq \lambda^k \rho_{\mathcal{C}_\lambda}(x_0)$, and the control signal satisfies $|u(k)|_\infty \leq \lambda^k \rho_{\mathcal{C}_\lambda}(x_0)$.*

Proof: This theorem can be proved by induction. Clearly, the theorem is true for $k = 0$. Now assume that $x(k) \in \lambda^k \rho_{\mathcal{C}_\lambda}(x_0)\overline{\mathcal{C}_\lambda}$. Then $\rho_{\mathcal{C}_\lambda}[x(k)] \leq \lambda^k \rho_{\mathcal{C}_\lambda}(x_0)$ and there exists $v_k \in \ell_\infty^m$ with $\|v_k\|_\infty = \rho_{\mathcal{C}_\lambda}[x(k)]$ and $v_k(0) = h[x(k)]$ such that

$$x(k) = -\sum_{l=0}^{\infty} \lambda^l A^{-l-1} B v_k(l).$$

Hence

$$\begin{aligned} x(k+1) &= Ax(k) + Bh[x(k)] \\ &= -\sum_{l=0}^{\infty} \lambda^l A^{-l} B v_k(l) + Bv_k(0) \\ &= -\sum_{l=1}^{\infty} \lambda^l A^{-l} B v_k(l) \\ &= -\lambda \sum_{l=0}^{\infty} \lambda^l A^{-l-1} B v_k(l+1) \\ &\in -\lambda \rho_{\mathcal{C}_\lambda}[x(k)]\overline{\mathcal{C}_\lambda} \\ &\subset \lambda^{k+1} \rho_{\mathcal{C}_\lambda}(x_0)\overline{\mathcal{C}_\lambda}. \end{aligned}$$

The theorem then follows by induction. □

This theorem shows that there is a uniform bound on the rate of convergence of the state and control for all initial conditions in $\overline{\mathcal{C}_\lambda}$, which is given by λ^k. Hence the number λ gives a tradeoff between the rate of convergence of the closed loop response and the domain of attraction.

Conceptually, the controller is formed in the following way: at time k find an admissible control sequence v_k such that it steers $x(k)$ to the origin, then we just implement the first term in this sequence, i.e., let

$u(k) = v_k(0)$. This is exactly the idea used in the popular receding horizon or model predictive control (MPC) scheme [5]. However, an important difference between the scheme above and the traditional MPC is that the admissible control sequence v_k here is assumed to have infinite horizon, whereas the traditional MPC uses only a finite horizon admissible control sequence.

At each step, the controller has to compute $\rho_{\mathcal{C}_\lambda}(\zeta)$ and $h(\zeta)$ for some $\zeta \in \overline{\mathcal{C}_\lambda}$. Here $h(\zeta)$ is defined as the first term of any sequence $v \in \ell_\infty^m$ with the property that $\|v\|_\infty \le \rho_{\mathcal{C}_\lambda}(\zeta)$ and

$$\zeta = -\sum_{l=0}^{\infty} \lambda^l A^{-l-1} B v(l). \tag{8}$$

It follows from Lemma 3 that

$$\rho_{\mathcal{C}_\lambda}(\zeta) = \left[\min_{\xi \in \zeta^\perp} \| \left\{ -\left(\frac{\zeta}{\zeta'\zeta} + \xi \right)' \lambda^l A^{-l-1} B \right\}_{l=0}^{\infty} \|_1 \right]^{-1}. \tag{9}$$

There are a couple of ways to compute $h(\zeta)$. The first way is actually to carry out the Hahn-Banach extension. Consider v as a linear functional on $\ell_1^{m\times 1}$. The constraint (8) specifies the linear functional on the space spanned by the rows of

$$M = \left\{ -A^{-1}B, -\lambda A^{-2}B, -\lambda^2 A^{-3}B, \dots \right\}.$$

This restricted linear functional has induced norm $\rho_{\mathcal{C}_\lambda}(\zeta)$. The required $h(\zeta)$ is then the first m coordinates of any extension of v to the whole $\ell_1^{m\times 1}$ space. Hence it can be computed by extending v to the space spanned by the rows of M and rows of

$$E = \{I, 0, 0, \dots\},$$

without increasing its norm. The constructive proof of the Hahn-Banach theorem [16, pp. 187–188] provides a way to carry out this extension. Denote the i-th row of E by e_i. Assume that $h_j(\zeta), 1 \le j \le i-1$, have been computed. Define

$$\begin{aligned} \underline{h}_i(\zeta) &= \sup_{\substack{\phi\in\mathbb{R}^n \\ \alpha_j\in\mathbb{R}, j<i}} \Bigg[-\rho_{\mathcal{C}_\lambda}(\zeta)\|\phi' M + \sum_{j=1}^{i-1} \alpha_j e_j - e_i\|_1 \\ &\qquad + \phi'\zeta + \sum_{j=1}^{i-1} \alpha_j h_j(\zeta) \Bigg] \end{aligned} \tag{10}$$

$$\begin{aligned} \overline{h}_i(\zeta) &= \inf_{\substack{\phi\in\mathbb{R}^n \\ \alpha_j\in\mathbb{R}, j<i}} \Bigg[\rho_{\mathcal{C}_\lambda}(\zeta)\|\phi' M + \sum_{j=1}^{i-1} \alpha_j e_j + e_i\|_1 \\ &\qquad - \phi'\zeta - \sum_{j=1}^{i-1} \alpha_j h_j(\zeta) \Bigg]. \end{aligned} \tag{11}$$

Then $\underline{h}_i(\zeta) \le \overline{h}_i(\zeta)$ and $h_i(\zeta)$ can be computed as

$$\underline{h}_i(\zeta) \le h_i(\zeta) \le \overline{h}_i(\zeta).$$

In a practical implementation of the controller, the sequence M in (10) and (11) has to be truncated, i.e., M has to be approximated by

$$\left\{ -A^{-1}B, -\lambda A^{-2}B, \dots, -\lambda^L A^{-L-1}B, 0, \dots \right\}$$

for a large enough L. In this case, the expression inside the square brackets in the right hand side of (10), considered as a function of ϕ and α_j, is a continuous polyhedral function. Hence the supremum in (10) is actually a maximum. Similarly, the infimum in (11) is actually a minimum. Hence, the computation of $\underline{h}_i(\zeta)$ and $\overline{h}_i(\zeta)$ are $m+i-1$ dimensional convex optimization problems. If we simply choose $h_i(\zeta)$ to be either of its bounds, then we need only compute one bound.

The feedback control function h is obviously nonlinear and there is certain nonuniqueness in its design. Observe that $\underline{h}_i$ and $\overline{h}_i$ are homogeneous in the sense that

$$\underline{h}_i(\alpha\zeta) = \alpha\underline{h}_i(\zeta) \quad \text{and} \quad \overline{h}_i(\alpha\zeta) = \alpha\overline{h}_i(\zeta)$$

for $\alpha \ge 0$. Since the expression inside the square brackets in the right hand side of (11), after M being replaced by its truncated version and considered as a function of ϕ and α_j, is a continuous polyhedral function, the minimum is attained in a compact set, namely the set of all points corresponding to the extreme points of the epigraph of the function, which depends only on M. This implies that $\overline{h}_i$ is a continuous function. Similarly $\underline{h}_i$ is a continuous function. A reasonable choice of $h(\zeta)$ will also keep the homogeneity and continuity properties. Furthermore, since $\underline{h}_i(-\zeta) = -\overline{h}_i(\zeta)$, if we choose

$$h_i(\zeta) = \frac{\underline{h}_i(\zeta) + \overline{h}_i(\zeta)}{2}$$

or

$$h_i(\zeta) = \arg \min_{\eta\in[\underline{h}_i(\zeta), \overline{h}_i(\zeta)]} |\eta|,$$

then we will have $h(\alpha\xi) = \alpha h(\xi)$ for all $\alpha \in \mathbb{R}$.

Although the function h computed in the above way has the nice homogeneity and continuity properties, experience show that the optimization problems in (10) and (11) have very bad numerical properties. Continuous research is underway to address the computational issues of the optimization problems. An alternative way to compute h is given as follows. Let

$$\mu^* = \arg \min_{|\mu|_\infty \le \rho_{\mathcal{C}_\lambda}(\zeta)} \rho_{\mathcal{C}_\lambda}(A\zeta + B\mu). \tag{12}$$

Theorem 2 *For each $\zeta \in \overline{\mathcal{C}_\lambda}$, there exists $v \in \ell_\infty^m$ with $\|v\|_\infty = \rho_{\mathcal{C}_\lambda}(\zeta)$ and $v(0) = \mu^*$ such that*

$$\zeta = -\sum_{l=0}^{\infty} \lambda^l A^{-l-1} B v(l).$$

Proof: From Theorem 1, we know that

$$\rho_{\mathcal{C}_\lambda}(A\zeta + B\mu^*) \le \lambda\rho_{\mathcal{C}_\lambda}(\zeta).$$

Suppose that

$$\rho_{\mathcal{C}_\lambda}(A\zeta + B\mu^*) < \lambda \rho_{\mathcal{C}_\lambda}(\zeta).$$

Then there exists $\tilde{v} \in \ell_\infty^m$ with $\|\tilde{v}\|_\infty = \rho_{\mathcal{C}_\lambda}(A\zeta + B\mu^*)$ such that

$$A\zeta + B\mu^* = -\sum_{l=0}^{\infty} \lambda^l A^{-l-1} B\tilde{v}(l).$$

Then

$$\begin{aligned} \zeta &= -A^{-1}\sum_{l=0}^{\infty} \lambda^l A^{-l-1} B\tilde{v}(l) - A^{-1}B\mu^* \\ &= -\sum_{l=1}^{\infty} \lambda^l A^{-l-1} B\frac{\tilde{v}(l-1)}{\lambda} - A^{-1}B\mu^*. \end{aligned}$$

Let

$$v = \left\{\mu^*, \frac{\tilde{v}(0)}{\lambda}, \frac{\tilde{v}(1)}{\lambda}, \dots\right\}$$

Then

$$\zeta = -\sum_{l=0}^{\infty} \lambda^l A^{-l-1} Bv(l)$$

and $|v(0)|_\infty \le \rho_{\mathcal{C}_\lambda}(\zeta)$ but $|v(l)|_\infty < \rho_{\mathcal{C}_\lambda}(\zeta)$ for all $l > 0$. However Lemma 4 says that $|v(l)|_\infty = \rho_{\mathcal{C}_\lambda}(\zeta)$ must hold for infinitely many l, which is a contradiction. Hence we have shown that

$$\rho_{\mathcal{C}_\lambda}(A\zeta + B\mu^*) = \lambda \rho_{\mathcal{C}_\lambda}(\zeta).$$

Then v can be constructed using the same way as above. $\square$

This theorem shows that $h(\zeta) = \mu^*$ defines a feedback controller needed by Theorem 1. This controller also has an MPC interpretation: it is obtained from a one-step MPC with the cost function being the Minkowski functional of the state at the next step. This theorem effectively shows that in the particular case studied in this paper, an infinite horizon MPC is equivalent to a one-step horizon MPC.

Now let us turn to the general case when A has both stable and antistable eigenvalues. In this case, we can assume, without loss of generality, that A and B have the following form

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \tag{13}$$

with A_1 having all eigenvalues on or outside the unit circle, and A_2 having all eigenvalues inside the unit circle; notice that (13) and (2) are partitioned differently. For the subsystem

$$x_1(k+1) = A_1 x_1(k) + B_1 u(k),$$

the method in the last section can be used to design a controller

$$u(k) = h[x_1(k)]$$

to ensure that $x_1(k)$ goes to the origin exponentially for all initial conditions $x_1(0)$ in a prescribed compact set $\mathcal{S} \subset \mathcal{C}(A_1, B_1)$. If we apply the same controller to the whole system, then $x_1(k)$ goes to the origin exponentially for all initial conditions $x_1(0)$ in a prescribed compact set $\mathcal{S} \subset \mathcal{C}(A_1, B_1)$ and x_2 satisfies

$$x_2(k+1) = A_2 x_2(k) + B_2 h[x_1(k)], \quad x_2(0) = x_{20}.$$

Now since A_2 is stable and $\|h[x_1(k)]\|_\infty \le \lambda^k \rho_{\mathcal{C}_\lambda(A_1,B_1)}(x_0)$, it follows that $x_2(k)$ also converges to the origin exponentially. This shows that for every initial condition

$$x_{10} \in \mathcal{S}, \quad x_{20} \in \mathbb{R}^{n_2},$$

the nonlinear feedback control $u(k) = h[x_1(k)]$ makes $x(k)$ converge to the origin exponentially.

4 An Example

The following system is a transformed version of a system considered in [15]:

$$x(k+1) = \begin{bmatrix} 2 & 2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 0.5 \end{bmatrix} x(k) + \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} u(k)$$

This system has poles at $2 \pm j2$ and 0.5. Its antistable subsystem is given by

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \end{bmatrix} u(k).$$

Figure 1 shows the null-controllable regions

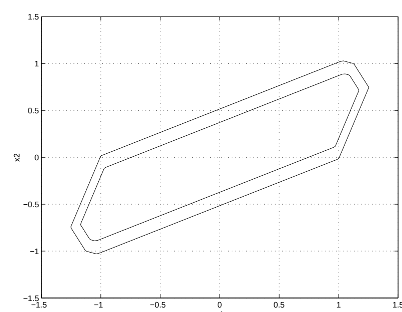

Figure 1: Null-controllable regions of the antistable subsystem

$\mathcal{C}\left(\begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \end{bmatrix}\right)$ (the region inside the outer curve) and $\mathcal{C}_{0.8}\left(\begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \end{bmatrix}\right)$ (the region inside the inner curve). The controller given in terms of Theorem 1 and Theorem 2 can then guarantee exponential internal stability for all initial conditions satisfying

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \in \mathcal{C}_{0.8}\left(\begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \end{bmatrix}\right)$$

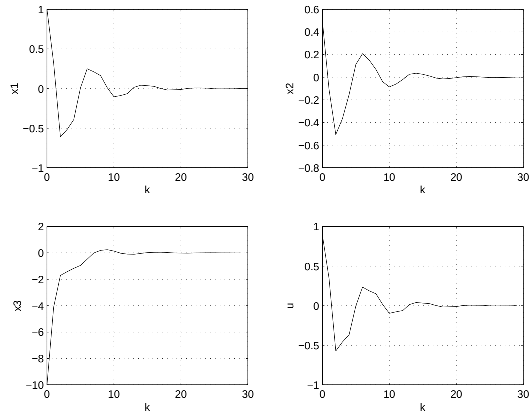

Figure 2: A typical closed loop response

and $x_3(0) \in \mathbb{R}$. The rate of convergence is bounded by 0.8^k for all such initial conditions. Figure 2 shows the response of the system for the initial condition

$$\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0.5 \\ -10 \end{bmatrix}.$$

5 Concluding Discussions

In the paper we constructed a state feedback control law satisfying a prespecified constraint that makes the state and control variables converge exponentially to zero if the initial state lies in any prespecified compact subset S of the null-controllable region $\mathcal{C}(A, B)$.

An important implication of this paper, in the perspective of the popular model predictive control, is that infinite horizon open loop control sequences can be assumed at each step for the optimization purpose. This paper provides a way to turn the resulting infinite dimensional optimization (or feasibility) problem into an equivalent finite dimensional optimization problem by using the Hahn-Banach theorem. This has the potential to overcome the difficulty of the traditional model predictive control in addressing the stability issues. The development along this line is currently being undertaken.

References

[1] D.S. Berstein and A.N. Michel (Editors), *Special Issue on Saturating Actuators, Int. Journal of Robust and Nonlinear Control*, vol. 5, 1995.

[2] G. Bitsoris and M. Vassilaki, "Constrained regulation of linear systems", *Automatica*, vol. 31, pp. 223–227, 1995.

[3] F. Blanchini, "Set invariance in control — a survey", *Automatica*, vol. 35, pp. 1747–1767, 1999.

[4] F. Blanchini, F. Mesquine, and S. Miani, "Constrained stabilization with an assigned initial condition set", *International Journal of Control*, vol. 62, pp. 601–617, 1995.

[5] E. F. Camacho and C. Bordons, *Model Predictive Control*, Springer, 1999.

[6] P. Gutman and M. Cwikel, "Admissible sets and feedback control for discrete-time linear dynamical systems with bounded controls and states", *IEEE transactions on Automatic Control*, vol. AC-31, pp. 373-376, 1986.

[7] P. Gutman and P. Hagander, "A new design of constrained controllers for linear systems", *IEEE transactions on Automatic Control*, vol. AC-30, pp. 22–33, 1985.

[8] O. Hájek, *Control Theory in the Plane*, Springer, 1991.

[9] T. Hu and L. Qiu, "Controllable regions of linear systems with bounded inputs", *Systems & Control Letters*, vol. 33, pp. 55–61, 1998.

[10] T. Hu, Z. Lin and L. Qiu, "Null controllability and stabilization of exponentially unstable linear systems with saturation actuators", submitted to *IEEE Transactions on Automatic Control*, 1998.

[11] Z. Lin, *Low Gain Feedback*, Springer-Verlag, 1999.

[12] Z. Lin and A. Saberi, "Semi-global exponential stabilization of linear systems subject to 'input saturation' via linear feedbacks", *Systems and Control Letters*, 21, pp. 225–239, 1993.

[13] Z. Lin and A. Saberi, "Semi-global exponential stabilization of linear discrete-time systems subject to input saturation via linear feedbacks", *Systems and Control Letters*, 24, pp. 125–132, 1995.

[14] Z. Lin and A. Saberi, "A semi-global low-and-high design for linear systems with input saturation: stabilization and disturbance rejection", *Int. J. of Robust and Nonlinear Control*, 5, pp. 381-398, 1995.

[15] D. G. Meyer, "On finding the minimum possible peak value of multiinput infinite-horizon steering controls", *IEEE Transactions on Automatic Control*, vol. 35, pp. 243–246, 1990.

[16] H. L. Royden, *Real Analysis*, McMillan, 1968.

[17] A. Saberi, Z. Lin and A. R. Teel, "Control of linear systems with saturating actuators", *IEEE Trans. Automat. Contr.* AC-41, pp. 368-378, 1996.

[18] A. Saberi, A. A. Stoorvogel and P. Sannuti, *Control of Linear Systems with Regulation and Input Constraints*, Springer-Verlag, 1999.

[19] H. J. Sussmann, E. D. Sontag, and Y. Yang, "A general result on the stabilization of linear systems using bounded controls", *IEEE Trans. Automat. Contr.* AC-39, pp. 2411-2425, 1994.

[20] S. Tarbouriech and G. Garcia (Editors), *Control of Uncertain Systems with Bounded Inputs*, Springer-Verlag, 1997.

[21] G. M. Ziegler, *Lectures on Polytopes*, Springer-Verlag, 1995.