Embedding for exponential observers of nonlinear systems

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Abstract

For nonlinear systems in $\mathbb{R}^n$ which admit an observability index m strictly larger than n, we show that under the existence of an injective immersion onto a manifold of $\mathbb{R}^m$, one can build an exponential observer. The main point concerns the determination of a Lipschitz extension to $\mathbb{R}^m$ of the dynamics, which is defined only on a manifold of dimension n. We propose some constructive tools and illustrate their utility on a simple biological model.

1 Introduction

Consider a dynamical system defined on an open connected subset $\mathcal{M}$ of $\mathbb{R}^n$, with a scalar output $y \in \mathbb{R}$:

$$(\mathcal{S}) : \left\{ \begin{array}{rcl} \dot{x} & = & f(x) \\ y & = & h(x) \end{array} \right.$$

where $f : \mathcal{M} \mapsto \mathbb{R}^n$ and $h : \mathcal{M} \mapsto \mathbb{R}$ are assumed to be smooth (*i.e.* C^r relatively to $\mathcal{M}$ with large enough r).
Definition : The *observability index* $i_{f,g}(\mathcal{M})$ of $(\mathcal{S})$ on $\mathcal{M}$ is the smallest integer m (if it exists) such that the map :

$$q_m : x \longrightarrow \left[\begin{array}{c} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{m-1} h(x) \end{array} \right] \qquad (1)$$

is injective from $\mathcal{M}$ onto $\mathcal{N} = q_m(\mathcal{M}) \subset \mathbb{R}^m$.
For conditions of existence of such a index in the analytic case, see the "ascending chain property" studied in [2].
The questions of observability and construction of observers for such nonlinear systems have been extensively studied in the literature. Nevertheless, the problem of explicit construction of exponential observers seems to have been addressed only for the case of observability index equal to n (see for instance [1]).
When the system admits a finite observability index m strictly larger than n (*i.e.* the knowledge of $y(t)$ and of at least its $m-1$ time derivatives allows to reconstruct $x(t)$), one may expect to embed the system in $\mathbb{R}^m$ and face again an observable system which observability index is equal to the dimension of the state. In this work, we address the question of the explicit construction of such an embedding (with the purpose of the synthesis of an exponential observer) which does not seem to have been studied in the literature, apart some very particular cases [3].

2 Existence of an exponential observer

Proposition 1 : If $(\mathcal{S})$ admits a finite observability index $m = i_{f,g}(\mathcal{M})$ and $dq_m(x)$ has rank n for any $x \in \mathcal{M}$, then for any (relatively) compact sub-manifold $\mathcal{K}$ of $\mathcal{M}$, positively invariant by the dynamics f, there exist maps $F : \mathbb{R}^m \mapsto \mathbb{R}^m$ and $l : \mathbb{R}^m \mapsto \mathbb{R}^n$ (globally) Lipschitz such that :

$$l(\xi) = q_m^{-1}(\xi), F(\xi) = \left[\begin{array}{c} \xi_2 \\ \vdots \\ \xi_m \\ L_f^m h(q_m^{-1}(\xi)) \end{array} \right], \ \forall \xi \in q_m(\mathcal{K})$$

and, for any $\beta > 0$, there exists θ large enough such that the dynamical system in $\mathbb{R}^m$:

$$(\mathcal{O}) : \left\{ \begin{array}{rcl} \widehat{x} & = & l(\widehat{\xi}) \\ \dot{\widehat{\xi}} & = & F(\widehat{\xi}) + S(\theta)^{-1} C'(y - \widehat{\xi}_1) \end{array} \right.$$

is a β-exponential observer for $(\mathcal{S})$ on $\mathcal{K}$, with $S(\theta)$ solution of the Riccati equation : $A'S(\theta) + S(\theta)A - CC' + \theta S(\theta) = 0$, where (A, C) is the Brunovsky observable pair of matrices in dimension m.
Sketch of proof : It consists in considering Lipschitz extensions of the maps $\psi = q_m^{-1}$ and $\varphi = L_f^m h$ which are defined and (relatively) Lipschitz only on the compact set $\mathcal{Q} = q_m(\mathcal{K})$ of $\mathbb{R}^m$. Then, we use the "high-gain" observer [1].

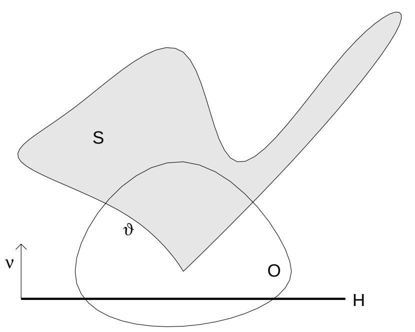
3 Construction of Lipschitz extensions

We consider the problem of the explicit construction of a Lipschitz extension of a function $\phi : \mathbb{R}^m \mapsto \overline{\mathbb{R}}$ which is Lipschitz relatively to a closed nonempty set S, with Lipschitz rank λ.
Lemma 1 (convex case) : If S is convex, then the function : $\widetilde{\phi}(x) = \phi \circ \pi_S(x), \quad \forall x \in \mathbb{R}^m$, where π_S is

the Euclidean projection on S, agrees with ϕ on S and is Lipschitz with rank λ on $\mathbb{R}^m$.

Lemma 2 (directional Lipschitz extension) : If S can be seen as the epigraph of a Lipschitz function on an open convex set $\mathcal{O}$ (with non empty intersection with S)[1], i.e. such that there exist $\nu \in \mathbb{R}^m$ with $|\nu| = 1$, $\overline{x} \in \mathcal{O}$ and a Lipschitz function ϑ with rank κ defined on the hyperplane $H = \{x \in \mathbb{R}^m \mid \; < x - \overline{x}/\nu >= 0\}$ such that :

$$S \cap \mathcal{O} = \{\, h + \zeta\nu \mid h \in H \text{ and } \zeta \in [\vartheta(h), +\infty) \,\} \cap \mathcal{O},$$



then, for any number γ, the function :

$$\widetilde{\phi}(x) = \left| \begin{array}{l} \phi \circ (\pi_H + (\vartheta \circ \pi_H)\nu)(x) + \gamma(\vartheta \circ \pi_H(x) - < x, \nu >), \\ \hfill x \in \mathcal{O} \setminus S \\ \phi(x), \quad x \in S \cap \mathcal{O} \end{array} \right.$$

is Lipschitz on $\mathcal{O}$ with rank $\lambda(1+\kappa)(1+|\gamma|)$.

Proposition 2 : If $K = \overline{\text{conv}}(S) \setminus S$ is a nonempty compact set that can be covered by a finite family of open convex sets $\mathcal{O}_i$, such that K can be seen as the epigraph of a Lipschitz function on each $\mathcal{O}_i$, then the function :

$$\widetilde{\phi}(x) = \left| \begin{array}{ll} \displaystyle\sum_{i=1}^{N} \alpha_i(x)\widetilde{\phi}_i(x) & x \in \overline{\text{conv}}(S) \setminus S \\ \phi(x) & x \in S \end{array} \right.$$

where $\{\alpha_i\}_{i=1}^n$ is an unity partition associated with the family $\{\mathcal{O}_i\}_{i=1}^n$ and $\widetilde{\phi}_i$ are Lipschitz extensions of ϕ on $\mathcal{O}_i$ for some parameters γ_i (see Lemma 2), is a Lipschitz extension of ϕ on $\overline{\text{conv}}(S)$. Then, Lemma 1 provides a Lipschitz extension of ϕ on all $\mathbb{R}^m$.

4 An example

Consider the dynamics of bio-reaction involving a biomass (of concentration x_1) and a substrate (of concentration x_2) inside a reactor of constant volume :

$$(\mathcal{S}) : \left\{ \begin{array}{l} \dot{x} = \begin{pmatrix} 1 \\ -1/k \end{pmatrix} \mu(x_2)x_1 \\ y = x_1 \end{array} \right.$$

[1]this property is known in the literature as "epi-Lipschtizian" and is characterized by a pointed normal cone of S at $\overline{x}$ (see [5]).

k is a positive parameter, μ is the "growth function" (assumed to be smooth, non-negative and s.t. $\mu(0) = 0$). It is straightforward to see that q_2 is non injective as soon as μ is non monotonic[2]. If we consider the logistic law : $\mu(x_2) = rx_2(1 - x_2/c)$ (r and c are positive parameters), it is easy to exhibit a compact invariant set $\mathcal{K}$ on which q_3 is injective and full rank. Consider then the new coordinates :

$$X = l(x) = \begin{bmatrix} \ln(x_1) \\ \mu(x_2) \\ \mu'(x_2)\mu(x_2) \end{bmatrix}, \quad x = \begin{bmatrix} e^{X_1} \\ \dfrac{c}{2}\left(1 - \dfrac{X_3}{rX_2}\right) \end{bmatrix}$$

the dynamics of $(\mathcal{S})$, well defined on $\mathcal{Q} = l(\mathcal{K})$ is :

$$\dot{X} = F(X) = \begin{bmatrix} X_2 \\ -\dfrac{X_3}{k}e^{X_1} \\ \dfrac{e^{X_1}}{k}\left(\dfrac{2r}{c}(X_2)^2 - \dfrac{(X_3)^2}{X_2}\right) \end{bmatrix}$$

but outside $\mathcal{Q}$, the function $\phi(X_2, X_3) = X_3/X_2$ presents a singularity at $X_2 = 0$. So, for building an observer which has to be defined in all $\mathbb{R}^m$, we need to define a Lipschitz extension of the function ϕ outside $\mathcal{Q}$, for which we apply Proposition 2.

The simulations show that the additional parameter γ (cf Lemma 2) helps to compensate the peaking phenomenon [4] during the transient stage.

References

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[2]This is often the case in biological applications, when a saturation effect is present.