# Inversion of Nonlinear Stochastic Models for Parameter Estimation

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**Abstract**

Prediction error and maximum likelihood estimation of nonlinear stochastic models requires inversion of the model, a step which may require substantial efforts, either in terms of manual calculations or through the use of software capable of symbolic computations. In this paper we show that model inversion can be easily implemented in numerical software such as, e.g., Simulink and Matrix$_X$, by means of a feedback connection based on the model. We derive sufficient conditions for the existence of a stable causal inverse as well as sufficient conditions for the initial transient to decay. These conditions are given in terms of properties for a linear time-varying system associated with the nonlinear model. The method is illustrated on a numerical example.

## 1 Introduction

The objective of this paper is to study the problem of model inversion that occurs in parameter estimation in discrete-time nonlinear stochastic models of the type

$$\begin{aligned} x_{t+1} &= g_t(x_t, e_t, u_t, \theta), \quad x_0 = x_0(\theta) \\ y_t &= p_t(x_t, e_t, \theta) \end{aligned}$$

where $\theta \in D_{\mathcal{M}} \subset \mathbb{R}^n$ is an unknown parameter vector, where $x_0 \in \mathbb{R}^m$ is the initial state, where $e_t \in \mathbb{R}$ is the unknown stochastic input which is assumed to be white, where $u_t \in \mathbb{R}^q$ is a measured input and where $y_t \in \mathbb{R}$ is the output.

Defining $g_t(x_t, e_t, \theta) = g_t(x_t, e_t, u_t, \theta)$ transforms the model above into

$$x_{t+1} = g_t(x_t, e_t, \theta), \quad x_0 = x_0(\theta) \tag{1}$$

$$y_t = p_t(x_t, e_t, \theta) \tag{2}$$

and since it turns out that the measured input $u_t$ does not play a role in the context below we will use (1)–(2) as our basic model.

The study will be limited to invertible models, thus, models with a one-to-one relationship between the stochastic input and the output.

It is assumed that the probability density function (PDF) of the stochastic input is known save for possibly some unknown parameters. In Figure 1, a schematic of the problem set-up is shown.

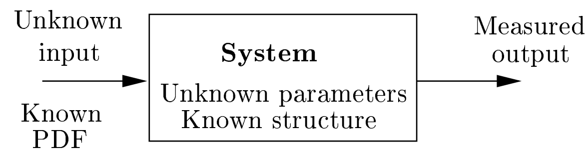


**Figure 1:** A schematic of the problem, where the unknown parameters of the system are to be estimated.

For general theory on identification we refer to (Ljung 1999) and (Söderström and Stoica 1989). In (Sjöberg *et al.* 1995, Juditsky *et al.* 1995) an overview and the mathematical background to the identification of nonlinear black-box models are given. A thorough treatment of grey-box modelling, where physical modelling and parameter identification is combined, is given in (Bohlin 1991). Time-series analysis, when only the output is observed, and the identification of nonlinear models is also discussed in (Tong 1990).

A necessary step in computationally efficient prediction error and maximum likelihood estimation of nonlinear stochastic models is that the model has to be inverted. For a linear model this is trivial but for nonlinear models this can be a quite difficult, or even impossible, task if done symbolically. This problem of inversion is the theme of this paper.

**Example 1.1** *Consider the static system*

$$y_t = \exp(e_t) + \sin(e_t) \tag{3}$$

*where* $|e_t| < 0.5$. *The function* $P(x) = \mathrm{e}^x + \sin(x)$ *is convex on* $[-0.5, 0.5]$ *and hence invertible. There is, however, no explicit expression for the inverse so trying to find the inverse using symbolic software such as Mathematica is futile.*

In the previous example it was not possible to do the inversion symbolically. Instead, one has to use a numerical equation solver to solve (3) for $e_t$. As we will show later, the same situation arises when causal inversion of discrete time dynamical systems is considered.

This means that model inversion can be computationally very demanding since one algebraic equation has to be solved numerically at each point in time.

This paper starts with a review of the ML method in Section 2. In Section 3 it is clarified when an algebraic equation solver has to be employed in causal inversion of (1)–(2). A feedback characterization of the inverse is also derived. This configuration enables simple implementation of the inverse in simulation environments such as Matlab/Simulink and Matrix$_X$. Sufficient conditions for the existence of a causal inverse are derived in Section 4, where also sufficient conditions are provided for when the transient due to the initial state vanishes. Finally, a numerical example is shown in Section 5. Some of the results in this paper are presented in (Markusson and Hjalmarsson 1998).

## 2 The maximum likelihood method

The maximum likelihood method is a general method for parameter estimation, (Ljung 1999). For many problems the ML method is efficient and it can be applied in many situations.

We will assume that the model class is given by (1)–(2). We consider $N$ observations of the output $y_t$ and denote all observations $y^N = (y_{N-1}, y_{N-2}, ..., y_0)$ and similarly we use $e^N$ for $e^N = (e_{N-1}, e_{N-2}, ..., e_0)$.

For notational brevity we shall sometimes use the following notation for the model (1)–(2)

$$y_t = P_t(e^t, \theta), \tag{4}$$

where we have suppressed the dependence of $y_t$ on the initial condition $x_0$.

### 2.1 The ML criterion

The principle of the ML method is based on treating the observed variable, $y_t$, as a random variable with a PDF, denoted $p_y(y \mid \theta)$. This PDF depends on the system and the PDF of the stochastic input. The probability of $y_t$ thus depends on the parameter vector $\theta$. To estimate $\theta$ from the observations of $y_t$, the method chooses a $\theta$ which maximizes the likelihood

$$L(\theta \mid y^N) \stackrel{\triangle}{=} p_y(y^N \mid \theta). \tag{5}$$

The PDF $p_y(y^N \mid \theta)$ can be computed using the inverse of the model which will be shown below. Let us first define the model residuals $w_t(\theta)$ by

$$y_t = P_t(w^t(\theta), \theta), \quad t = 0, 1, \dots N-1. \tag{6}$$

Assuming that $y_t$ has been generated by (1) – (2) for some $\theta = \theta_0$, note that the model residuals $w_t(\theta)$ will be equal to the true input $e_t$ only if $\theta = \theta_0$ and the model uses the same initial state as the true system.

We denote the relationship between the sequences $w^N(\theta)$ and $y^N$ by $W(y^N, \theta)$

$$w^N(\theta) = W(y^N, \theta). \tag{7}$$

As shown in (Jazwinski 1970), the PDF $p_y(y^N \mid \theta)$ is given by

$$p_y(y^N \mid \theta) = p_e(W(y^N, \theta)) \left\| \frac{\partial[W(y^N, \theta)]}{\partial y^N} \right\|, \tag{8}$$

where $p_e(\cdot, \cdot)$ is the PDF of the white input $e_t$ and $\|\cdot\|$ denotes the absolute value of the determinant.

Due to the causality of the model, the matrix $\frac{\partial[W(y^N,\theta)]}{\partial y^N}$ is triangular and therefore the determinant is the product of all the diagonal elements. The diagonal element for sample $t$ is given by

$$\frac{\partial w_t(\theta)}{\partial y_t}. \tag{9}$$

Assuming that $p_e(\cdot)$ is a Gaussian PDF with known variance the cost function is described by

$$V(\theta) = \frac{1}{N} \sum_{t=0}^{N-1} \left( \frac{1}{2} |w_t(\theta)|^2 - \log \left| \frac{\partial w_t(\theta)}{\partial y_t} \right| \right). \tag{10}$$

Finally, the ML estimate of $\theta$ is obtained as

$$\hat{\theta} = \arg\min_{\theta} V(\theta).$$

*Remark:* The log term in (10) does not appear for models of the type

$$y_t = f(y^{t-1}, e^{t-1}, \theta) + e_t. \tag{11}$$

In this case $e_t$ appears explicitly in the right hand side of (11) and the model residuals are easily given by

$$w_t(\theta) = y_t - f(y^{t-1}, w^{t-1}, \theta). \tag{12}$$

### 2.2 Unknown initial states

In ML estimation, the initial state $x_0$ should be treated as an unknown parameter and estimated together with $\theta$. However, it is well known, see, e.g., (Ljung 1999), that initial states cannot be identified consistently. Hence, it is common to fix the initial state in the model, e.g. to zero. The algorithm is then commonly referred to as conditional ML (Söderström and Stoica 1989). Provided that the transient effect of the initial state vanishes sufficiently fast, the conditional ML inherits the statistical properties of the ML method such as consistency and efficiency (Söderström and Stoica 1989).

## 3 Inversion of nonlinear stochastic models

From (10) and (6) it is clear that in ML (and PEM) estimation of nonlinear stochastic models, model inversion is crucial since the cost function (10) is based on the model input. Inversion of discrete-time nonlinear models is treated in (Kotta 1990).

### 3.1 Model description

We will study discrete-time systems where the scalar output $y_t$ is generated by a system of the type (1)–(2). We also recall the notation (4)

$$y_t = P_t(e^t), \tag{13}$$

where now, in addition to the initial state, we have suppressed the parameter dependence to indicate that we are considering one particular model corresponding to some arbitrary $\theta$. For a particular initial state $x_0$, the input $\{e_t\}_{t=0}^{N-1}$ generates the output sequence $y^N$. Inputs that correspond to other initial conditions but which generate the same output $y^N$, collectively, will be denoted by $w_t$. We also assume that the model is invertible, i.e. for a given initial state, the map $W$ in (7) exists and is unique. Further we assume that the state vector $\{x_t\}_{t=0}^{\infty}$ and the output $\{y_t\}_{t=0}^{\infty}$ in (1)–(2) are bounded when the input sequence $\{e_t\}_{t=0}^{\infty}$ is bounded.

### 3.2 Inversion by equation solving

Let us return again to the simple static case in Example 1.1. Determining the input $e_t$ for a particular time $t$ and a particular output $y_t$ amounts to solving

$$y_t = P(e_t)$$

for $e_t$ which, of course, is equivalent to solving

$$y_t - P(e_t) = 0. \tag{14}$$

Equation (14) can be displayed in block diagram form as in Figure 2.a where $\varepsilon_t = y_t - P(e_t)$. Here, the algebraic equation solver finds the value $e_t$ which satisfies $0 = \varepsilon_t \triangleq \varepsilon_t(e_t)$, which is the same as computing the inverse of $P(e_t)$.

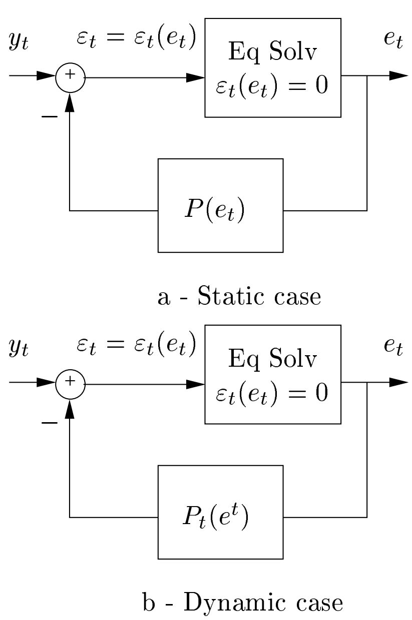


**Figure 2:** Inversion by equation solving.

Thus for static systems, the entire input trajectory can be determined by simulating the feedback system in Figure 2.a forward in time. This suggests a very easy way to obtain the input trajectory for static nonlinear systems using simulation environments capable of handling algebraic equations such as Matlab/Simulink and Matrix$_X$. The key observation is now that for dynamical causal nonlinear systems, $y_t = P_t(e^t)$, *exactly* the same simulation set-up can be employed, see Figure 2.b. Let us describe the dynamical case by an example.

**Example 3.1** *Consider an extension of Example 1.1 where dynamics is added to the system*

$$x_{t+1} = \sin(x_t e_t) \tag{15}$$

$$y_t = e_t + \exp(e_t) + \sin(e_t) + x_t. \tag{16}$$

*If we now apply the setting in Figure 2.b then*

$$\varepsilon_t = y_t - e_t - \exp(e_t) - \sin(e_t) - x_t \tag{17}$$

*Assume that the initial state $x_0$ is known. For $t = 0$, the algebraic equation $\varepsilon_0 = 0$ reads $y_0 - e_0 - \exp(e_0) - \sin(e_0) - x_0 = 0$ and can be solved for $e_0$ since all other quantities are known. Hence, the algebraic equation solver gives $e_t$ for $t = 0$. Using $e_0$ and $x_0$ in (15) then gives $x_1$. Now, the algebraic equation $\varepsilon_1 = 0$ contains no other unknowns than $e_1$ which can be solved for. The procedure can now be repeated for $t = 2, 3, \ldots$ and the inverse $e_t$, $t = 0, 1, 2, \ldots$ is obtained.*

We thus have a very simple way of computing the inverse of nonlinear causal dynamical models: Implement the system in Figure 2.b in a simulation environment that is capable of handling algebraic loops such as Matlab/Simulink and Matrix$_X$. Thus, the burden of solving the algebraic equation is delegated to the simulation software.

### 3.3 Solving algebraic equations

As we have seen above, model inversion for systems with a scalar output and one unknown input requires solving one algebraic equation for each point in time. This can be done numerically using standard iterative search algorithms such as Newton's method (Dennis and Schnabel 1996). Many of these algorithms are available in numerical software packages such as, e.g., LINPACK, Matlab/Simulink and Mathematica. For example, Matlab/Simulink uses Newton's method with weak line search and rank one updates of a Jacobian approximation.

### 3.4 Model inversion by feedback

Let us define an arbitrary linear time-invariant system which is stable and minimum-phase and denote the system by

$$L(q^{-1}) = \sum_{l=0}^{\infty} c_l \, q^{-l} \tag{18}$$

where $q^{-1}$ is the time-shift operator, i.e., ($q^{-1}e_t = e_{t-1}$). Now we can write (13) as

$$\begin{aligned} P_t(e^t) &= L(q^{-1})e_t + (P_t(e^t) - L(q^{-1})e_t) \\ &\triangleq L(q^{-1})e_t + N_t(e^t) \end{aligned} \tag{19}$$

We have thus partitioned the system into two parallel parts as shown in Figure 3.

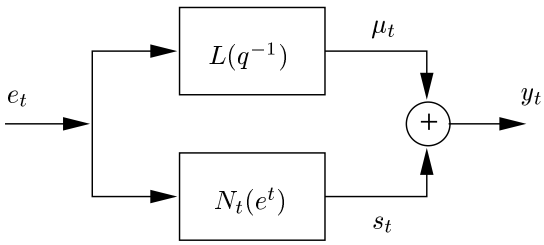


**Figure 3:** Block diagram of a partitioned parallel system, $P(e^t)$.

Following (Doyle *et al.* 1995), we can also write $P_t$ as

$$P_t = L(I + L^{-1}N_t)$$

This gives that the inverse of $P_t$ can be expressed as

$$(I + L^{-1}N_t)^{-1}L^{-1}. \tag{20}$$

In Figure 4, (20) is shown in block diagram form where the input to the inverse is $y_t$ and the output is $w_t$. The reason for denoting the output of the inverse by $w_t$ is that, as pointed out in Section 2.2, it will not be exactly equal to $e_t$ if the initial state is different in the model inverse compared to the model. We see that the causal inverse is obtained from a feedback system consisting of the inverse of the linear part $L$ in the forward path and the nonlinear part $N$ in the feedback path. Notice that $N_t$, and hence $P_t$, *does not have to be inverted.* The feedback structure in Figure 4 thus suggests an alternative way of obtaining the causal inverse without explicit computation of the inverse of the nonlinear part $N_t$ which also easily can be implemented in simulation environments such as Matlab/Simulink and Matrix$_{\mathrm{X}}$.

*Remark 1:* In general there is an algebraic loop in the set-up in Figure 4 which has to be solved by the simulation software. This loop corresponds to the equation solver in Figure 2.b.
*Remark 2:* The existence of a causal inverse depends on the stability of the feedback loop in Figure 4. Notice that since this feedback loop is explicit as opposed to the loop in Figure 2.b, the analysis of the inverse is simplified as we will see in Section 4.

Let us now study an example where the feedback structure shown in Figure 4 can be applied to a general nonlinear model.

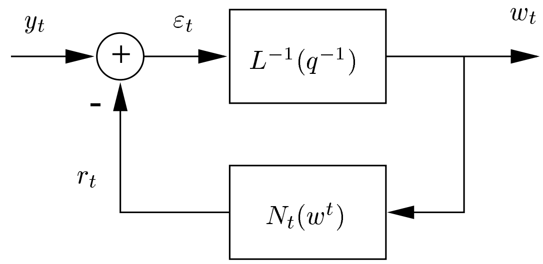


**Figure 4:** Realization of the nonlinear model inverse.

**Example 3.2** *To illustrate the procedure let us return Example 3.1. Rewriting (15) according to (19) using* $L(q^{-1}) = 1$ *gives that the model is given by*

$$x_{t+1} = \sin(x_t e_t) \tag{21}$$
$$s_t = \exp(e_t) + \sin(e_t) + x_t \tag{22}$$
$$\mu_t = e_t \tag{23}$$
$$y_t = s_t + \mu_t \tag{24}$$

*where (21) and (22) represents the nonlinear part and (23) represents the linear part. The inverse is illustrated in Figure 4 and, denoting the state by* $z_t$*, we have*

$$z_{t+1} = \sin(z_t w_t) \tag{25}$$
$$r_t = \exp(w_t) + \sin(w_t) + z_t \tag{26}$$
$$\varepsilon_t = y_t - r_t \tag{27}$$
$$w_t = \varepsilon_t \tag{28}$$

*Assuming* $z_0 = x_0$*, we can identify* $z_t$ *with* $x_t$*,* $r_t$ *with* $s_t$ *and* $w_t$ *with* $e_t$*, i.e. the feedback loop in Figure 4 generates the inverse when the correct initial state is used.*

### 3.5 Partitioned models

Assume that $c_0 \neq 0$ in (18) and that $N_t(e^t)$ in (19) does not depend on $e_t$ but only on previous inputs $e_{t-1}, e_{t-2}, \ldots$, in other words, $P_t$ can be decomposed into a linear time-invariant part $L$ with a direct term and a nonlinear part $N$ with at least one time-delay

$$P_t(e^t) = L(q^{-1})e_t + N_t(e^{t-1})$$

For such partitioned models there will be no algebraic loop in Figure 4 which in its turn means that the inversion of such models is *extremely simple* which means significantly lower computational complexity.

## 4 Existence of the inverse and analysis of unknown initial conditions

In this section we will analyze the existence of a stable causal inverse. This issue is linked to the stability of the feedback system in Figure 4. Also linked to the stability of this feedback loop is the influence of an erroneous initial state on the inverse. In this section we will use the feedback structure in Figure 4 to establish sufficient conditions for 1) the existence of a stable causal inverse and 2) the influence of the initial conditions to vanish. The conditions will be based on a linear time-varying system related to the feedback system in Figure 4.

### 4.1 Model description

We will assume that the system is in the state space form (1)–(2) but we will partition it according to (19), cf. Figure 3. We assume relative degree zero of the linear system, and hence the whole system.
We introduce the following notation

$$\begin{aligned} \zeta_{t+1} &= A\zeta_t + Be_t \\ \mu_t &= C\zeta_t + De_t, \quad D \neq 0 \\ x_{t+1} &= f_t(x_t, e_t) \\ s_t &= h_t(x_t, e_t) \\ y_t &= \mu_t + s_t \end{aligned} \tag{29}$$

where $\zeta_t$ and $x_t$ are the state vectors of the linear part $L$ and the nonlinear part $N_t$, respectively. The input to both parts is denoted $e_t$ and the output of each subsystem is $\mu_t$ and $s_t$, respectively, summed together to give the total output $y_t$. Since we will analyze the stability and transient for one particular model we have dropped the $\theta$ dependence as well.

### 4.2 Inverse model description

In Figure 5 the forward and the inverse systems are shown with the internal signals presented. The states of each block are given within parenthesis.

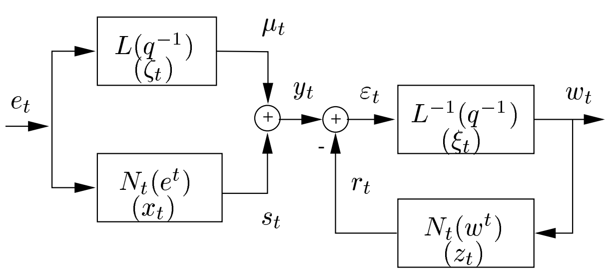


**Figure 5:** Block diagram of the forward and the inverse system.

With the inverse and the notation as in Figure 5, the state-space equation of the inverse system is

$$\begin{aligned} \xi_{t+1} &= (A - BD^{-1}C)\xi_t + BD^{-1}\varepsilon_t \\ w_t &= -D^{-1}C\xi_t + D^{-1}\varepsilon_t \\ z_{t+1} &= f_t(z_t, w_t) \\ r_t &= h_t(z_t, w_t) \\ \varepsilon_t &= y_t - r_t \end{aligned} \tag{30}$$

assuming $D$ is nonzero. However, since the states of the inverse model are unknown, $\zeta_0 \neq \xi_0$ and $x_0 \neq z_0$, the output of the inverse $w_t$ will not be equal to $e_t$. These initial errors will be analyzed in the next section.

### 4.3 Error analysis

To analyze the dependence of erroneous initial state of the inverse model, we consider the differences between $\zeta_t$, $s_t$, $e_t$, and $x_t$ versus $\xi_t$, $r_t$, $w_t$ and $z_t$ which we define as

$$\begin{aligned} \rho_t &\triangleq \zeta_t - \xi_t \quad & \sigma_t &\triangleq s_t - r_t \\ \delta_t &\triangleq w_t - e_t \quad & \Delta_t &\triangleq z_t - x_t. \end{aligned} \tag{31}$$

We will use the following assumptions

- $A1$ : The input signals of the system $e_t$ are bounded as well as the internal dynamics of the system $\zeta_t, x_t$ and the output of the system $y_t$.
- $A2$ : The nonlinear functions $f_t(z_t, w_t)$ and $h_t(z_t, w_t)$ have continuous derivatives wrt $z_t$ and $w_t$ up to order 2.

**Theorem 4.1** *Under Assumptions $A1$ and $A2$, the inverse will exist and the errors (31) due to unknown initial conditions will vanish if the linear time-varying system*

$$\begin{aligned} \Delta_{t+1} &= \Gamma_t^{\Delta}\Delta_t + \Gamma_t^{\rho}\rho_t \\ \rho_{t+1} &= \Sigma_t^{\Delta}\Delta_t + \Sigma_t^{\rho}\rho_t \end{aligned} \tag{32}$$

*is asymptotically stable where,*

$$\Gamma_t^{\Delta} = \frac{\partial f_t}{\partial z_t} + \frac{\partial f_t}{\partial w_t}\left(-D^{-1}\frac{\partial h_t}{\partial z_t}\left(1 + D^{-1}\frac{\partial h_t}{\partial w_t}\right)^{-1}\right) \tag{33}$$

$$\Gamma_t^{\rho} = \frac{\partial f_t}{\partial w}D^{-1}C\left(1 + D^{-1}\frac{\partial h_t}{\partial w_t}\right)^{-1} \tag{34}$$

$$\Sigma_t^{\Delta} = BD^{-1}\frac{\partial h_t}{\partial z_t}\left(1 + D^{-1}\frac{\partial h_t}{\partial w_t}\right)^{-1} \tag{35}$$

$$\begin{aligned} \Sigma_t^{\rho} &= A - BD^{-1}C \\ &\quad + BD^{-1}\frac{\partial h_t}{\partial w_t}D^{-1}C\left(1 + D^{-1}\frac{\partial h_t}{\partial w_t}\right)^{-1} \end{aligned} \tag{36}$$

*under the assumption that $D \neq 0$. In (33) to (36) the derivatives wrt $z_t$ and $w_t$ are computed along the trajectory $z_t = x_t$ and $w_t = e_t$, i.e. e.g.,*

$$\frac{\partial f_t}{\partial z_t} = \left.\frac{\partial f_t(z_t, w_t)}{\partial z_t}\right|_{\substack{z_t = x_t \\ w_t = e_t}}$$

The proof is shown in (Markusson and Hjalmarsson 2000).
According to Theorem 4.1, $w_t$ will converge towards $e_t$ if the linear time-varying system (32) is asymptotically stable. Thus, we are now concerned with the stability analysis of a LTV system with zero-input.
Let us define

$$F_t = \begin{pmatrix} \Gamma_t^{\Delta} & \Gamma_t^{\rho} \\ \Sigma_t^{\Delta} & \Sigma_t^{\rho}) \end{pmatrix} \tag{37}$$

such that (32) can be written as

$$\begin{pmatrix} \Delta_{t+1} \\ \rho_{t+1} \end{pmatrix} = F_t \begin{pmatrix} \Delta_t \\ \rho_t \end{pmatrix}. \tag{38}$$

The stability of (38) can be analyzed with a number of methods, see e.g. (Rugh 1996). As an example the LTV system in (38) is uniformly exponentially stable if there exists a finite constant $\gamma$ and a constant $0 \le \nu \le 1$ such that the largest pointwise eigenvalue of $F_t^T F_t$, denoted $\lambda_{max}(t)$, satisfies

$$\prod_{i=j}^{t} \lambda_{max}^{1/2}(i) \le \gamma\, \nu^{t-j} \tag{39}$$

for all $t, j$ such that $t \ge j$ (Rugh 1996). This stability result and Theorem 4.1 gives the following corollary.

**Corollary 4.1** *The inverse of the nonlinear system (29) exists and the output of the inverse $w_t$ converges towards $e_t$ as $t \to \infty$ exponentially fast if there exists a finite constant $\gamma$ and a constant $0 \le \nu \le 1$ such that the largest pointwise eigenvalue of $F_t^T F_t$ obeys the inequality in (39), where $F_t$ is given by (37).*

Following the discussion in Section 2.2 we conclude that consistency and asymptotic efficiency in nonlinear dynamic ML estimation is normally ensured if the conditions of the corollary above are satisfied (and the system is identifiable).

## 5 Numerical example

This example is a model with an unmeasured stochastic input, partitioned as a linear model in parallel with a nonlinear model. The example was simulated using Matlab 5.3 and Simulink 3.0. The initial value of the algebraic equation solver was zero in all cases. The linear part is described by a first order model, $\mu_t = L(q^{-1}, \theta) e_t$, where

$$L(q^{-1}, \theta) = \frac{1 + bq^{-1}}{1 + aq^{-1}},$$

where $a$ and $b$ are real valued parameters. The nonlinear part is described by

$$\begin{aligned} x_{t+1} &= c x_t + e_t/(1 + d e_t^2) \\ s_t &= x_t + g e_t \exp(-h e_t^2) \end{aligned}$$

and the measured output is $y_t = s_t + \mu_t$. The input, $e_t$, was white Gaussian distributed noise with unit variance. The parameters, $\theta = (a, b, c, d, g, h)^T$, were simultaneously estimated using the ML method. The true value of $\theta$ is given in Table 1 where also the estimation results are shown. One hundred input realizations were used, each containing 1000 samples. The log-term in (10) is given by $\frac{\partial w_t}{\partial y_t} = \left(1 + g \exp\{-h w_t^2\}(1 - 2h w_t^2)\right)^{-1}$.

## 6 Conclusions

In this paper we have characterized model inversion in terms of a feedback loop. One of the benefits of the feedback configuration is that it can be directly implemented in simulation tools like e.g., Matlab/Simulink or Matrix$_X$. Also, this feedback formula was used to derive sufficient conditions for the existence of a stable causal inverse and for the decay of initial errors. Finally, the feedback configuration was used in ML estimation on a numerical example.

**Table 1:** *Estimation results (100 trials).*

| | True value $\theta_0$ | $mean(\hat{\theta})$ | $std(\hat{\theta})$ |
|---|---|---|---|
| $a$ | 0.90 | 0.89 | 0.02 |
| $b$ | 0.60 | 0.59 | 0.04 |
| $c$ | 0.80 | 0.79 | 0.03 |
| $d$ | 1.00 | 1.01 | 0.20 |
| $g$ | 0.70 | 0.71 | 0.06 |
| $h$ | 0.20 | 0.20 | 0.03 |

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