On discrete time nonnegative storage functions and state functions

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Abstract

In this paper, we study discrete time dissipativeness. Particularly, we focus on storage functions. Differently from the continuous time case, every storage function is not necessarily a state function in discrete time. For this problem, this paper shows that a every *nonnegative* storage function is a state function in discrete time. In addition, we provide some of the necessary and sufficient conditions for the existence of the nonnegative storage function.

Keywords: dissipativeness, behavioral approach, discrete time, storage function, state function.

1 Introduction

1.1 Backgrounds and Motivations

Dissipativeness is one of the most important properties in dynamical systems (cf. [9],[7],[12],[8]). The reason for this is that various important system characteristics can be formalized as dissipativeness. Physical intuitively, it means that a dynamical system always dissipates energy (*dissipation rate*) for a given supplied power (*supply rate*), which is equivalent to saying that increase of stored energy (*storage function*) in the system can not exceed the supplied power. As for a storage function, it measures the amount of energy stored in the internal of the system, so it is naturally described by using internal variables of the system, that is, state variables. In fact, Trentelman and Willems proved that every storage function is a state function in continuous time (cf.[7],[12]). This result plays crucial roles from the view point of not only theoretical importance but also practical importance. At first grance, it seems easy to prove that every storage function is a state function in discrete time similarly as in the continuous time case. However, differently from the continuous time case, every storage function is not necessarily a state function in discrete time. For this problem, this paper shows that a every *nonnegative* storage function is a state function in discrete time. In addition, we provide a necessary and sufficient condition for the existence of the nonnegative storage function.

1.2 Notations

Let $\mathbf{Z}$, $\mathbf{R}$, and $\mathbf{C}$ denote the set of integers, real numbers and complex numbers, respectively. The notation $\mathbf{R}^q$ ($\mathbf{C}^q$) denotes the set of real (complex, respectively) vectors of size q. For $\lambda \in \mathbf{C}$, $\bar{\lambda}$ denotes the conjugate of λ. For $a \in \mathbf{R}^q$, $\|a\|^2 := a^T a$. For $a \in \mathbf{C}^q$, a^* denotes the conjugated transpose of a and $\|a\|^2 := a^* a$. Let $\mathbf{R}^{p\times q}$ denote the set of real matrices of size $p \times q$. Let $(\mathbf{R}^q)^Z$ denote the set of real time series vectors of size q. For $w \in (\mathbf{R}^q)^Z$ the shift operator σ is defined by $(\sigma w)(t) := w(t+1)$. The notation l_2^q is the set of square summable time series vectors of size q, i.e., $w \in l_2^q$ means that $\sum_{t=-\infty}^{t=\infty} \|w(t)\|^2 < \infty$. Let $\mathbf{R}[\lambda]$ denote the set of polynomials in the indeterminate λ with coefficients in $\mathbf{R}$. Similarly, $\mathbf{R}[\zeta, \eta]$ denote the set of two-variable polynomials in the indeterminates ζ and η with coefficients in $\mathbf{R}$. The powers of these indeterminates may be not only nonnegative but also negative in discrete time. Particularly, $\mathbf{R}[\lambda^{-1}, \lambda]$ denote the set of two-sided polynomials which are obtained by regarding the indeterminates ζ and η in $\mathbf{R}[\zeta, \eta]$ as λ^{-1} and λ, respectively. Similarly, the set of matrix version of them are written respectively by $\mathbf{R}^{q\times q}[\lambda]$, $\mathbf{R}^{q\times q}[\zeta, \eta]$, and $\mathbf{R}^{q\times q}[\lambda^{-1}, \lambda]$ for real coefficient matrices of size $p \times q$. For a nonsingular polynomial matrix $D(\lambda)$, we call it Hurwitz (anti-Hurwitz) if $\det(D(\lambda)) \neq 0$ for all $\lambda \in \mathbf{C}$ such that $|\lambda| \geq 1$ ($|\lambda| \leq 1$, respectively). For a difference equation $v = M(\sigma)l$ induced by $M(\lambda) \in \mathbf{R}^{*\times *}[\lambda]$, in order to represent that v is restricted in this difference equation, we use the notation $v \in Im(M(\sigma))$. For a constant matrix A, let $rank(A)$ denote the rank of A. Finally, let I_q and $0_{p\times q}$ denote the unit matrix of size $q \times q$ and the zero matrix of size $p \times q$, respectively.

2 Preliminaries

2.1 Discrete time state functions

Consider a linear time-invariant discrete time dynamical system $\Sigma := (\mathbf{Z}, \mathbf{R}^q, \mathcal{B})$, where $\mathbf{Z}$ is the time axis (the set of integers), $\mathbf{R}^q$ is the signal space, and $\mathcal{B} \subseteq (\mathbf{R}^q)^Z$ is the (manifest) behavior. Moreover, we assume that the system is controllable. Then, the behavior $w \in \mathcal{B}$ is described by the image representation (cf. [10],[11]) $w = M(\sigma)l$,where $M(\lambda) \in \mathbf{R}^{q\times d}[\lambda]$ and $l \in (\mathbf{R}^d)^Z$. If $M(\lambda)$ is column full rank for all $\lambda \in \mathbf{C}$, i.e., the system is observable, then multiplying $M(\lambda)$ by an appropriate nonsingular matrix $P \in \mathbf{R}^{q\times q}$ yields $PM(\lambda)$ $= \left[\begin{array}{cc} U(\lambda)^T & Y(\lambda)^T \end{array}\right]^T$ with $\det(U(\lambda)) \neq 0$ and $Y(\lambda)U(\lambda)^{-1}$ is a proper rational function. Thus, in the case of observable image representations, we can assume without loss of generality that the mathematical model is described by

$$w = M(\sigma)l = \left[\begin{array}{c} U(\sigma) \\ Y(\sigma) \end{array}\right] l \quad (1)$$

where $\det(U(\lambda)) \neq 0$ and $Y(\lambda)U(\lambda)^{-1}$ is proper.

Similarly to the classical control theory, one can consider state variables in the behavioral setting. Consider a dynamical system $\Sigma = (\mathbf{Z}, \mathbf{R}^q, \mathcal{B})$. Let $\Sigma_s := (\mathbf{Z}, \mathbf{R}^q \times \mathbf{R}^n, \mathcal{B}_f)$ denote this system with latent variables, where $\mathbf{R}^n$ is the latent variable space, and $\mathcal{B}_f \subseteq (\mathbf{R}^q \times \mathbf{R}^n)^Z$ is the set of full behaviors. Then Σ_s is said to be a system with state variables if $\{(w_1, x_1), (w_2, x_2) \in \mathcal{B}_f$ and $x_1(0) = x_2(0)\} \Longrightarrow \{(\tilde{w}(t), \tilde{x}(t)) \in \mathcal{B}_f\}$ where, for $t < 0$, $(\tilde{w}(t), \tilde{x}(t))$ is equal to $(w_1(t), x_1(t))$, and, for $t \geq 0$, $(\tilde{w}(t), \tilde{x}(t))$ is equal to $(w_2(t), x_2(t))$. This is called the axiom of state (cf.[10]) and a latent variable x is said to be a state variable of $\mathcal{B}$. By using polynomial matrices in Eq.(1), state variables can be characterized as follows. The proof is straightforward from that for the continuous time case (cf. [7],[5],[12]), so we omit it here.

Proposotion 2.1 *Assume that a dynamical system $\Sigma = (\mathbf{Z}, \mathbf{R}^q, \mathcal{B})$ is controllable, and its observable image representation is described by Eq.(1). Let $x \in (\mathbf{R}^n)^Z$ denote a state variable of $\mathcal{B}$, and $\mathcal{B}_f$ denote the full behavior of this system with state variables. Let $F(\lambda)$ be a polynomial matrix whose column size is equal to the size of l and induce a new variable $f = F(\sigma)l$. Moreover, let $\mathcal{B}_{ext}$ be the system with image representation*

$$\left[\begin{array}{c} w \\ f \end{array}\right] = \left[\begin{array}{c} U(\sigma) \\ Y(\sigma) \\ F(\sigma) \end{array}\right] l. \quad (2)$$

Then, there exists a real constant matrix H such that $f = Hx$ for all f and x for which there exists $w \in \mathcal{B}$ such that $(w, f) \in \mathcal{B}_{ext}$ and $(w, x) \in \mathcal{B}_f$, iff $F(\lambda)U(\lambda)^{-1}$ is a strictly proper rational function.

2.2 Quadratic difference forms

Quadratic difference forms are appropriate mathematical tools related to discrete time dissipativeness. They are used throughout this paper, so we intordUce some necessary definitions and properties briefly. See [12] and [1] for more details of the continuous time case and the discrete time case, respectively.

An element of $\mathbf{R}^{q\times q}[\zeta, \eta]$ is described by

$$\Phi(\zeta, \eta) = \sum_{k,l} \Phi_{kl} \zeta^k \eta^l. \quad (3)$$

The sum in Eq.(3) ranges over the integers and is assumed to be finite, and $\Phi_{kl} \in \mathbf{R}^{q\times q}$. For $\Phi(\zeta, \eta)$ $\in \mathbf{R}^{q\times q}[\zeta, \eta]$, let $\mathbf{R}_s^{q\times q}[\zeta, \eta]$ denote the set of two-variable polynomial matrices satisfying $\Phi(\zeta, \eta) = \Phi(\eta, \zeta)^T$. For all $w \in (\mathbf{R}^q)^Z$, $\Phi(\zeta, \eta) \in \mathbf{R}_s^{q\times q}[\zeta, \eta]$ induces a quadratic difference form $Q_\Phi : (\mathbf{R}^q)^Z \to \mathbf{R}^Z$ as defined by

$$Q_\Phi(w)(t) := \sum_{k,l} w(t+k)^T \Phi_{kl} w(t+l). \quad (4)$$

For a given arbitrary $\Phi(\zeta, \eta) \in \mathbf{R}_s^{q\times q}[\zeta, \eta]$, we define the following infinite matrix

$$\tilde{\Phi} := \left[\begin{array}{ccccc} & \vdots & & \vdots & \\ \cdots & \Phi_{0,0} & \cdots & \Phi_{0,l} & \cdots \\ & \vdots & & \vdots & \\ \cdots & \Phi_{k,0} & \cdots & \Phi_{k,l} & \cdots \\ & \vdots & & \vdots & \end{array}\right]. \quad (5)$$

Here all but a finite number of the elements of $\tilde{\Phi}$ are zero. Since we concentrate on the finite nonzero block of $\tilde{\Phi}$ in this paper, we regard $\tilde{\Phi}$ as a finite square matrix. For example, if $\Phi(\zeta, \eta) \in \mathbf{R}_s^{q\times q}[\zeta, \eta]$ is defined as $\sum_{k,l=0}^n \Phi_{kl} \zeta^k \eta^l$ explicitly, we assume $\tilde{\Phi} \in \mathbf{R}^{q(n+1)\times q(n+1)}$.

In a similar way to the constant symmetric matrix case, the nonnegativity of quadratic difference form induced by $\Phi(\zeta, \eta) \in \mathbf{R}_s^{q\times q}[\zeta, \eta]$ is defined as $Q_\Phi(w)(t) \geq 0$ for all $w \in (\mathbf{R}^q)^Z$ and for all $t \in Z$, and denoted by $\Phi(\zeta, \eta) \geq 0$. As shown in Proposition 2.1 in [1], $\Phi(\zeta, \eta) \geq 0$ is equivalent to $\tilde{\Phi} \geq 0$.

For $\Phi(\zeta, \eta) = \sum_{k,l=0}^n \Phi_{kl} \zeta^k \eta^l \in \mathbf{R}_s^{q\times q}[\zeta, \eta]$, factorize $\tilde{\Phi} = \tilde{M}^T \Sigma_\Phi \tilde{M}$ such that $\tilde{M} \in \mathbf{R}^{*\times q(n+1)}$ is row full rank and $\det(\Sigma_\Phi) \neq 0$, i.e., rank(Σ_Φ) = rank $(\tilde{\Phi})$. With such a factorization of $\tilde{\Phi}$, we obtain the canonical factorization (cf.[12, 7]) of $\Phi(\zeta, \eta)$ described by

$$\Phi(\zeta, \eta) = M(\zeta)^T \Sigma_\Phi M(\eta) \quad (6)$$

where

$$M(\lambda) := \tilde{M} \left[\begin{array}{cccc} I & \lambda I & \cdots & \lambda^n I \end{array}\right]^T \in \mathbf{R}^{*\times q}[\lambda].$$

2.3 Discrete time Dissipativeness

By using quadratic difference forms and two-variable polynomial matirces, we can formalize the notion of discrete time dissipativeness as follows. Of course, these notions are parallel with the continious time case (cf. [12] and [7]). In order to make discussion easy, we assume that $\mathcal{B} = (\mathbf{R}^q)^Z$ throughout the paper. This standpoint is equivalent to that stated in Section 5 in [12].

At first, we can regard $Q_\Phi(w)$ induced by $\Phi(\zeta, \eta) \in \mathbf{R}_s^{q\times q}[\zeta, \eta]$ as the power entering into the dynamical system, i.e., a *suppy rate*. Then, we can formalize dissipativeness of discrete time dynamical system as follows (cf. Definition 3.1 in [1] and Definition 5.1 in [12]).

Definition 2.1 *Let $\Phi(\zeta, \eta) \in \mathbf{R}_s^{q\times q}[\zeta, \eta]$ induce a quadratic supply rate $Q_\Phi(w)$.*

1. *$Q_\Psi(w)$ induced by $\Psi(\zeta, \eta) \in \mathbf{R}_s^{q\times q}[\zeta, \eta]$ is said to be a storage function for the supply rate $Q_\Phi(w)$, if $Q_\Psi(w)(t + 1) - Q_\Psi(w)(t) \leq Q_\Phi(w)(t)$ for all t and for all $w \in l_2^q$.*
2. *$Q_\Delta(w)$ induced by $\Delta(\zeta, \eta) \in \mathbf{R}_s^{q\times q}[\zeta, \eta]$ is said to be a dissipation rate for the supply rate $Q_\Phi(w)$, if $\sum_{t=-\infty}^{t=\infty} Q_\Phi(w)(t) = \sum_{t=-\infty}^{t=\infty} Q_\Delta(w)(t)$, and $Q_\Delta(w)(t) \geq 0$ for all t and for all $w \in l_2^q$.*□

As for dissipation rates, it follows from Lemma 3.1 in [1] that $\sum_{t=-\infty}^{t=\infty} Q_\Phi(w)(t) = \sum_{t=-\infty}^{t=\infty} Q_\Delta(w)(t)$ for all $w \in l_2^q$ is equivalent to saying

$$\Phi(\lambda^{-1}, \lambda) = \Delta(\lambda^{-1}, \lambda) \tag{7}$$

for all nonzero $\lambda \in \mathbf{C}$.

The relation between a supply rate, a storage function, and a supply rate can be formalized as follows (cf. Proposition 3.3 in [1], Theorem 4.3 in [7], and Proposition 5.2 in [12]).

Theorem 2.1 *Let $\Phi(\zeta, \eta) \in \mathbf{R}_s^{q\times q}[\zeta, \eta]$ induce a supply rate. Then, the following four conditions are equivalent.*

1). For all $w \in l_2^q$, $\sum_{t=-\infty}^{\infty} Q_\Phi(w)(t) \geq 0$.

2). $\Phi(e^{-j\omega}, e^{j\omega}) \geq 0$ for all $\omega \in [0, 2\pi)$.

3). $\Phi(\zeta, \eta)$ admit a storage function.

4). $\Phi(\zeta, \eta)$ admit a dissipation rate.

Moreover, for the supply rate induced by $\Phi(\zeta, \eta)$ there is a one-one relation between storage functions $Q_\Psi(w)$ induced by $\Psi(\zeta, \eta)$ and dissipation rates $Q_\Delta(w)$ induced by $\Delta(\zeta, \eta)$, which is described by

$$Q_\Psi(w)(t + 1) - Q_\Psi(w)(t) = Q_\Phi(w)(t) - Q_\Delta(w)(t)(\zeta, \eta)) \tag{8}$$

for all time $t \in \mathbf{Z}$, or equivalently,

$$\Psi(\zeta, \eta) = (\Phi(\zeta, \eta) - \Delta(\zeta, \eta))/(\zeta\eta - 1). \quad \square \tag{9}$$

Consider the supply rate $Q_\Phi(w)$ induced by $\Phi(\zeta, \eta) \in \mathbf{R}_s^{q\times q}[\zeta, \eta]$ with its canonical factorization $\Phi(\zeta, \eta) = M(\zeta)\Sigma M(\eta)$. In the following, we regard $\Sigma = (\mathbf{Z}, (\mathbf{R})^*, Im(M(\sigma)))$ as a dynamical system. In addition, assume that $M(\lambda)$ is column full rank for all $\lambda \in \mathbf{C}$. Related to Proposition 2.1, $\Psi(\zeta, \eta) \in \mathbf{R}_s^{q\times q}[\zeta, \eta]$ that induces a storage function for $Q_\Phi(w)$ is said to be a state function of this dynamical system Σ if $U(\zeta)^{-T}\Psi(\zeta, \eta)U(\eta)^{-1}$ is strictly proper with respect to ζ and η, or equivalently $F(\lambda)U(\lambda)^{-1}$ is proper, where $F(\lambda)$ is obtained by the canonical factorization $\Psi(\zeta, \eta) = F(\zeta)^T\Sigma_\Psi F(\eta)$. It then follows from Proposition 2.1 that $\Psi(\zeta, \eta)$ can be described by $\Psi(\zeta, \eta) = x^T Kx$ where x is a state variable of $\mathcal{B} = Im(M(\sigma))$ and $K := H^T\Sigma_\Psi H$.

In addtion to Section 2.1, we again define some notations related to the above stand points. Let $\Phi(\zeta, \eta)$ induce a supply rate $Q_\Phi(w)$. For this supply rate $Q_\Phi(w)$, define $\mathcal{S}(\Phi) := \{\Psi(\zeta, \eta) \in \mathbf{R}_s^{q\times q}[\zeta, \eta]|\ Q_\Psi(w)$ induced by $\Psi(\zeta, \eta)$ is a storage function for $Q_\Phi(w)$ and $\mathcal{D}(\Phi) := \{\Delta(\zeta, \eta) \in \mathbf{R}_s^{q\times q}[\zeta, \eta]|\ Q_\Delta(w)$ induced by $\Delta(\zeta, \eta)$ is a dissipation rate for $Q_\Phi(w)$. Next, fix one of $\Delta(\zeta, \eta) \in \mathcal{D}(\Phi)$ $(\Psi(\zeta, \eta) \in \mathcal{S}(\Phi))$ $S(\Phi, \Delta) := \{\Psi(\zeta, \eta) \in \mathcal{S}(\Phi)|\ \Psi(\zeta, \eta)$ satisfies Eq.(9) for a fixed $\Delta(\zeta, \eta) \in \mathcal{D}(\Phi)\}$. Similarly, for a fixed $\Psi \in \mathcal{S}(\Phi)$, define $D(\Phi, \Psi) := \{\Delta(\zeta, \eta) \in \mathcal{D}(\Phi)|\ \Delta(\zeta, \eta)$ satisfies Eq.(9) for a fixed $\Psi(\zeta, \eta) \in \mathcal{S}(\Phi)\}$.

3 Nonnegative storage functions and state functions (Main results)

As stated in the introduction, every storage function is not necessarily a state function in discrete time.

Example 3.1 *Consider a supply rate induced by*

$$\Phi(\zeta, \eta) = \begin{bmatrix} 1 & \zeta \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ \eta \end{bmatrix} \tag{10}$$

with its canonical factorization $\Phi(\zeta, \eta) = M(\zeta)^T \Sigma_\Phi M(\eta)$, where $M(\lambda) := \begin{bmatrix} 1 + \lambda & \lambda \end{bmatrix}^T$ and $\Sigma_\Phi := I_2$. It is clear that $M(\lambda)$ is observable and defining $U(\lambda) := 1 + \lambda$ allows us to observe that $M(\lambda)U(\lambda)^{-1}$ is proper, which implying $U(\zeta)^{-1} \Phi(\zeta, \eta)\ U(\eta)^{-1}$ is proper with respect to ζ and η. Moreover, $U(\lambda^{-1})^{-T}M(\lambda^{-1})^T$ is also proper, so $U(\lambda^{-1})^{-1}\Phi(\lambda^{-1}, \lambda)U(\lambda)^{-1}$ is proper. Here, one *can take*

$$\Delta(\zeta,\eta) = \begin{bmatrix} 1 & \zeta & \zeta^2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \eta \\ \eta^2 \end{bmatrix} \quad (11)$$

with its canonical factorization $\Delta(\zeta,\eta) = D(\zeta)^T D(\eta)$ *, where* $D(\lambda) := \begin{bmatrix} 1+\lambda & \lambda^2 \end{bmatrix}^T$*. Indeed, it is easy to verify that* $\Delta(\zeta,\eta) \geq 0$ *and* $\Delta(\lambda^{-1},\lambda) = \Phi(\lambda^{-1},\lambda)$*. However,* $D(\lambda)U(\lambda)^{-1}$ *is not proper. At the same time, it follows from Eq.(9) that* $\Psi(\zeta,\eta) := S(\Phi,\Delta)$ *can be described by*

$$\Psi(\zeta,\eta) = \begin{bmatrix} 1 & \zeta \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ \eta \end{bmatrix} \quad (12)$$

with its canonical factorization $\Psi(\zeta,\eta) = F(\zeta)\ \Sigma_\Psi\ F(\eta)$*, where* $F(\lambda) := \lambda$*, and* $\Sigma_\Psi := -1$*. It is clear that the storage function* $Q_\Psi(w)$ *induced by* $\Psi(\zeta,\eta)$ *is not a state function bacause* $F(\lambda)U(\lambda)^{-1} = \lambda/(1+\lambda)$ *is not stricyly proper.* □

The above simple example illustrates that the other additional conditions on supply rates, storage functions, or dissipation rates are required. For this problem, the folloing theorem provides one of the sufficient conditions.

Theorem 3.1 *Let* $\Phi(\zeta,\eta) \in \mathbf{R}_s^{q\times q}[\zeta,\eta]$ *induce a supply rate. Assume that there exists a nonnegative storage function for this supply rate* $Q_\Phi(w)$*. Then, every nonnegative storage function is a state function of the supply rate* $Q_\Phi(w)$*.* □

Proof: Before going to the proof, we show the following two lemmas. See [2] for their proofs.

Lemma 3.1 *Let* $\Phi(\zeta,\eta) \in \mathbf{R}_s^{q\times q}[\zeta,\eta]$ *induce a supply rate. For this supply rate, let* $\Delta(\zeta,\eta) \in \mathcal{D}(\Phi)$*. Assume that*

$$rank(\tilde{\Delta}) = rank(\Delta(0,0)), \quad (13)$$

where, $\tilde{\Delta}$ *is defined for* $\Delta(\zeta,\ \eta)$ *in the same way as Eq.(5). Then, the storage function induced by* $S(\Phi,\ \Delta)$ *is a state function of the supply rate* $Q_\Phi(w)$*.* □

Lemma 3.2 *Let* $\Phi(\zeta,\eta) \in \mathbf{R}_s^{q\times q}[\zeta,\eta]$ *induce a supply rate. Assume that* $\Phi(e^{-j\omega},e^{j\omega}) > 0$ *for all* $\omega \in [0,2\pi)$*. Then, there exist a storage function induced by* $\Psi^+(\zeta,\eta)$ *that satisfies*

$$\Psi(\zeta,\eta) \leq \Psi^+(\zeta,\eta) \quad (14)$$

for any other $\Psi(\zeta,\eta) \in \mathcal{S}(\Phi)$*. In addition,* $\Psi^+(\zeta,\eta)$ *is written by*

$$\Psi^+(\zeta,\eta) = (\Phi(\zeta,\eta) - A(\zeta)^T A(\eta))/(\zeta\eta - 1) \quad (15)$$

where $A(\lambda)$ *is an anti-Hurwitz spectral factor of* $\Phi(\lambda^{-1},\lambda) = A(\lambda^{-1})^T A(\lambda)$*.* □

We now return to the proof of Theorem 3.1. In order to simplify the proof, we assume the following conditions for the supply rate induced by $\Phi(\zeta,\eta) \in \mathbf{R}_s^{q\times q}[\zeta,\eta]$ whose canonical factorization is described by $\Phi(\zeta,\eta) = M(\zeta)^T \Sigma_\Phi M(\eta)$.

(a). $M(\lambda)$ is column full rank for all $\lambda \in \mathbf{C}$

(b). $\Phi(e^{-j\omega},e^{j\omega}) > 0$ for all $\omega \in [0,2\pi)$.

Firstly, we provide the proof under the above assumptions. Using (b), we can obtain $\Phi(\lambda^{-1},\lambda) = A(\lambda^{-1})^T A(\lambda)$, where $A(\lambda)$ is an anti-Hurwitz spectral factor. It then follows from Lemma 3.2 that the maximum storage function is induced by $\Psi^+(\zeta,\eta)$ described by Eq.(15). In addition, the existence of the nonnegative storage function yields $\Psi^+(\zeta,\eta) \geq 0$, that is, its canonical factorization is described by $\Psi^+(\zeta,\eta) = F^+(\zeta)^T F^+(\eta)$, where $F^+(\lambda) \in \mathbf{R}^{*\times q}[\lambda]$.

Consider an arbitrary nonnegative storage function induced by $\Psi(\zeta,\eta) \in \mathcal{S}(\Phi)$. It is also clear that its canonical factorization can be also described by $\Psi(\zeta,\eta) = F(\zeta)^T F(\eta)$, where $F(\lambda) \in \mathbf{R}^{*\times q}[\lambda]$. Define $\Psi_d(\zeta,\eta) = \Psi^+(\zeta,\eta) - \Psi(\zeta,\eta)$ and $\tilde{\Psi}_d$ for this $\Psi_d(\zeta,\eta)$ as in the same way to Eq.(5). It follows from $\Psi^+(\zeta,\eta) \geq \Psi(\zeta,\eta)$ that $\tilde{\Psi}_d \geq 0$. Here, for an arbitrary $\lambda \in \mathbf{C}$ and $\alpha \in \mathbf{C}^q$, define an complex vector

$$a := \begin{bmatrix} I_q \\ \lambda I_q \\ \vdots \\ \lambda^{h^*} I_q \end{bmatrix} \alpha \in \mathbf{C}^{q(h^*+1)}$$

where h^* is the degree of $\Psi_d(\zeta,\eta)$. The quadratic form $a^* \tilde{\Psi}_d a$ is nonnegative, so $\alpha^* \Psi_+(\bar{\lambda},\lambda)\alpha - \alpha^* \Psi(\bar{\lambda},\lambda)\alpha \geq 0$, i.e.,

$$\alpha^* F^+(\bar{\lambda})^T F^+(\lambda)\alpha \geq \alpha^* F(\bar{\lambda})^T F(\lambda)\alpha \geq 0 \quad (16)$$

for all $\lambda \in \mathbf{C}$ and $\alpha \in \mathbf{C}^q$.

Next, by using $U(\lambda)$ obtained for $M(\lambda)$ as in the same way to Eq.(1) and an arbitrary $u \in \mathbf{C}^q$, define a complex vector described by $\alpha' := U(\lambda)^{-1}u$. In Eq.(16) we can apply an arbitrary complex vector α, so substituting α' into α in Eq.(16) leads to

$$\|F^+(\lambda)U(\lambda)^{-1}u\|^2 \geq \|F(\lambda)U(\lambda)^{-1}u\|^2 \geq 0 \quad (17)$$

for almost every $\lambda \in \mathbf{C}$. Since $\det|A(0)^T A(0)| \neq 0$, we can observe that $A(\zeta)^T A(\eta)$ satisfies Eq.(13). Hence, it then follows from $\Psi^+(\zeta,\eta) = F^+(\zeta)^T F^+(\eta) = S(\Phi, A(\zeta)^T A(\eta))$ and Lemma 3.1 that $F_+(\lambda)U(\lambda)^{-1}$ is strictly proper. By using this fact, $|\lambda| \to \infty$ in Eq.(17) leads to $\lim_{|\lambda|\to\infty} \|F(\lambda)U(\lambda)^{-1}u\|^2 = 0$. Due to the arbitrariness of u, we can conclude that the all columns of $F(\lambda)U(\lambda)^{-1}$ are strictly proper. This completes the proof under the assumptions (a) and (b).

Secondly, we suppose that (b) does not hold. Define new supply rate induced by

$$|\Phi|(\zeta,\eta) := M(\zeta)^T M(\eta). \tag{18}$$

Note that Eq.(18) also describes the canonical factorization of $|\Phi|(\zeta,\eta)$. Under the assumption (a), it is clear that $|\Phi|(e^{-j\omega}, e^{j\omega}) > 0$ for all $\omega \in [0, 2\pi)$, so this new supply rate satisfies the assumption (b). Thus the maximum storage function for the supply rate $Q_{|\Phi|}(w)$ is induced by

$$\Psi^+_{|\Phi|}(\zeta,\eta) = \frac{|\Phi|(\zeta,\eta) - A_{|\Phi|}(\zeta)^T A_{|\Phi|}(\eta)}{\zeta\eta - 1}$$

where $A_{|\Phi|}(\lambda)$ is an anti-Hurwitz spectral factor of $|\Phi|(\lambda^{-1},\lambda)$. From these facts, the same discussion as stated in the previous subsection allows us to observe that $U(\zeta)^{-T}\Psi^+_{|\Phi|}(\zeta,\eta)\ U(\eta)^{-1}$ is strictly proper with respect to ζ and η.

Here, consider

$$(\zeta\eta - 1)\Psi(\zeta,\eta) \le \Phi(\zeta,\eta) \le |\Phi|(\zeta,\eta). \tag{19}$$

The first inequality is the dissipation inequality for the original supply rate induced by $\Phi(\zeta,\eta)$ and the second one is transparent from the definition of $|\Phi|(\zeta,\eta)$. Eq.(19) also means that $\mathcal{S}(\Phi) \subseteq \mathcal{S}(|\Phi|)$. Thus, $\Psi(\zeta,\eta) \le \Psi^+_{|\Phi|}(\zeta,\eta)$ holds for all $\Psi(\zeta,\eta) \in \mathcal{S}(\Phi)$. Assume that $\Psi(\zeta,\eta) \in \mathcal{S}(\Phi)$ is nonnegative. In addition, let $\Psi(\zeta,\eta) = F(\zeta)^T F(\eta)$ and $\Psi^+_{|\Phi|}(\zeta,\eta) = F^+_{|\Phi|}(\zeta)^T F^+_{|\Phi|}(\eta)$ denote their canonical factorizations. Then, similarly to Eq.(17), $\|F^+_{|\Phi|}(\lambda)U(\lambda)^{-1}u\|^2 \ge \|F(\lambda)U(\lambda)^{-1}u\|^2 \ge 0$ holds for all $u \in \mathbf{C}^q$ and almost every $\lambda \in \mathbf{C}$, i.e., λ satisfying $\det(U(\lambda)) \ne 0$. Since $F_{|\Phi|}(\lambda)U(\lambda)^{-1}$ is strictly proper, taking $|\lambda| \to \infty$ leads to $\lim_{|\lambda|\to\infty} \|F(\lambda)U(\lambda)^{-1}u\|^2 = 0$. Due to the arbitrariness of u, $F(\lambda)U(\lambda)^{-1}$ is strictly proper. This completes the proof under the assumption (a).

Finally, what we have to do is only to eliminate the assumption (a). This elimination process is similar to the proof of Theorem 6.1 in [7] in continuous time. This completes the proof of Theorem 3.1 □.

Remark 3.1 *For a trivial zero matrix, so the assumption (13) holds automatically. This implies that Lemma 3.1 includes the lossless case of* $\tilde{\Delta} = 0$ *which implies* $\Delta(0,0) = 0$. *That is to say, every storage fucntion is a state function of a supply rate in the discrete time lossless case.* □

Both Theorem 3.1 and Lemma 3.1 provide sufficient conditions for a storage fucntion being a state fucntion. In Example 3.1, from the view point of Theorem 3.1, the nonnegative storage function is not a state function. From the view point of Lemma 3.1, $\Psi(\zeta,\eta)$ whose corresponding $\Delta(\zeta,\eta)$ violates Eq.(13) is not a state function. Thus the assertions in both theorems may be also necessary condition. These points are further issues.

Example 3.2 *Again, consider the supply rate induced by Eq.(10). From the viewpoint of Theorem 3.1, one can take* $\Psi(\zeta,\eta) = 1/2 = F(\zeta)^T F(\eta) \ge 0$, *where* $F(\lambda) := 1/\sqrt{2}$ *as nonnegative one of* $\mathcal{S}(\Phi)$. *Indeed, we can observe that* $(\zeta\eta - 1)\Psi(\zeta,\eta) \le \Phi(\zeta,\eta)$ *as follows*

$$\Phi(\zeta,\eta) - (\zeta\eta - 1)\Psi(\zeta,\eta) = \\ = \begin{bmatrix} 1 & \zeta \end{bmatrix} \begin{bmatrix} 3/2 & 1 \\ 1 & 3/2 \end{bmatrix} \begin{bmatrix} 1 \\ \eta \end{bmatrix} \ge 0 \tag{20}$$

Thus, $\Psi(\zeta,\eta) \in \mathcal{S}(\Phi)$. *At the same time, it is clear that* $F(\lambda)U(\lambda)^{-1} = 1/(\sqrt{2}(1+\lambda))$, *is strictly proper, so the storage function induced by* $\Psi(\zeta,\eta)$ *is a state function for* $\Phi(\zeta,\eta)$.

Example 3.3 *Again, consider the supply rate induced by Eq.(10). From the viewpoint of Lemma 3.1, one can take*

$$\Delta(\zeta,\eta) = \begin{bmatrix} 1 & \zeta \end{bmatrix} \begin{bmatrix} \frac{3+\sqrt{5}}{2} & 1 \\ 1 & \frac{3-\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ \eta \end{bmatrix}$$

with its canonical factorization $\Delta(\zeta,\eta) = D(\zeta)^T D(\eta)$ *where* $D(\lambda) := ((\sqrt{5}+1) + (\sqrt{5}-1)\lambda)/2$ *as one of* $\mathcal{D}(\Phi)$ *satisfying Eq.(13). Indeed, we can observe that* $\tilde{\Delta} \ge 0$, *i.e.,* $\Delta(\zeta,\eta) \ge 0$, *and* $\Delta(\lambda^{-1},\lambda) = \Phi(\lambda^{-1},\lambda)$, *so* $\Delta(\zeta,\eta) \in \mathcal{D}(\Phi)$. *In addtion, it is easy to verify that* rank$(\tilde{\Delta}) = 1$ *implying that Eq.(13) holds. We can also observe that* $D(\lambda)U(\lambda)^{-1}$ *is proper. At the same time, by using Eq.(9),* $\Psi(\zeta,\eta) := D(\Phi,\Delta)$ *for this* $\Delta(\zeta,\eta)$ *can be described by* $\Psi(\zeta,\eta) = (1+\sqrt{5})/2 = F(\zeta)\Sigma_\Psi F(\eta)$ *where* $F(\lambda) := ((1+\sqrt{5})/2)^{\frac{1}{2}}$ *and* $\Sigma_\Psi = 1$. *Since* $F(\lambda)U(\lambda)^{-1} = ((1+\sqrt{5})/2)^{\frac{1}{2}}U(\lambda)^{-1}$ *is strictly proper, the storage function induced by* $\Psi(\zeta,\eta)$ *is a state function.*

4 The existence of nonnegative storage functions

In Theorem 3.1, the existence of the nonnegative storage functions is assumed, so we need to consider when there exists a nonnegative storage function for a given supply rate.

Proposotion 4.1 *Let* $\Phi(\zeta,\eta) \in \mathbf{R}_s^{q\times q}[\zeta,\eta]$ *induce a supply rate. Then, the following two conditions are equivalent.*

1). $\sum_{t=\infty}^{T} Q_\Phi(w)(t) \geq 0$ for all $T \in \mathbf{Z}$ and $w \in l_2^q$.

2). There exists a nonnegative storage function induced by $\Psi(\zeta,\eta) \in \mathcal{S}(\Phi)$. □

See [2] for the proof. In continuous time case, as stated in [6], the existence of the nonnegative storage function correponds to the existence of the nonnegative solution of the algebraic Riccati equation. The same result may hold in discrete time. This is one of the further studies.

Here, we consider when a storage function is nonnegative. It follows from Eq.(9) that a storage function induced by $\Psi \in \mathbf{R}_s^{q\times q}[\zeta,\eta]$ is nonnegative if and only if $\Delta(\zeta,\eta) = D(\Phi,\Psi)$ satisfies

$$\sum_{t=\infty}^{T} Q_\Phi(w)(t) \geq \sum_{t=-\infty}^{T} Q_\Delta(w)(t), \qquad (21)$$

for all $T \in \mathbf{Z}$ and $w \in l_2^q$. This means that the storage function, i.e., the stored energy, is nonnegative at any time T if and only if the correponding (total) dissipation energy does not exceed the (total) supplied energy. It follows from Theorem 3.1, that we can observe that a storage function is a state function in the above situation. It is not unnatural to consider that this is a general situation. If the difference between supplied energy and dissipated energy is positive (or nonnegative) the stored energy must to be stored as *net energy* in the dynamical system, i.e., positive (or nonnegative) quadratic forms of state variables.

5 Conclusion

In this paper, for a given supply rate, we consider when a discrete time storage function is a state function of the supply rate with a dynamical system. For this problem, we have shown that every nonnegative storage function is a state function. The detailed proof will be shown in our forthcoming paper ([2]).

Further stuidies are as follows. Firstly, we have to consider what physical meaning the assumption Eq.(13) imposed to dissipation rates have. Secondly, we should claim that a storage function is a state function under the condition which is more general than that in this paper. Finally, we are going to develop the discrete time dissipativeness based on the above studies.

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