A Comparative Study of 7 Algorithms for Model Reduction[1]

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Abstract

In this note, we compare seven model reduction algorithms by applying them to four different dynamical systems. There are four SVD based methods, and three moment matching based methods. The results illustrate that overall, balanced reduction and approximate balanced reduction are the best when we consider whole frequency range. Moment matching methods always lead to higher error norms than SVD based methods due to their local nature; but they are numerically more efficient. Among them, the rational Krylov algorithm gives the best results.

1 Introduction

Many algorithms for model reduction have been developed. One of the most common approaches is balanced model reduction; first introduced by Moore [9]. This method transforms the system to a basis where the states are equally difficult to reach and observe. The reduced model is obtained simply by truncating the states which are the most difficult to reach and observe. Two other closely related model reduction techniques are Hankel norm approximation [10] and the singular perturbation approximation [8], [1]. When applied to stable systems, all of these three approaches are guaranteed to preserve stability and provide bounds on the approximation error. Recently much research has been done to establish a connection between Krylov subspace projection methods used in numerical linear algebra and model reduction [2], [3], [4], [5], [7], [12]. The implicitly restarting algorithm [11] has been applied to obtain stable reduced models [6]. The approximate balancing method introduced in [2] iteratively computes a k^{th} order approximately balanced system without computing the full order balanced model.

In this note, we compare seven model reduction algorithms; namely:

- Balanced Model Reduction
- Approximate Balanced Reduction
- Singular Perturbation Method
- Hankel Norm Approximation
- Arnoldi Procedure
- Lanczos Procedure
- Rational Krylov Method

The methods are applied to four different dynamical systems: We examine the advantages and drawbacks of each algorithm, and also their applicability in different cases.

2 Model Reduction Methods

Throughout this section, let

$$\Sigma = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \tag{2.1}$$

be the full order model where $A \in \mathbb{R}^{n\times n}$ is stable, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{p\times n}$; and let $\Sigma_k = \left[\begin{array}{c|c} A_k & B_k \\ \hline C_k & D_k \end{array}\right]$ be the k^{th} order reduced model where $A_k \in \mathbb{R}^{k\times k}$, $B_k \in \mathbb{R}^{k\times m}$, and $C_k \in \mathbb{R}^{p\times k}$. In this section we briefly describe the algorithms used in this paper (see above.) The first four make use of the Hankel singular values (defined below) and the latter three are based on matching the moments, i.e. the coefficients of the Laurent expansion of the transfer function around some point of the complex plane.

2.1 SVD Based Methods

In this section we will discuss approximation methods which are based on the Hankel singular values of the to-be-approximated system Σ.

Let $\mathcal{P}$ and $\mathcal{Q}$ be the unique Hermitian positive definite solutions to

$$A\mathcal{P} + \mathcal{P}A^T + BB^T = 0, \qquad A^T\mathcal{Q} + \mathcal{Q}A + C^TC = 0.$$

The hankel singular values $\sigma_i(\mathbf{\Sigma})$ of the system Σ are the square roots of the eigenvalues of the product $\mathcal{PQ}$:

$$\sigma_i(\Sigma) = \sqrt{\lambda_i(\mathcal{PQ})}.$$

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2.1.1 Balanced Model Reduction: Let $\mathcal{P} = UU^T$ and $\mathcal{Q} = LL^T$ where U and L are upper and lower triangular matrices respectively. Let $U^T L = ZSY^T$ be the singular value decomposition (SVD) of $U^T L$. A balancing transformation is given by $T_b = S^{\frac{1}{2}} Z^T U^{-1} = S^{-\frac{1}{2}} Y^T L^T$. Then, the Lyapunov equations become

$$A_b S + S A_b^T + B_b B_b^T = 0, \qquad A_b^T S + S A_b + C_b^T C_b = 0.$$

where $A_b = T_b A T_b^{-1}$, $B_b = T_b B$, $C_b = C T_b^{-1}$. We partition $S = \mathrm{diag}(\sigma_i) = \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix}$ with the entries of Σ_2 are small; we also partition the balanced system conformally:

$$\Sigma_b = \left[\begin{array}{c|c} A_b & B_b \\ \hline C_b & D \end{array}\right] = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D_b \end{array}\right] \tag{2.2}$$

where $A_{11} \in \mathbb{R}^{k\times k}$, $B_1 \in \mathbb{R}^{k\times m}$, and $C_1 \in \mathbb{R}^{p\times k}$. The reduced system $\Sigma_k = \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & 0 \end{array}\right]$, of order k has the following properties: A_{11} is stable and the $\mathcal{H}_\infty$ norm of the error system satisfies

$$\|\Sigma - \Sigma_k\|_\infty \le 2(\sigma_{k+1} + \cdots + \sigma_n). \tag{2.3}$$

2.1.2 Approximate Balanced Method: The approximate balanced method [2] solves a Sylvester Equation to obtain a reduced order almost balanced system iteratively without computing the full order balanced realization Σ_b in (2.2).

In the SISO (single-input single-output) case, the cross Grammian X is computed as the solution to the Sylvester equation

$$AX + XA + BC = 0. \tag{2.4}$$

It can be shown that $X^2 = \mathcal{P}\mathcal{Q}$ for this case. In the MIMO case, (2.4) is not even defined unless m=p. Hence, we proceed by embedding the system Σ in a system $\hat{\Sigma}$ which has the same order, is square, and symmetric:

$$\hat{\Sigma} = \left[\begin{array}{c|c} A & \hat{B} \\ \hline \hat{C} & \end{array}\right] = \left[\begin{array}{c|cc} A & JC^T & B \\ \hline C & & \\ B^T J^{-1} & & \end{array}\right], \quad J = J^T.$$

J is called the *symmetrizer*, and is chosen so that $AJ = JA^T$ and $\lambda_i(\mathcal{P}\mathcal{Q}) \approx \lambda_i(\hat{\mathcal{P}}\hat{\mathcal{Q}}) = \lambda(\hat{X})^2$ where $\hat{X}$ is the cross grammian of $\hat{\Sigma}$; for details see [2]. Let $\hat{X} = U\Pi V^T$ be the SVD of $\hat{X}$ and define $\hat{X}_k = U_k \Pi_k V_k^T$ where U_k and V_k are the leading k columns of U, V, respectively, and Π_k is the leading $k \times k$ block of Π; i.e. $\hat{X}_k$ is the best rank k approximation to $\hat{X}$. Let $\hat{X}_k = Z_k D_k W_k^*$ where $W_k^* Z_k = I_k$ be a partial eigenvalue decomposition of $\hat{X}_k$. Then an approximately balanced system is obtained as $\Sigma_k = \left[\begin{array}{c|c} W_k^* A_k Z_k & W_k^* B_k \\ \hline C Z_k & \end{array}\right]$. The advantage of the approximate balancing method is that it computes an *almost* balanced reduced system *iteratively* without computing a balanced realization of the full order system first, followed by truncation. For details on the implementation of this algorithm, see [2].

2.1.3 Singular Perturbation Approximation: From (2.2) follows the singular perturbation approximation:

$$\Sigma_k = \left[\begin{array}{c|c} A_{11} - A_{12}A_{22}^{-1}A_{21} & B_1 - A_{12}A_{22}^{-1}B_2 \\ \hline C_1 - C_2 A_{22}^{-1} A_{21} & D - C_2 A_{22}^{-1} B_2 \end{array}\right].$$

The $\mathcal{H}_\infty$ norm of the error system satisfies the same inequality (2.3) as in balanced reduction case.

2.1.4 Hankel Norm Approximation: Let Σ be given as in (2.1). Then, there exist a dynamical system $\tilde{\Sigma}$ so that the system $\Sigma - \tilde{\Sigma}$ is all pass, and $\|\Sigma - \tilde{\Sigma}\|_{\mathcal{L}_\infty} = \sigma_{k+1}(\Sigma)$. $\tilde{\Sigma}$ has exactly k stable poles. Decompose $\tilde{\Sigma}$ in a stable part Σ_+ and anti-stable part Σ_-: $\tilde{\Sigma}$=Σ_++Σ_-. Then Σ_+ is an optimal Hankel norm approximation of order k of the system Σ. The $\mathcal{H}_\infty$ norm of the error system satisfies (2.3).

2.2 Moment matching methods

Given Σ as in (2.1), we expand the transfer function $G(s) = C(sI - A)^{-1}B + D$ around some $s_0 \in \mathbb{C}$:

$$G(s) = \eta_0 + \eta_1(s - s_0) + \eta_2(s - s_0)^2 + \eta_3(s - s_0)^3 + \cdots.$$

The η_j, $j \ge 0$'s are called the *moments* of Σ at s_0. Then the moment matching approximation problem consists in finding a reduced order model Σ_k with

$$G_k(s) = \hat{\eta}_0 + \hat{\eta}_1(s - s_0) + \hat{\eta}_2(s - s_0)^2 + \hat{\eta}_3(s - s_0)^3 + \cdots.$$

such that for an appropriate k there holds: $\eta_j = \hat{\eta}_j$ $j = 0, 1, 2, \cdots, k$. This problem can be solved in a recursive and numerically reliable way. For the special case where $s_0 = \infty$; i.e. the expansion is around infinity, the moments are indeed the Markov parameters of Σ and the moment matching problem is the problem of partial realization. In this case solutions can be computed by means of the Lanczos and Arnoldi procedures briefly described below. For an arbitrary $s_0 \in \mathbb{C}$, the problem becomes the Rational Interpolation problem and solutions can be computed by means of the rational Krylov method described in Section 2.2.3 below.

2.2.1 The Arnoldi Procedure: Let the full order model Σ in (2.1) be SISO, i.e., $p = m = 1$. The k step Arnoldi process generates matrices $V \in \mathbb{R}^{n\times k}$, $H \in \mathbb{R}^{k\times k}$, and $f \in \mathbb{R}^n$ so that

$$AV = VH + f e_k^T, \; V^T V = I_k, \; V^T f = 0$$

where H is an upper Hessenberg matrix, $Ve_1 = \frac{B}{\|B\|}$ and e_k denotes the k^{th} unit vector. The reduced model is obtained by the following projection:

$$\Sigma_k = \left[\begin{array}{c|c} V^T A V & V^T B \\ \hline C V & D \end{array}\right]$$

2.2.2 The Lanczos Procedure: Again, let the full order model Σ in (2.1) be SISO, i.e., $p = m = 1$. Then the two-sided Lanczos algorithm generates matrices $V, W \in \mathbb{R}^{n \times k}$, $H \in \mathbb{R}^{k \times k}$, and $f \in \mathbb{R}^n$ so that

$$AV = VT + fe_k^T,\ A^T W = WT^T + ge_k^T\ ,$$
$$V^T W = I_k,\ V^T g = W^T f = 0$$

where T is a tridiagonal matrix. The reduced model is obtained by the following projection:

$$\hat{\Sigma} = \left[\begin{array}{c|c} V^T A W & V^T B \\ \hline C W & D \end{array}\right]$$

We note that both the Arnoldi and Lanczos methods do not guarantee stability. But using the implicit restarting algorithm [11], we can obtain stable approximants. Also there exist no global error bounds.

Remark: There are block versions of the previous two algorithms for MIMO systems. Moreover, for the Lanczos procedure to overcome break-downs due to zero k^{th} moment, there exists a so-called "look-ahead Lanczos" algorithm. We do not make use of these in this note. In case of break-downs, the shifted Lanczos and Arnoldi algorithms can be used instead.

2.2.3 The Rational Krylov Method: The rational Krylov Method is a generalized version of the standard Lanczos method. Given a dynamical system Σ, a set of interpolation points $w_1, \cdots, w_l$, and an integer N, the rational Krylov algorithm produces a reduced order system Σ_k that matches N moments of Σ at $w_1, \cdots, w_l$. The reduced system is not guaranteed to be stable and no global error bounds exist. Moreover the selection of interpolation points which determines the reduced model is not an automated process and has to be figured out by the user using trial and error. However, this algorithm can be applied to systems of very high order.

3 Analysis of the Reduction Algorithms

In this section we apply the algorithms mentioned above to four different dynamical systems: a structural model, a heat transfer model, a clamped beam, and a low-pass Butterworth filter. We reduce the order of the models approximately to the $\frac{1}{10}$ except for the Butterworth example. Table-1 shows the order of the systems, n; the number of inputs, m, and outputs, p; the order of reduced systems, k and the corresponding tolerance[1] value ρ. Moreover, the normalized[2] Hankel singular values of each model are depicted in Figure 1. To make a better comparison between the systems, in Figure 2 we also show relative degree reduction $\frac{k}{n}$ vs a given error tolerance $\frac{\sigma_k}{\sigma_1}$. This figure shows how much the order can be reduced for the given tolerance: the lower the curve, the easier to approximate. It can be seen from Figure 2 that among all models for a fixed tolerance value less than 1.0×10^{-5}, the structural model is the hardest one to approximate. One should notice that specification of the tolerance value ρ determines everything in all of the methods except the rational Krylov method. The order of the reduced model and the eigenvalue placement are completely automatic. On the other hand, in the rational Krylov method, one has to choose the interpolation points and the the number of moments N which are matched per point. In each subsection below, we briefly describe the systems and then apply the algorithms. For each example the Hankel singular values, the largest singular value of the frequency response of the full order model, the reduced order model , and of the corresponding error systems are shown. Moreover, the relative $\mathcal{H}_\infty$ and $\mathcal{H}_2$ norms of the error systems are tabulated.

	n	m	p	k	ρ
Structural Model	270	3	3	30	4.30×10^{-3}
Heat Model	197	2	2	20	2.37×10^{-8}
Clamped Beam	348	1	1	40	6.55×10^{-6}
Butterworth Flt.	100	1	1	35	1.00×10^{-3}

Table 1: Systems for Testing

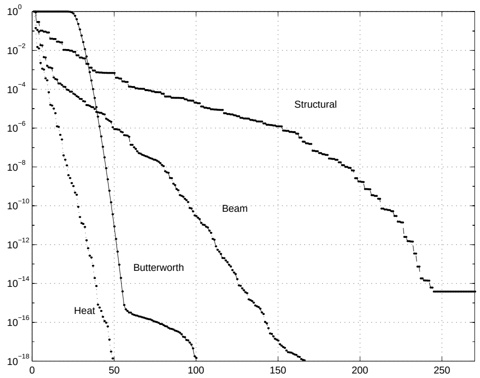

Figure 1: Normalized Hankel singular values of all models

3.1 Structural Model

This is a model of component 1r (Russian service module) of the International Space Station. It has 270 states, 3 inputs and 3 outputs. The real part of the pole closest to imaginary axis is -3.11×10^{-3}. The normalized Hankel singular values of the system are

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[1] Tolerance value corresponding to a k^{th} order reduced system is given by the ratio $\frac{\sigma_k}{\sigma_1}$ where σ_1 and σ_k are the largest and k^{th} singular value of the system respectively.

[2] For comparison, we normalize the highest Hankel singular value of each system to 1.

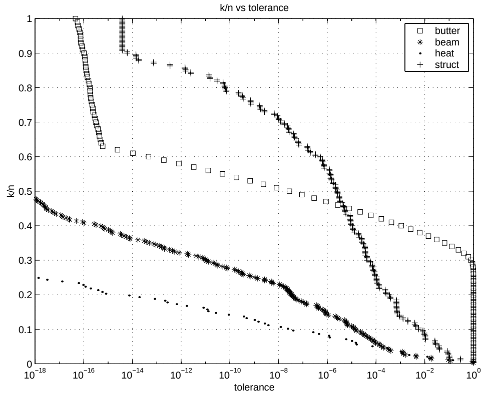

Figure 2: Relative degree reduction $\frac{k}{n}$ vs tolerance $\frac{\sigma_k}{\sigma_1}$

shown in Figure 1-a. We approximate the system with reduced models of order 30. This corresponds to a tolerance values of 4.3×10^{-3}. Since the system is MIMO, the Arnoldi and Lanczos Algorithms do not apply. The resulting reduced order systems are shown in Figure 3-a. We note that since the reduced models obtained by singular perturbation and balanced reduction are very close to each other, we do not depict the singular perturbation approximation. As seen from the figure, all the models work quite well. The peaks, especially the ones at the lower frequencies, are well approximated. Figure 3-b shows the largest singular value of the frequency response of the error systems. Rational Krylov does a perfect job at the lower and higher frequencies. But for the moderate peak frequency levels, it has the highest error amplitude. This is because of the fact that the selection of interpolation point is not an automated process and relies on ad-hoc specification by the user. Hankel norm approximation seems to be the worst for low and higher frequencies. On the other hand, at moderate frequencies the error systems for SVD based methods are very close. Table 2 lists the relative[3] $\mathcal{H}_\infty$ and $\mathcal{H}_2$ norms of the errors system. As seen from the figure, rational Krylov has the highest error norms. Hankel norm approximation and approximate balancing are the best in terms of $\mathcal{H}_\infty$ and $\mathcal{H}_2$ norms respectively. Considering the whole frequency range, balanced reduction and approximate balanced reduction are the best.

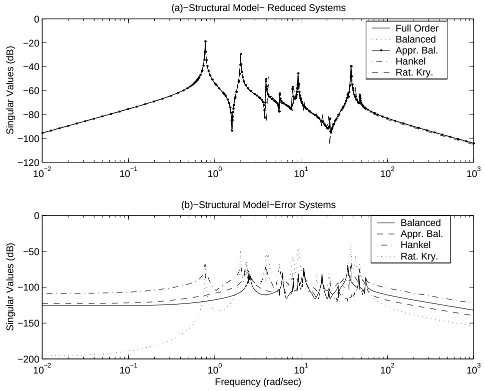

Figure 3: The largest singular value of the frequency response of (a) the reduced systems (b) the error systems of the structural model

	$\mathcal{H}_\infty$ norm	$\mathcal{H}_2$ norm
Balanced	3.89×10^{-3}	2.18×10^{-2}
Approx. Balanced	4.17×10^{-3}	9.98×10^{-3}
Hankel Norm App.	3.86×10^{-3}	4.10×10^{-2}
Singular Pertur.	3.89×10^{-3}	2.18×10^{-2}
Rational Krylov	2.45×10^{-2}	4.17×10^{-1}

Table 2: Relative error norms for the structural model

3.2 Heat Transfer Model

The original system is a plate with two heat sources and measurements at two points on the plate. It is described by the heat equation. A model of order 197 is obtained by spatial discretization. The system is stable. The real part of the pole closest to imaginary axis is -1.52×10^{-2}. It is observed from Figure 1-a that this system is very easy to approximate since the Hankel singular values decay very rapidly. We approximate the model with a model of order 20 which corresponds to a tolerance value of 2.37×10^{-8}. Since this is a MIMO system, Lanczos and Arnoldi do not apply. As expected due to the very low tolerance value, all methods generate satisfactory approximants matching the full order model through the whole frequency range (see Figure 4). Only the rational Krylov method has some problems for moderate frequencies due to the unautomated choice of interpolation points.

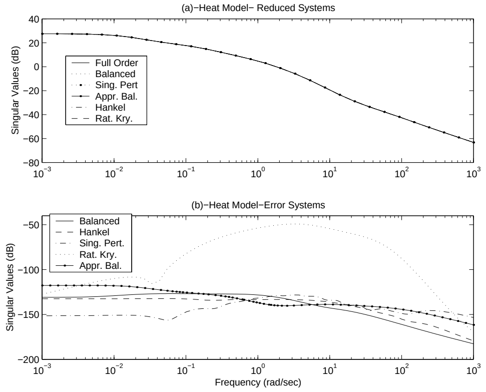

Figure 4: The largest singular value of the frequency response of (a) the reduced systems (b) the error systems of the heat model

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[3] To find the relative error, we divide the norm of the error system with the corresponding norm of the full order system.

	$\mathcal{H}_\infty$ norm	$\mathcal{H}_2$ norm
Balanced	2.12×10^{-8}	1.22×10^{-7}
Approx. Balanced	5.49×10^{-8}	2.35×10^{-7}
Hankel Norm App.	1.01×10^{-8}	1.91×10^{-6}
Singular Pertur.	1.63×10^{-8}	3.64×10^{-6}
Rational Krylov	1.46×10^{-4}	1.86×10^{-3}

Table 3: Relative error norms of the heat model

3.3 Clamped Beam Model

It is the model of a beam clamped at one end. The force applied at the free end is the input and the resulting displacement is the output. The model we used has 348 states, is SISO, and is again obtained by spatial discretization of an appropriate partial differential equation. For this example, the real part of the pole closest to imaginary axis is -5.05×10^{-3}. We approximate the system with a model of order 40 corresponding to a tolerance value of 6.55×10^{-6} as can be seen from Figure 1-a. Approximate balanced method results in almost the same reduced model as balanced reduction. Hence we show the plots only for balanced reduction. The plot of the largest singular value of the frequency response of the approximants and error systems are shown in Figure 5-a and 5-b respectively. Since $CB = 0$, we expand the transfer function $G(s)$ of the original system around $s_0 = 0.1$ instead of $s_0 = \infty$ to prevent the breakdown of Lanczos. We also use the shifted version of Arnoldi procedure with $s_0 = -1$ rad/sec. As can be seen, all the algorithms generate reduced order models which approximate the full order model quite well for the whole frequency range. Rational Krylov is again the best one for both $s = 0$ and $s = \infty$. Indeed except for the frequency range between 0.6 and 30 rad/sec, this method gives the best approximant. The Arnoldi and Lanczos procedures also lead to very good approximants especially for the frequency range $0 - 1$ rad/sec. This is due to the choice of s_0 as a low frequency point. Among the SVD based methods, Hankel norm approximation has the highest error around $s = 0$. Balanced model reduction is the best one when we consider the whole frequency range as can be seen from the error curves. In terms of error norms, SVD based methods are much better than moment matching based eethods. In spite of having the least $\mathcal{H}_\infty$ error norm, Hankel norm approximation has again the highest $\mathcal{H}_2$ error norm among the SVD based methods.

3.4 Low-Pass Butterworth Filter

The full order model is a low-pass Butterworth filter of order 100 with cutoff frequency 1 rad/sec. The normalized Hankel singular salues corresponding to this system are shown in Figure 1-a and Figure 1-b. It should be noticed that unlike the other systems, Hankel singular vaalues stay constant at the beginning, and then start to decay. Therefore, we cannot reduce the model to order less than 25 using an SVD based method. Hence we approximate the system with a model of order 35 corresponding to a tolerance value of 1×10^{-3}. One should notice that the transfer function of this example has no zeros. Thus Arnoldi and Lanczos procedures do not work if we expand the transfer function $G(s)$ around $s = \infty$. Instead, we expand $G(s)$ around $s_0 = 0.1$. As Figure 6-a illustrates, all the moment matching based methods have difficulty especially around the cutoff frequency. Among them, Lanczos and Arnoldi show very similar results and are better than rational Krylov method. On the other hand, SVD based methods work without any problem producing quite good approximants for the whole frequency range. Rational Krylov has the highest $\mathcal{H}_\infty$ and $\mathcal{H}_2$ error norms. And, Hankel norm approximation is again the best in terms $\mathcal{H}_\infty$ norm but the worst in terms of $\mathcal{H}_2$ norm among the SVD based methods.

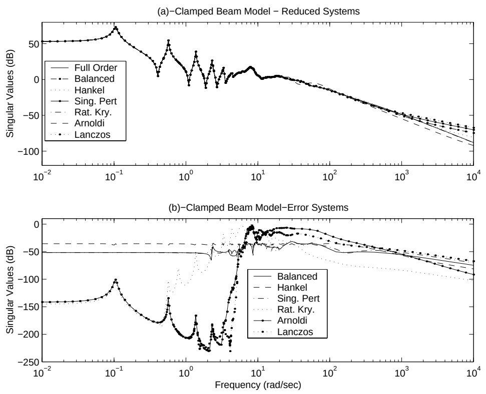

Figure 5: The largest singular value of the frequency response of (a) the reduced systems (b) the error systems of the clamped beam model

	$\mathcal{H}_\infty$ norm	$\mathcal{H}_2$ norm
Balanced	5.88×10^{-6}	2.91×10^{-4}
Approx. Balanced	5.88×10^{-6}	2.91×10^{-4}
Hankel Norm App.	4.17×10^{-6}	4.44×10^{-4}
Singular Pertur.	6.81×10^{-6}	3.57×10^{-4}
Rational Krylov	2.34×10^{-4}	1.91×10^{-3}
Arnoldi	1.65×10^{-4}	5.14×10^{-3}
Lanczos	1.19×10^{-4}	1.90×10^{-4}

Table 4: Relative error norms of the clamped beam model

	$\mathcal{H}_\infty$ norm	$\mathcal{H}_2$ norm
Balanced	6.29×10^{-4}	5.19×10^{-4}
Approx. Balanced	6.29×10^{-4}	5.19×10^{-4}
Hankel	5.68×10^{-4}	1.65×10^{-3}
Singular Pertur.	6.33×10^{-4}	5.21×10^{-4}
Rational Krylov	1.02×10^{0}	4.44×10^{-1}
Arnoldi	1.02×10^{0}	5.38×10^{-1}
Lanczos	1.04×10^{0}	3.68×10^{-1}

Table 5: Relative error norms of the Butterworth filter

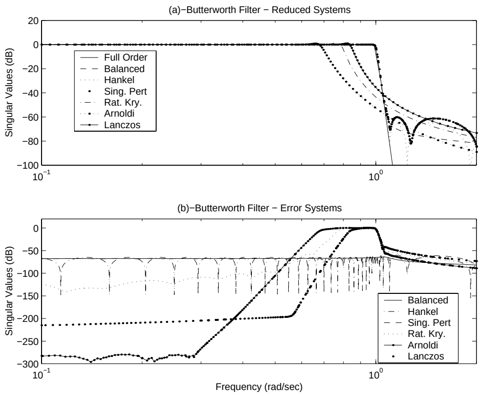

Figure 6: The largest singular value of the frequency response of (a) the reduced systems (b) the error systems of the Butterworth filter

4 Conclusion

In this note we presented a comparative study of seven algorithms for Model Reduction: balanced model reduction, approximate balanced reduction, singular perturbation method, Hankel norm approximation, Arnoldi procedure, Lanczos procedure, and rational Krylov method- by applying them to four different dynamical systems. The first four make use of Hankel singular values and the latter three are based on matching the moments; i.e. the coefficients of the Laurent expansion of the transfer function around some point of the complex plane. The results show that balanced reduction and approximate balanced reduction are the best when we consider the whole frequency range. Between these two, approximate balancing has the advantage that it computes an *almost* balanced reduced system *iteratively* without obtaining a balanced realization of the full order system first, and truncating subsequently, thus reducing the computational cost and storage requirements. Hankel norm approximation gives the worst approximation around $s = 0$ among the SVD based methods. Although it has the lowest $\mathcal{H}_\infty$ error norm in most of the cases, it leads to the highest $\mathcal{H}_2$ error norm. Being local in nature moment matching methods always lead higher error norms than SVD based methods; but they reduce the computational cost and storage requirements remarkably when compared to the latter. Among them, the rational Krylov algorithm gives better results with the flexibility of the selection of interpolation points. However, the selection of the points which determines the reduced model is not an automated process and has to be specified by the user unlike the other methods where a given tolerance value determines everything.

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