Mixed H_∞/H_2 Design of Digital Phase Locked Loops with Polytopic Type Uncertainties

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Abstract

A robust H_∞ control method is applied to the design of loop filters for digital phase locked loop carrier phase tracking. The proposed method successfully copes with large S-curve slope uncertainty and with a significant decision delay in the closed loop that may stems from the decoder and/or the equalizer there. The design problem is transformed into a state-feedback control problem where phase and gain margins should be guaranteed in spite of the uncertainty. Of all the loop filters that achieve the required margins the one that minimizes an upper-bound on the effect of the phase and the measurement noise signals is derived.
Keywords:PLL design, robust synchronization, mixed H_2/H_∞ control for polytopic uncertainty

1 Introduction

Digital Phase Locked Loop (DPLL) has been successfully applied to tracking of carrier phase, symbol (bit) synchronization systems and frequency synthesis [9],[10]. For example, in all applications pertaining to coherent telecommunications it is necessary to first reconstruct a carrier reference from a noise corrupted version of received signal [11]. DPLL that are based on sampling in a frequency much higher than the loop band-width can be approximately designed like continuous-time filters [14]. A continuous time approximation is not valid if the loop band-width approaches the sampling frequency. In that case the discrete-time nature of the loop must be recognized. One of the problems that arise in PLL design is phase noise effect due to the oscillator instability [12],[13]. Another problem in the design of DPLL is uncertainty that occurs in the loop gain. The problem then becomes one of robust control, where a controller should be applied on the output of the phase detector so as to satisfy the stability and the noise attenuation requirements, in spite of the uncertainties in the loop gain.
Recently, a simple design technique has been introduced which achieves the above requirements in presence of large parameter uncertainties [2]. In the present note we apply this technique on the loop filter design combined with new technique of [3]. We show that the resulting filter can achieve excellent noise suppression and preassigned stability margins in spite of the large gain uncertainty. We apply a new iterative algorithm to the results of [2] and [3] in order to further improve the performance index. The paper is organized as follows: Section 2 describes the problem to be solved. In Section 3 we bring the theory behind our method and in Section 4 we apply this theory to the design of specific DPLLs.

2 Statement of the Problem

We consider the linear model of the PLL feedback system that is schematically depicted in Fig. 1. where:

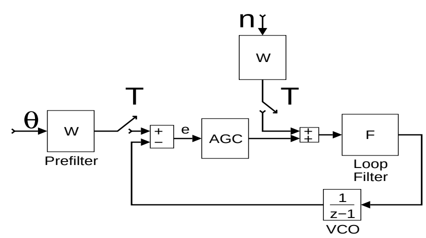

Figure 1: The PLL model

W denotes a simple average integral reset prefilter $W = \frac{1}{T}\int_{(n-1)T}^{nT} f(t)dt$, the sampling period is T, F is the digital filter to be designed, n is a zero mean, Gaussian, white noise vector with the spectral density Φ_n, and θ is the phase-noise with the spectral density Φ_θ. It is assumed that the phase-noise is generated by a linear time-invariant system driven by white noise, and that it is uncorrelated with n. The gain of the AGC (Automatic Gain Control) is the gain g of the phase-detector. It is fixed, but 'unknown', and is assumed to belong to a given interval $g \in [g_1, g_2]$.
The objective is to minimize the RMS value of e

$$\overline{e} = var\{e(kT)\}. \qquad (1)$$

The open loop response should have some gain and phase margins in order to guarantee a well damped closed loop response, defined here by a constant $\gamma > 0$ such that:

$$\left|\frac{L(e^{j\omega})}{1+L(e^{j\omega})}\right| \preceq \gamma \quad \forall\omega \in [02\pi] \ \ and \ \ g \in [g_1, g_2]. \qquad (2)$$

where $L(z) = gF(z)\frac{1}{z-1}$ is the open loop transfer function.

3 The Solution Method

The problem of minimizing of the worst possible $\overline{e}$ of (1) over all $g \in [g1, g2]$, while satisfying (2), is a mixed

H_2/H_∞ robust output-feedback control problem. Unlike the corresponding state-feedback problem, which exhibits convex properties, the output-feedback problem has no apparent solution. The method of [2] circumvents the difficulty of solving the output-feedback problem by transforming the latter into an equivalent state-feedback problem. This is done by transforming the difference equation that describes the relation between an input and an output of a plant to an augmented, non minimal, state-space model on which a state-feedback design technique can be applied. Namely, denote by z^{-1} the Z-transform of a unit delay. We consider a single-input-single-output, linear, discrete, time-invariant plant whose transfer function is described by:

$$y(z) = \frac{b_0 z^m + b_1 z^{m-1} + \ldots b_{m-1} z + b_m}{z^n + a_1 z^{n-1} + \ldots a_{n-1} z + a_n} u(z) \tag{3}$$

where y is the measurement output and u is the control input. Our purpose is to design an output-feedback controller, $u = H(z)y$, that will achieve prescribed performance requirements. Consider the case when $n - m = 1$ (other cases can be treated by taking the appropriate coefficients equal to zero). Multiplying (3) by z^{-n+1}, in both the denominator and the numerator, we readily obtain the following difference equation that describes the relation between y and u:

$$\begin{gathered} y_{k+1} = -a_1 y_k - a_2 y_{k-1} - \ldots - a_n y_{k-n+1} + \\ b_0 u_{k+m-n+1} + b_1 u_{k+m-n} + \ldots + b_m u_{k-n+1} \end{gathered} \tag{4}$$

A phase-canonical state-space representation to the system of (4) is achieved by applying the state vector: $\xi = [y_k \quad y_{k-1} \ldots y_{k-n+1} \quad u_{k-1} \ldots u_{k-n+1}]^T$ and writing the state-space model:

$$\xi_{k+1} = A\xi_k + B_2 u_k + B_1 \omega_k, \quad z_k = Lx_k + Du_k \tag{5}$$

where: ω describes an exogenous disturbance in L_2 and

$$A = \begin{bmatrix} -a_1 & -a_2 & \ldots & -a_n & b_1 & \ldots & b_{m-1} & b_m \\ 1 & 0 & 0 & \ldots & & & & 0 \\ 0 & 1 & 0 & \ldots & & & & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & & \ldots & & & & 0 \\ 0 & 0 & & \ldots & 1 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & & 0 & 1 & 0 \end{bmatrix},$$

$$B_1^T = [\ 1 \quad 0 \quad 0 \quad \ldots \quad 0 \quad 0 \quad \ldots \quad 0\ ].$$

$$B_2^T = [\ b_0 \quad 0 \quad 0 \quad \ldots \quad 1 \quad 0 \quad \ldots \quad 0\ ].$$

The matrices A, B_1, B_2, C, L and D are usually functions of the uncertain parameters in the system model. The matrices L and D define an objective vector z_k according to the solved problem.

On the resulting model the standard H_∞ state-feedback design technique of [1] can be applied. Assuming that, due to uncertainty the parameters of the system, the quintet (A, B_1, B_2, L, D) is known to lie in a polytope Ω with N vertices, we obtain the following result:

Lemma 3.1 *[1] The H_∞-norm of the system (5) is less than a prescribed scalar $\gamma > 0$ for the all the points in Ω if there exists $P \in R^{(2n-1)\times(2n-1)}$, $P = P^T > 0$, that satisfies for $i = 1, \ldots, N$:*

$$\begin{bmatrix} -P & A_i P - B_{2i} Y & 0 & 0 & B_{1i} \\ P^T A_i^T - Y^T B_{2i}^T & -P & P^T L_i^T & Y^T D_i^T & 0 \\ 0 & L_i P & -\gamma I & 0 & 0 \\ 0 & D_i Y & 0 & -\gamma I & 0 \\ B_{1i}^T & 0 & 0 & 0 & -\gamma I \end{bmatrix} < 0, \tag{6}$$

where (A, B_1, B_2, L, D) correspond to the i-th vertex of the polytope Ω. If such P exists, the state-feedback control low that achieves for all the points in Ω

$$\|z\|_2^2 - \gamma^2 \|\omega\|_2^2 \le 0 \quad \forall \omega \in L_2 \tag{7}$$

is obtained by:

$$u = -K\xi \tag{8}$$

with

$$K \triangleq [k_1 \ldots k_{2n-1}] = YP^{-1}.$$

The feedback rule of (8) can be easily transformed into an output-feedback controller:

$$H(z) = \frac{k_1 z^{n-1} + k_2 z^{n-2} + \ldots + k_n}{z^{n-1} + k_{n+1} z^{n-2} + \ldots + k_{2n-1}}. \tag{9}$$

The LMI (Linear Matrix Inequalities) result of (6) demands (because of the requirement for a constant P for all i=1,...,N) the quadratic stability of the system [6]. This constraint of quadratic stability may be quite conservative. It can be eleviated by applying the technique of [3]. It is shown in [3] that instead of solving for a constant P in (6) it is enough to look for $P_i = P_i^T > 0$, $Z \in R^{(2n-1)\times(2n-1)}$ and $Y \in R^{2n-1}$ that satisfy for $i = 1, \ldots N$:

$$\begin{bmatrix} -P_i & A_i Z - B_{2i} Y & 0 & 0 & B_{1i} \\ Z^T A_i^T - Y^T B_{2i}^T & P_i - Z - Z^T & Z^T L_i^T & Y^T D_i^T & 0 \\ 0 & L_i Z & -\gamma I & 0 & 0 \\ 0 & D_i Y & 0 & -\gamma I & 0 \\ B_{1i}^T & 0 & 0 & 0 & -\gamma I \end{bmatrix} < 0, \tag{10}$$

If a solution to the latter exists then the required state-feedback is given by:

$$K = YZ^{-1}. \tag{11}$$

In spite of the fact that (10) is based on an 'improved' BRL, it still entails an over design since (10) requires Z to be constant. Assuming that (10) has a solution for a minimum gamma of γ_z, a further improvement of the minimum achievable γ is achieved by the following method. We first choose a finite subset of Ω that consists of the points $(A_\theta, B_{1\theta}, B_{2\theta}, L_\theta, D_\theta)$, $\theta = 1, \ldots, \bar{N}$. We denote this subset by Γ. The method bellow will guarantee that (7) is satisfied for all the points in Γ. Like other control design techniques (e.g. QFT [8] or gridding in gain scheduling [15]), if Γ is chosen dense enough one can achieve the prescribed performance for all points in the set.

In order to guarantee the performance of (7) for all the points in Γ, for a minimal possible $\gamma_z \ge \gamma > 0$, we require, similarly to (6), that for $\theta = 1, \ldots, \bar{N}$:

$$\begin{bmatrix} -P_\theta & A_\theta - B_{2\theta} K & 0 & 0 & B_{1\theta} \\ A_\theta^T - K^T B_{2\theta}^T & -P_\theta^{-1} & L_\theta^T & K^T D_\theta^T & 0 \\ 0 & L_\theta & -\gamma I & 0 & 0 \\ 0 & D_\theta K & 0 & -\gamma I & 0 \\ B_{1\theta}^T & 0 & 0 & 0 & -\gamma I \end{bmatrix} < 0.$$

with the objective $J = \min \gamma$. (12)

Unfortunately, the matrix inequality in (12) is bilinear in P_θ and K (it becomes bilinear by multiplying the second row and column of (12) by from both sides by P_θ). We resolve this difficulty using the following lemma on (12):

Lemma 3.2 *Given symmetric matrices P and X such that*

$$P + X > 0, \quad and \quad det\{P\} \neq 0.$$

Then,

$$(P+X)^{-1} \geq P^{-1} - P^{-1}XP^{-1} > 0$$

Proof: $(P+X)^{-1} = (I + P^{-1}X)^{-1}P^{-1} = (I - P^{-1}X(I + P^{-1}X)^{-1})P^{-1}$

$$= (I - P^{-1}X(I - P^{-1}X(I + P^{-1}X)^{-1}))P^{-1}$$

$$= P^{-1} - P^{-1}XP^{-1} + P^{-1}X(P+X)^{-1}XP^{-1} \geq P^{-1} - P^{-1}XP^{-1}.$$

It follows from the Lemma 2.2 that a solution to (12) exists if there exist P_θ, X_θ that satisfy for $\theta = 1, ..., \bar{N}$:

$$\Omega(P_\theta, X_\theta, K) =$$

$$\begin{bmatrix} -(P_\theta + X_\theta) & A_\theta - B_{2\theta}K & 0 & 0 & B_{1\theta} \\ A_\theta^T - K^T B_{2\theta}^T & -(P_\theta^{-1} - P_\theta^{-1}X_\theta P_\theta^{-1}) & L_\theta^T & K^T D_\theta^T & 0 \\ 0 & L_\theta & -\gamma I & 0 & 0 \\ 0 & D_\theta K & 0 & -\gamma I & 0 \\ B_{1\theta}^T & 0 & 0 & 0 & -\gamma I \end{bmatrix} < 0,$$

$$J = \min \gamma. \qquad (13)$$

The latter inequality is not linear in P_θ. It has, however, the advantage that it does not require a constant P and that is amenable for iterations. The following convergent procedure is suggested for finding P_θ, K, $\theta = 1, ..., \bar{N}$ that satisfy (12).

3.1 The Proposed Algorithm:

- Step 1: Derive an initial value for all P_θ that minimizes, under the requirement of quadratic stability, the value of γ using the methods of either [1] or [3]. If the method of [3] is used, then the initial value P_θ^0 for P_θ, for any point in Γ is obtained by the appropriate combination of the P_i^0 that were achieved for the vertices of Ω. We denote the stae-feedback gain that is obtained in (11) by K^0.
- Step 2: Solve (13), in say the i-th iteration step, based on P_θ^{i-1} that has been found previously. Find K^i and X_θ^i for all $\theta = 1, .., \bar{N}$ that satisfy (13) and achieve a minimum value of γ_i for γ.
- Step 3: Calculate $P_\theta^i = P_\theta^{i-1} + X_\theta^i$
- Step 4: Return to step no 2, and increase i by one, if a prescribed convergence tolerance test fails.
- Step 5: When the tolerance test is successful, use the resulting K^i in (9) to provide the output-feedback that solves the problem for all the points in Γ.

The latter iteration converges and provides K for a minimum value of γ that less or equal to γ_z. This is due the fact that $\{\gamma_i\}$ is a nonincreasing sequence that is bounded from below by zero.

The above algorithm, combined with [3], provides a robust output-feedback control design solution that is less conservative than those suggested in [1].

Remark 3.3 *The proposed method can be used not only to reduce the minimum achievable disturbance attenuation level γ, but also to minimize bound on the H_2-norm in a mixed H_∞/H_2 problem (see eq. (22) below).*

Remark 3.4 *The controller H that is obtained by (9) is not necessarily stable. Denoting*

$$K^i \triangleq \begin{bmatrix} k_{n+1}^i & k_{n+2}^i & . & . & k_{2n-1}^i \end{bmatrix}$$

where K^i is the state-feedback gain obtained in the i-th iteration step above, and defining

$$A_k^i = \begin{bmatrix} -k_{n+1}^i & -k_{n+2}^i & \dots & -k_{2n-2}^i & -k_{2n-1}^i \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} \qquad (14)$$

one can augment the above method to secure the stability of the controller if A_k^0 is a stability matrix, namely, if there exists $\tilde{P}^0 = (\tilde{P}^0)^T \in \mathcal{R}^{(n-1)\times(n-1)}$ that satisfies

$$\begin{bmatrix} -\tilde{P}^0 & A_k^0 \\ A_k^{0\,T} & -(\tilde{P}^0)^{-1} \end{bmatrix} < 0. \qquad (15)$$

In the latter case, the stability of the controller that is achieved in the i-th iteration step of our algorithm is maintained if, in addition to (13), we also find $\tilde{x}^i$ that satisfies the following LMI:

$$\begin{bmatrix} -(\tilde{P}^{i-1} + \tilde{X}^i) & A_k^i \\ (A_k^i)^T & -(\tilde{P}^{i-1})^{-1} + (\tilde{P}^{i-1})^{-1}\tilde{X}^i(\tilde{P}^{i-1})^{-1} \end{bmatrix} < 0 \qquad (16)$$

where, for $i \geq 1$, $\tilde{P}^i = \tilde{P}^{i-1} + \tilde{X}^i$.

Details of the proposed method are described in the following section, where a specific PLL design method is introduced that minimizes a bound on the variance of the effect the phase and the measurement noise signals have on the loop.

4 Robust Design of a PLL with AGC

In this section we apply the method of Section 2 to design two PLL systems with AGC. The first problem considers a PLL without delay, while the second copes with a delay of three sampling periods.

In both problems we treat the system of [4] where $\Phi_\theta = (2.75 \cdot 10^7/\omega^2)^2$, $\Phi_n = 10^{-9.3}$, and $T = 0.5 \cdot 10^{-6}$. The parameters in this example are given in units of radians and seconds. The gain g is assumed to attain any value in the interval $\Omega = [1, 4]$. The stability margins constraint of (2) requires $\gamma < 3dB$, which guarantees at least $\frac{\pi}{4}$ phase- margin and a $5dB$ gain margin for all the allowed values of the parameter g.

In the Appendix we find that the sequence $\{\theta_k\}$ in Fig 1 can be modeled as an output of a three-input-one-output discrete time system that is driven by a white noise (standard) in R^3 (A.1). Substituting in (A.1) the values of T and ν that correspond to our problem we obtain the following transfer function matrix between θ_k and ψ_k:

$$T_\theta(z) = [T_{\theta 1}(z)\ T_{\theta 2}(z)\ T_{\theta 3}(z)] =$$

$$= 10^{-3} \begin{bmatrix} \frac{1.1z^2 + 0.3z - 1.4}{z(z-1)^2} & \frac{1.6z^2 + 6.5z - 1.6}{z(z-1)^2} & \frac{899.5z^2 - 662.6z - 236.9}{z(z-1)^2} \end{bmatrix}.$$

We denote

$$T_{\theta i} = \frac{N_{i1}z^2 + N_{i2}z + N_{i3}}{z(z-1)^2}, \quad i = 1, 2, 3. \qquad (17)$$

Based on standard results [5], $\{n_k\}$ in Fig 1 is white with the variance: $var\{n_k\} = \frac{\Phi_n}{T}$. The relation between $\{u_k\}$ and $\{y_k\}$, in the absence of $\{\theta_k\}$, is given by:

$$y_{k+1} = y_k + g u_k, \tag{18}$$

where u_k is the output of the loop-filter F. To achieve a finite variance for $\{e_k\}$ one has to cope with the double summation in T_Θ. We, therefore, apply one summation on the output y_k and relay on the other one step delayed summation that is provided by the VCO. The state-space description of the plant of (18), combined with the additional summation, is given by:

$$\xi = [y_k \quad y_{k-1} \quad u_{k-1}],$$

$$A = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \text{ and } B_2 = \begin{bmatrix} g \\ 0 \\ 1 \end{bmatrix}. \tag{19}$$

where B_1 represents a situation where the exogenous disturbance ω is operating on the closed loop at the input of the VCO. The new system is depicted in Fig 2. Let assume that a third order filter is required. A

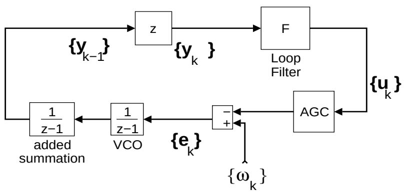

Figure 2: The extended PLL model

first order component of this filter has already been determined by applying the summation on y_k. However, instead of choosing H in (9) to be a second order in order to provide a third order F(z), we seek H(z) of a third order with one pole at the origin. This pole will cancel the zero introduced by the summation that we have added and the resulting F will then be free of zeros and poles at the origin. To achieve a third order H(z) in (9) with a pole at the origin one should take n=4 and $k_{2n-1} = 0$. This means that additional components should be added to ξ of (19): y_{k-2}, y_{k-3} and u_{k-2} (u_{k-3} is not required due to the fact that $k_7 = 0$). We obtain the new augmented system:

$$\xi = [y_k \; y_{k-1} \; y_{k-2} \; y_{k-3} \; u_{k-1} \; u_{k-2}],$$

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \; B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \; B_2 = \begin{bmatrix} g \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}. \tag{20}$$

In order to achieve the design requirements we consider first the problem of ensuring that the H_∞-norm of the transfer function that relates ω with z, is less than a prescribed γ for all $g \in \Gamma$, where we have $z_k = g u_k$, namely, L of Section 2 is $g[k_1 \quad k_2 \quad ... \quad k_6]$. The latter z_k implies that we investigate the transmission from ω_k to the output of AGC.

The resulting design problem can be solved by applying the method suggested in Section 2. The polytope Ω in our problem is defined by g that can lie anywhere between $g_1 = 1$ and $g_2 = 4$. The problem we have to solve is, in fact, a mixed H_2/H_∞ problem where in addition to the requirement of (10) we seek to minimize an upper-bound $\bar{e}$ of (1).

We start by solving the problem described in (10). Namely, we look first for constant matrices $P_i \in R^{6\times 6}$, $Z \in R^{6\times 6}$ and $Y \in R^{1\times 6}$ with $P_i > 0$, that satisfy the following set of inequalities for the two vertices of the polytope $\Omega (i = 1, 2)$:

$$\begin{bmatrix} -P_i & A_i Z - B_{2i} Y & 0 & B_{1i} \\ Z^T A_i^T - Y^T B_{2i}^T & P_i - Z - Z^T & Z^T C_{1i}^T & 0 \\ 0 & C_{1i} Z & -\gamma I & 0 \\ B_{1i}^T & 0 & 0 & -\gamma I \end{bmatrix} < 0. \tag{21}$$

It follows from the development in the Appendix (especially (A.1)) that the system of Fig 2 is driven by a sum of three independent white noise sequences $\{\psi_{1k}, \psi_{2k}, \psi_{3k}\}$ where $\psi_k = [\psi_{1k} \quad \psi_{2k} \quad \psi_{3k}]^T$. Denoting the effect of ψ_{ki} on e_k by $\hat{e}_k^i$, i=1,2,3 we have that $\hat{e}_k = \hat{e}_k^1 + \hat{e}_k^2 + \hat{e}_k^3$ where $\hat{e}_k^i$ are independent. Fig 3 is

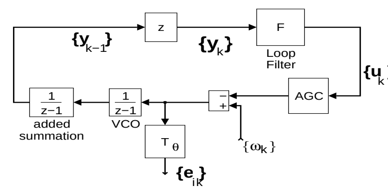

Figure 3: The equivalent PLL model for ψ_{ik}

equivalent to the one described in Fig 2 for $\omega_k = \psi_{ik}$.

We denote $\hat{e}_k = [\hat{e}_k^1, \hat{e}_k^2, \hat{e}_k^3]^T$ and find that: $\hat{e}_k = C_1 \xi_k$ where $C_1 = \begin{bmatrix} N_{11} & N_{12} & N_{13} & 0 & 0 & 0 \\ N_{21} & N_{22} & N_{23} & 0 & 0 & 0 \\ N_{31} & N_{32} & N_{33} & 0 & 0 & 0 \end{bmatrix}$ and where N_{ij} are defined in (17). The variance of $\{\hat{e}_k\}$ represents only the effect of $\omega_k = \theta_k$ on $\{e_k\}$ of Fig 2. To incorporate the effect of the measurement noise $\omega_k = n_k$ on the input of AGC we augment the above $\hat{e}_k$ and find that $e_k^* = \begin{bmatrix} \hat{e}_k^T & \hat{u}_k^T \end{bmatrix}^T = C_2 \xi$ where: $C_2 \triangleq diag\{I_3, \sqrt{\frac{\phi_n}{T}}\} \begin{bmatrix} & & C_1 & & & \\ k_1 & k_2 & k_3 & k_4 & k_5 & k_6 \end{bmatrix}$

Our objective is to find $K = [k_1, k_2, ..., k_6]$ that achieves the requirement of (2) for $\gamma = 3$dB and minimizes an upper-bound on the variance of $\{e_k^*\}$.

This mixed H_2/H_∞ problem is solved by looking for (P_i, Z, Y, V_i) (in the first step of our new method) that will satisfy (21) and (22) for:

$$\begin{bmatrix} V & C_{2i} Z \\ Z^T C_{2i}^T & Z^T + Z - P_i \end{bmatrix} > 0, \quad i = 1, 2 \tag{22}$$

so that $trace\{V\}$ is minimized.The latter inequality guarantees that $trace\{V\} > trace\{C_{2i} P_i C_{2i}^T\}$ i=1,2 which mean that $trace\{V\}$ is a bound on the variance of $\{e_k^*\}$. The solution to the above (taking $\gamma = 3dB$) yields a maximum variance of 0.0190. The actual H_∞-norm of the resulting system is 1.1064.

One simple way to improve the resulting L_2 behaviour of the loop, is to select a higher value for γ that may still lead to the above margin. We solved the above set of LMIs for a maximum value of $\gamma = 2.206$ and obtain, for the worst case ($g = 1$),

$$var\{e_k^*\} = 0.0033 \tag{23}$$

In order to further minimize the latter variance, we apply next the iterative procedure of Section 2 on the solution we obtained for (21) and (22), by taking $\gamma = 3dB$.

We first choose Γ to be [1,2,3,4]$\in \Omega$. For each point θ in Γ we calculate the initial $P_\theta = \bar{P}_\theta$ from P_i, i=1,2 that was achieved in (21) and (22). In the i-th step of our algorithm we substitute P_θ^{i-1}, $\bar{P}_\theta^{i-1}$ and look for K^i and X_θ^i that solve the following two sets of LMI:

$$\begin{bmatrix} -(P_\theta^{i-1}+X_\theta^i) & A_\theta - B_{2\theta}K^i & 0 & B_{1\theta} \\ A_\theta^T - K^{iT}B_{2\theta}^T & -\{(P_\theta^{i-1})^{-1} - (P_\theta^{i-1})^{-1}X_\theta^i(P_\theta^{i-1})^{-1}\} & L_\theta^T & 0 \\ 0 & L_\theta & -\gamma I & 0 \\ B_{1\theta}^T & 0 & 0 & -\gamma I \end{bmatrix} < 0, \tag{24}$$

$$\begin{bmatrix} -(\bar{P}_\theta^{i-1}+\bar{X}_\theta^i) & A_\theta - B_{2\theta}K^i & B_{1\theta} \\ A_\theta^T - K^{iT}B_{2\theta}^T & -\{(\bar{P}_\theta^{i-1})^{-1} - (\bar{P}_\theta^{i-1})^{-1}\bar{X}_\theta^i(\bar{P}_\theta^{i-1})^{-1}\} & 0 \\ B_{1\theta}^T & 0 & -I \end{bmatrix} < 0,$$

$$\theta \in \Gamma.$$

so that $v_i = \max_\theta trace\{C_{2\theta}(\bar{P}_\theta^{i-1}+\bar{X}_\theta^i)C_{2\theta}^T\}$ is minimized. We note that unlike the case where we had to solve for (22) in order to minimize the bound V_i on the trace $\{C_2 P_i C_2^T\}$, in (24) we solve for the 'actual' variance of the resulting error.

We repeat the procedure until $\frac{v_{i-1}}{v_i} > 1.001$. We use the convergent result of K^i as in (9) and obtain that:

$$F = H = \frac{0.3336z^3 - 0.4774z^2 + 0.1686z - 0.0059}{(z-1)(z^2 - 0.6841z - 0.0008)},$$

where the worst value of v_i was achieved for $g = 1$ to be:

$$var\{e_k^*\} = 0.0027. \tag{25}$$

We simulate the behaviour of the resulting PLL using the Matlab program dcovar [7]. We obtain that indeed the worst achievable $var\{e_k^*\}$ is 0.0027. The latter result is 22 percent improvement on the result achieved in (23). In Fig. 4 Nichols plot of the resulting PLL is depicted. In this figure we describe the results for all g in Γ. It is clearly there that the required margin of 3db is achieved. We also verified that the $var\{e_k^*\}$ is

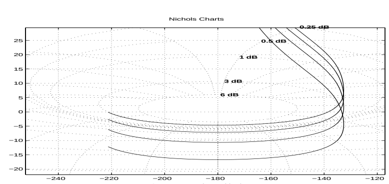

Figure 4: Nichols plots for example N^o 1

monotonously decreasing with g. An alternative solution to the above problem has been offered in [4] using the QFT method. There, an assumption is made that the worst case is the one that corresponds to g=1. The solution that is obtained the latter assumption leads to:

$$F_{QFT} = \frac{0.326(z-0.8942)(z-0.7223)(z-0.5186)}{(z-1)(z-0.7940)(z-0.6087)}.$$

The corresponding index of performance is $\max var\{e_k^*\} = 0.0028$. The latter is a little bit greater than the result obtained in (25).

5 Example 2

We consider the PLL design problem of Example 1 where a delay of three sampling periods is added to the loop and where we do not restrict *a priori* the order of the loop filter. The delay stems from a decision process that is applied in the PLL.

Also here we apply the algorithm of the Section 2. The difference equation that describes the behaviour of the system (with an additional summation) is now:

$$y_{k+1} = 2y_k - y_{k-1} + gu_{k-3}$$

Like the case in Example 2, we want to cancel the zero at the origin that stems from the summation. We thus take n=5 in (9) and require k_9 to be zero. We achieve the latter goal by choosing the following augmented state vector (the omission of u_{k-4} is equivalent to $k_9 = 0$):

$$\xi_k = [y_k \quad y_{k-1} \quad y_{k-2} \quad y_{k-3} \quad y_{k-4} \quad u_{k-1} \quad u_{k-2} \quad u_{k-3}].$$

We note that the latter choice for ξ_k implies a 4-th order filter.

The state space matrices are:

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & g \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}. \tag{26}$$

Our objective vector is $z = L\xi$, where $L = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad g]$ and

$$C_2 = \begin{bmatrix} 0.0011 & 0.0003 & -0.0014 & 0 & 0 & 0 & 0 & 0 \\ 0.0016 & 0.0065 & 0.0016 & 0 & 0 & 0 & 0 & 0 \\ 0.0009 & -0.0007 & -0.0002 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0317 \end{bmatrix}.$$

We solved (21), (22) for $\gamma = 3dB$. The resulting value of the index performance, for the worst case of $g = 1$, was found to be $var\{e_k^*\} = 4.1$. Taking $\gamma = 1.98$ we could still keep the required margin and reduce $\max var\{e_k^*\}$ to

$$var\{e_k^*\} = 0.3759. \tag{27}$$

Applying the iterative procedure of Section 2, for $\gamma = 3dB$, we obtained first

$$K^0 = 10^{-1}[1.858 \quad -1.849 \quad -1.95{\cdot}10^{-3} \quad 1.648{\cdot}10^{-9} \quad 1.015{\cdot}10^{-9} \quad 9.781 \; 9.501 \; 7.181] \tag{28}$$

and the iteration yielded the following loop filter:

$$F(z) = 0.3040\frac{(z-0.9548)(z+0.41)(z^2-0.4563z+0.1641)}{(z-1)(z+1.0194)(z^2-0.1017z+1.0396)}.$$

The resulting index of performance, for the worst case of $g = 1$, was then

$$var\{e_k^*\} = 0.1987. \tag{29}$$

An improvement of 89% is obtained with respect to the (27). In Fig. 5 the corresponding Nichols plot is depicted for all the points in Γ. It is clearly seen from the figure that the required margin of 3db is achieved. One can see that the above $F(z)$ possesses poles outside

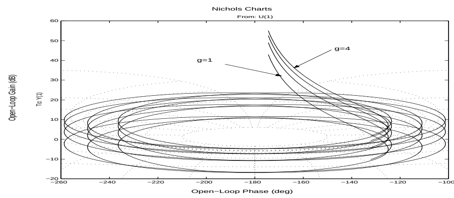

Figure 5: Nichols plots for example N^o 2

the unit circle. Using the fact that K^0 of (28) leads to a stability matrix A_k^0 in (14), we add the additional restriction of (16) to the iterative algorithm.
We obtained the following loop filter:

$$F(z) = 0.2850\frac{(z-0.9554)(z+0.1915)(z^2-0.3409z+0.1345)}{(z-1)(z+0.9676)(z^2-0.1981z+0.9999)}$$

and the resulting index of performance, for the worst case of $g=1$, was $var\{e_k^*\} = 0.2088$. The requirement for all poles inside the unit circle (except one of the discrete summation) leads to only slight increase in the value of our performance index.

6 Conclusions

A simple systematic procedure is introduced to design PLLs that entail parameter uncertainty and time delay. The procedure is based on translating the resulting output-feedback control problem into a problem of designing a state-feedback on an augmented non minimal system. A mixed H_2/H_∞ is used to obtain the required stability margin for the PLL while minimizing the effect of the measurement and the phase noise inputs. One of the main drawbacks of the above method of translating the output-feedback problem into a state-feedback one is that it is tuned to the exact model one assumes for the phase noise spectrum.

Appendix

We build a discrete-time model that produces $\{\theta_i\}$ in the system of Fig 1. The continuous-time system that produces the phase noise θ is:

$$\dot{\eta} = G\eta + \nu, \quad \theta = H\eta$$

where

$$G = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

and where ν is a white noise with the covariance $Q = diag\{0,\ (2.75\cdot 10^7)^2\}$.
The discrete-time model is derived by considering the augmented state vector $\bar{\eta} = [\eta^T\ \xi^T]^T$. We get:

$$\dot{\bar{\eta}} = \bar{G} + \bar{\nu}$$

where:

$$\bar{G} = \begin{bmatrix} G & 0 \\ \frac{1}{T}H & 0 \end{bmatrix} \quad and \quad \bar{\nu} = \begin{bmatrix} \nu \\ 0 \end{bmatrix}.$$

The latter leads to the following discrete-time model:

$$\bar{\eta}_{k+1} = e^{\bar{G}T}\bar{\eta}_k + \bar{\nu}_k$$

where

$$e^{\bar{G}T} = \begin{bmatrix} e^{GT} & 0 \\ \phi & 1 \end{bmatrix}, \ \phi \triangleq \begin{bmatrix} 1 & T/2 \end{bmatrix}$$

$$\bar{\eta}_{k+1} = \begin{bmatrix} \bar{\nu}_{k,1} \\ \bar{\nu}_{k,2} \end{bmatrix} \triangleq \int_{t_k}^{t_{k+1}} e^{\bar{G}(t_{k+1}-\tau)}\bar{\nu}(\tau)d\tau.$$

We readily find that $E\{\bar{\nu}_k\bar{\nu}_l\} = \bar{Q}_k\delta_{k,l}$ where:

$$\bar{Q}_k = \int_0^T e^{\bar{G}\tau}\bar{Q}e^{\bar{G}^T\tau}d\tau = (2.75\cdot 10^7)^2\begin{bmatrix} T^3/3 & T^2/2 & T^3/8 \\ . & T & T^2/6 \\ . & . & T^3/20 \end{bmatrix}.$$

The required $\{\theta_i\}$ is clearly obtained by:

$$\chi_{k+1} = \xi_{k+1} - \xi_k = \phi\eta_k + \nu_{k,2}$$

and thus:

$$\begin{bmatrix} \eta \\ \chi \end{bmatrix}_{k+1} = \begin{bmatrix} e^{GT} & 0 \\ \phi & 0 \end{bmatrix}\begin{bmatrix} \eta \\ \chi \end{bmatrix}_k + \bar{Q}_k^{\frac{1}{2}}\psi_k, \qquad (A.1)$$

$$\theta_k = [0 \quad 0 \quad 1]\begin{bmatrix} \eta \\ \chi \end{bmatrix}_k$$

where $\{\psi_k\}$ is a standard white noise sequence in R^3.

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