Synthesized Sliding Mode and Time-Delay Control for a Class of Matched and Unmatched Uncertain Systems

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Abstract

In this paper, the Sliding Mode Control (SMC) is incorporated with Time-Delay Control (TDC) during sliding phase to reduce the switching gain. TDC identifies the unknown system dynamics and disturbance directly every delay-time. For a system with a lumped perturbation which is relatively slow varying with respect to the sampling interval, a much low switching gain can be used if a reasonably good estimate of the derivative of system state can be obtained using the past information, hence chattering can be reduced or eliminated while retaining the tracking accuracy. This paper considers *unmatched* uncertainties. It is an extension of [1].

Keywords: Sliding Mode Control, Time-Delay Control, Lyapunov Direct Method, Chattering Elimination.

1 Introduction

Over the past two decades, there has been an increasing interest in Sliding Mode Control (SMC) owning to its insensitiveness to parametric uncertainties and disturbances. Essentially, Sliding Mode Control (SMC) utilizes a discontinuous control law to drive system's state trajectory onto a specified and user-chosen surface in the state space (called the sliding or switching surface), and to maintain the system's state trajectory on this surface for all subsequent time [2] [3]. The plant dynamics restricted to this surface represent the controlled system's behavior. Hence, VSC design follows two standard steps: (1) a sliding surface is designed such that the closed-loop system during sliding mode exhibits desired dynamic behavior; (2) a discontinuous control law is employed to force the system states to remain on the surface. These steps become two key issues to attain stable sliding motion satisfying some pre-specified specifications.

In this paper, we concentrate on the second design step. We consider a broader class of uncertain systems which may be contaminated by unmatched uncertainties. The presumption here is that the first step has been wound up such that the closed-loop system dynamics during the sliding mode is asymptotically stable irrespective of system unmatched uncertainties.

The ideal sliding mode, however, requires infinite switching frequency which in practice is not achievable due to the limited sampling rate in digital implementation. The finite switching frequency associated with large control gains will cause chattering which in most cases is undesirable because it may excite high-frequency unmodeled system dynamics and is harmful to actuator devices.

To overcome this problem, numerous schemes have been proposed. In [4], the method of boundary layer is introduced to obtain a smooth control law in the cost of tracking accuracy. In [5], the adaptive technique is used to estimate suitable upper bounds of the uncertainties and disturbances for the design of adaptive robust controllers. However, the adaptive type SMCs usually require that system be in the parameterization form and the controller design is rather complicated.

The idea of Time-Delay Control (TDC) is presented in [6], [7] which uses information in the recent past to directly estimate the unknown dynamics at any given instant through time-delay. TDC can work effectively in practice if the following two conditions are met: (1) the system disturbances are slow varying with respect to sampling interval, and (2) the derivatives of system states in recent past are available using appropriate schemes. The condition (1) holds for most mechanical systems with dominant inertia and for most process industrial control problems. Moreover, by virtue of the condition (2), TDC needs only to estimate system state derivatives in recent past instead of the current ones, numerous signal processing approaches such as zero-phase filter, Butterworth filter, etc, can be applied to achieve satisfactory derivative estimates. In [8], using the observations of the input and output of a plant with a white noise disturbance, the approximation of the derivatives of system states are obtained. Instead of using numerical differentiators, model reference observer is proposed in [9] and [10] to stably reconstruct state variables and their derivatives.

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To avoid high switching gain and severe chattering during sliding phase, in this paper TDC is incorporated with SMC for a class of uncertain systems which meets the two TDC conditions. It is demonstrated that the proposed method achieves much lower switching gain if the above conditions hold, hence chattering can be greatly reduced while retaining tracking accuracy.

The organization of this paper is as follows. In Section 2, problem formulation is introduced. In Section 3, SMC with TDC is proposed. In Section 4, simulation work is presented and a comparison between the proposed method with a typical SMC is made. To avoid any confusion, the implicit dependency of any functional $\mathbf{f}$ is defined as $\mathbf{f}(\mathbf{x},t) \overset{\triangle}{=} \mathbf{f}(\mathbf{x}(t),t)$ and $\mathbf{f}(\mathbf{x},\mathbf{u},t) \overset{\triangle}{=} \mathbf{f}(\mathbf{x}(t),\mathbf{u}(t),t)$.

2 Problem Formulation

Consider the following system

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x},t) + B(\mathbf{x},t)\mathbf{u}(\mathbf{x},t) + \mathbf{d}(\mathbf{x},\mathbf{u},t) \tag{1}$$

where $t \in \mathcal{R}^+$, $\mathbf{x}(t) \in \mathcal{R}^n$ is the measurable state. $\mathbf{u}(\mathbf{x},t) \in \mathcal{R}^m$ is the control. $\mathbf{f}(\mathbf{x},t) \in \mathcal{R}^n$, $B(\mathbf{x},t) \in \mathcal{R}^{n\times m}$ are known matrix functions. $\mathbf{d}(\mathbf{x},\mathbf{u},t)$ is a lumped perturbation which can be either matched or unmatched uncertainties.

In general the lumped system perturbation $\mathbf{d}$ can be classified into the following categories in term of its *priori* knowledge.

Category 1: $\mathbf{d}$ is bounded with a known bounding function

$$\|\mathbf{d}(\mathbf{x},\mathbf{u},t)\| \le d_{max}(\mathbf{x},t). \tag{2}$$

Category 2: The variation of $\mathbf{d}$ is bounded with a known bounding function

$$\|\mathbf{d}(\mathbf{x},\mathbf{u},t) - \mathbf{d}(\mathbf{x},\mathbf{u},t-lT_s)\| \le l_d(\mathbf{x},t) \tag{3}$$

where l is a small positive integer and T_s is a positive constant. T_s can be taken to be the sampling interval in a sampled-data system .

Note that to implement the SMC, Category 1 is usually required. While implementing the TDC, Category 2 is required. Here the objective is to use SMC first assuring the system robustness, meanwhile incorporating TDC whenever possible, to reduce switching gains. This can be achieved if in a system $l_d(\mathbf{x},t) \ll d_{max}(\mathbf{x},t)$, say $l_d(\mathbf{x},t) \le \alpha T_s$ where α is a positive constant with a moderate value, i.e., $\mathbf{d}$ is relatively slow varying with respect to T_s and $\alpha T_s \ll d_{max}(\mathbf{x},t)$ for a sufficiently small T_s.

The set of sliding surfaces is

$$\boldsymbol{\sigma}(\mathbf{x},t) = \begin{bmatrix} \sigma_1(\mathbf{x},t) & \sigma_2(\mathbf{x},t) & \cdots & \sigma_m(\mathbf{x},t) \end{bmatrix}^T = \mathbf{0}.$$

$\boldsymbol{\sigma}(\mathbf{x},t)$ is selected to be continuously differentiable with respect to $\mathbf{x}$ and t.*The closed-loop system dynamics during the sliding mode of $\boldsymbol{\sigma}(\mathbf{x},t)$ is asymptotically stable.* From the chain rule and (1),

$$\dot{\boldsymbol{\sigma}} = \frac{\partial\boldsymbol{\sigma}(\mathbf{x},t)}{\partial t} + \frac{\partial\boldsymbol{\sigma}(\mathbf{x},t)}{\partial\mathbf{x}}\left[\mathbf{f}(\mathbf{x},t) + B(\mathbf{x},t)\mathbf{u}(\mathbf{x},t) + \mathbf{d}(\mathbf{x},\mathbf{u},t)\right]. \tag{4}$$

A typical SMC can be expressed as

$$\mathbf{u}(\mathbf{x},t) = \mathbf{u}_{smc}(\mathbf{x},t) = \mathbf{u}_c(\mathbf{x},t) + \mathbf{u}_n(\mathbf{x},t) \tag{5}$$

where $\mathbf{u}_n(\mathbf{x},t)$ is a switching term to be defined later,

$$\mathbf{u}_c(\mathbf{x},t) = -\left[\frac{\partial\boldsymbol{\sigma}(\mathbf{x},t)}{\partial\mathbf{x}}B(\mathbf{x},t)\right]^{-1}\left[\frac{\partial\boldsymbol{\sigma}(\mathbf{x},t)}{\partial t} + \frac{\partial\boldsymbol{\sigma}(\mathbf{x},t)}{\partial\mathbf{x}}\mathbf{f}(\mathbf{x},t)\right].$$

A positive Lyapunov function $V = \frac{1}{2}\boldsymbol{\sigma}^T\boldsymbol{\sigma}$ is chosen. Using (4), (5) and the above we have

$$\begin{aligned} \dot{V} &= \boldsymbol{\sigma}^T\frac{\partial\boldsymbol{\sigma}(\mathbf{x},t)}{\partial\mathbf{x}}B(\mathbf{x},t)\left\{\mathbf{u}_n(\mathbf{x},t)\right. \\ &\quad \left. + \left[\frac{\partial\boldsymbol{\sigma}(\mathbf{x},t)}{\partial\mathbf{x}}B(\mathbf{x},t)\right]^{-1}\frac{\partial\boldsymbol{\sigma}(\mathbf{x},t)}{\partial\mathbf{x}}\mathbf{d}(\mathbf{x},\mathbf{u},t)\right\}. \end{aligned}$$

$\mathbf{u}_n$ can be chosen to be

$$\begin{aligned} \mathbf{u}_n(\mathbf{x},t) &= -\left[\frac{\partial\boldsymbol{\sigma}(\mathbf{x},t)}{\partial\mathbf{x}}B(\mathbf{x},t)\right]^{-1}K\boldsymbol{\sigma}(\mathbf{x},t) \\ &\quad -\rho_{con}(\mathbf{x},t)\frac{B^T(\mathbf{x},t)\left[\frac{\partial\boldsymbol{\sigma}(\mathbf{x},t)}{\partial\mathbf{x}}\right]^T\boldsymbol{\sigma}(\mathbf{x},t)}{\left\|B^T(\mathbf{x},t)\left[\frac{\partial\boldsymbol{\sigma}(\mathbf{x},t)}{\partial\mathbf{x}}\right]^T\boldsymbol{\sigma}(\mathbf{x},t)\right\|} \end{aligned}$$

where $K > 0$ and $\rho_{con}(\mathbf{x},t)$ is a positive scalar satisfying

$$\rho_{con}(\mathbf{x},t) \ge \left\|\left[\frac{\partial\boldsymbol{\sigma}(\mathbf{x},t)}{\partial\mathbf{x}}B(\mathbf{x},t)\right]^{-1}\frac{\partial\boldsymbol{\sigma}(\mathbf{x},t)}{\partial\mathbf{x}}\right\|d_{max}(\mathbf{x},t). \tag{6}$$

Under the above switching control and switching gain $\rho_{con}(\mathbf{x},t)$, it can be obtained that $\dot{V} \le -\boldsymbol{\sigma}^T K\boldsymbol{\sigma}$. Hence in the SMC, $\mathbf{u}_n(\mathbf{x},t)$ suppresses the system uncertainties (either matched or unmatched), in the sequel make $\dot{V}$ negative definite if $\boldsymbol{\sigma} \ne \mathbf{0}$. However, it results a discontinuous control which produces chattering motion in the neighborhood of the sliding surface. A straightforward smoothing method which eliminates chattering motion is given by [11]

$$\begin{aligned} \mathbf{u}_n(\mathbf{x},t) &= -\left[\frac{\partial\boldsymbol{\sigma}(\mathbf{x},t)}{\partial\mathbf{x}}B(\mathbf{x},t)\right]^{-1}K\boldsymbol{\sigma}(\mathbf{x},t) - \rho_{con}(\mathbf{x},t) \\ &\quad \times\frac{B^T(\mathbf{x},t)\left[\frac{\partial\boldsymbol{\sigma}(\mathbf{x},t)}{\partial\mathbf{x}}\right]^T\boldsymbol{\sigma}(\mathbf{x},t)\rho_{con}(\mathbf{x},t)}{\left\|B^T(\mathbf{x},t)\left[\frac{\partial\boldsymbol{\sigma}(\mathbf{x},t)}{\partial\mathbf{x}}\right]^T\boldsymbol{\sigma}(\mathbf{x},t)\rho_{con}(\mathbf{x},t)\right\| + \epsilon}. \end{aligned}$$

Nevertheless, chattering elimination is achieved in the cost of tracking accuracy. The larger ϵ, the smoother the control profile, but also the larger the tracking error.

3 SMC with Time-Delay Control

From the above discussions, it is clear that the switching gain of $\mathbf{u}_n$ can be reduced if the total uncertainty $\mathbf{d}$ can be estimated. A compensation $\mathbf{u}_{cmp}$ can be incorporated into $\mathbf{u}$ to cancel a part of $\mathbf{d}$, i.e.,

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{u}_{smc}(\mathbf{x}, t) + \mathbf{u}_{cmp}(\mathbf{x}, t).$$

Note that the uncertainty in (1) can be expressed by

$$\mathbf{d}(\mathbf{x}, \mathbf{u}, t) = \dot{\mathbf{x}}(t) - \mathbf{f}(\mathbf{x}, t) - B(\mathbf{x}, t)\mathbf{u}(\mathbf{x}, t).$$

However, obtaining $\mathbf{d}$ from $\dot{\mathbf{x}}(t)$ and $\mathbf{u}(\mathbf{x}, t)$ is not an easy task due to the delay of the sampled-data mechanism. Under the assumption that disturbances belong to category 2 and $l_d(\mathbf{x}, t) \leq \alpha T_s$ for a sufficiently small T_s and a moderate α, uncertainty can be estimated using past information as

$$\begin{aligned} \mathbf{d}(\mathbf{x}, \mathbf{u}, t) \approx \mathbf{d}(\mathbf{x}, \mathbf{u}, t - lT_s) &= \dot{\mathbf{x}}(t - lT_s) \\ -\mathbf{f}(\mathbf{x}, t - lT_s) - B(\mathbf{x}, t - lT_s)&\mathbf{u}(\mathbf{x}, t - lT_s). \end{aligned} \quad (7)$$

For practical consideration, $\dot{\mathbf{x}}(t - lT_s)$ may not be readily available. $\hat{\dot{\mathbf{x}}}(t - lT_s)$, the estimate of $\dot{\mathbf{x}}(t - lT_s)$ given data up to time t can be deduced from the history of the measurable system states, i.e., $\mathbf{x}(t - lT_s)$. The estimation of $\dot{\mathbf{x}}(t - lT_s)$ is easier than that of $\dot{\mathbf{x}}(t)$ as many effective signal processing methods can be applied, such as zero-phase filter, Butterworth filter, etc. The work in [9] and [10] also shows that system states and their derivatives can be stably reconstructed by using observer design. Thus, it is assumed that the measurement uncertainty of $\dot{\mathbf{x}}(t - lT_s)$ due to noise and estimation error is bounded by

$$\left\| \dot{\mathbf{x}}(t - lT_s) - \hat{\dot{\mathbf{x}}}(t - lT_s) \right\| \leq \gamma(\mathbf{x}, t) \quad (8)$$

and $\gamma(\mathbf{x}, t) \ll d_{max}(\mathbf{x}, t)$ when $\dot{\mathbf{x}}(t - lT_s)$ is estimated accurately enough.

Theorem: In system (1), $\boldsymbol{\sigma}$ will enter an ultimate bound $\|\boldsymbol{\sigma}\| \leq \sqrt{\frac{\epsilon}{\lambda_{min}(K)}}$ if the following control law is applied

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{u}_c(\mathbf{x}, t) + \mathbf{u}_n(\mathbf{x}, t) + \mathbf{u}_{cmp}(\mathbf{x}, t) \quad (9)$$

with

$$\begin{aligned} &\mathbf{u}_n(\mathbf{x}, t) \\ &= -\left[\frac{\partial \boldsymbol{\sigma}(\mathbf{x}, t)}{\partial \mathbf{x}} B(\mathbf{x}, t)\right]^{-1} K\boldsymbol{\sigma}(\mathbf{x}, t) - \rho_{tdc}(\mathbf{x}, t) \\ &\times \frac{B^T(\mathbf{x}, t)\left[\frac{\partial \boldsymbol{\sigma}(\mathbf{x}, t)}{\partial \mathbf{x}}\right]^T \boldsymbol{\sigma}(\mathbf{x}, t)\rho_{tdc}(\mathbf{x}, t)}{\left\| B^T(\mathbf{x}, t)\left[\frac{\partial \boldsymbol{\sigma}(\mathbf{x}, t)}{\partial \mathbf{x}}\right]^T \boldsymbol{\sigma}(\mathbf{x}, t)\rho_{tdc}(\mathbf{x}, t)\right\| + \epsilon}, \end{aligned}$$

$$\begin{aligned} \mathbf{u}_{cmp}(\mathbf{x}, t) = &\left[\frac{\partial \boldsymbol{\sigma}(\mathbf{x}, t)}{\partial \mathbf{x}} B(\mathbf{x}, t)\right]^{-1} \frac{\partial \boldsymbol{\sigma}(\mathbf{x}, t)}{\partial \mathbf{x}} \\ &\times \left[B(\mathbf{x}, t - lT_s)\mathbf{u}(t - lT_s) + \mathbf{f}(\mathbf{x}, t - lT_s)\right. \\ &\left. -\hat{\dot{\mathbf{x}}}(t - lT_s)\right], \quad (10) \end{aligned}$$

$$\begin{aligned} \rho_{tdc}(\mathbf{x}, t) = &\left\| \left[\frac{\partial \boldsymbol{\sigma}(\mathbf{x}, t)}{\partial \mathbf{x}} B(\mathbf{x}, t)\right]^{-1} \frac{\partial \boldsymbol{\sigma}(\mathbf{x}, t)}{\partial \mathbf{x}} \right\| \\ &\times \left[l_d(\mathbf{x}, t) + \gamma(\mathbf{x}, t)\right]. \quad (11) \end{aligned}$$

Proof of Theorem: Using control law (9), relation (4) and (7) one obtains

$$\begin{aligned} \dot{V} &= \boldsymbol{\sigma}^T \frac{\partial \boldsymbol{\sigma}(\mathbf{x}, t)}{\partial \mathbf{x}} B(\mathbf{x}, t) \left\{ \mathbf{u}_n(\mathbf{x}, t) + \mathbf{u}_{cmp}(\mathbf{x}, t) \right. \\ &\quad \left. + \left[\frac{\partial \boldsymbol{\sigma}(\mathbf{x}, t)}{\partial \mathbf{x}} B(\mathbf{x}, t)\right]^{-1} \frac{\partial \boldsymbol{\sigma}(\mathbf{x}, t)}{\partial \mathbf{x}} \mathbf{d}(\mathbf{x}, \mathbf{u}, t) \right\} \\ &= \boldsymbol{\sigma}^T \frac{\partial \boldsymbol{\sigma}(\mathbf{x}, t)}{\partial \mathbf{x}} B(\mathbf{x}, t) \left\{ \mathbf{u}_n(\mathbf{x}, t) + \left[\frac{\partial \boldsymbol{\sigma}(\mathbf{x}, t)}{\partial \mathbf{x}} B(\mathbf{x}, t)\right]^{-1} \right. \\ &\quad \times \frac{\partial \boldsymbol{\sigma}(\mathbf{x}, t)}{\partial \mathbf{x}} \left[\mathbf{d}(\mathbf{x}, \mathbf{u}, t) - \mathbf{d}(\mathbf{x}, \mathbf{u}, t - lT_s) + \mathbf{x}(t - lT_s)\right. \\ &\quad \left.\left. -\hat{\dot{\mathbf{x}}}(t - lT_s)\right]\right\} \end{aligned}$$

According to (3) in Category 2, (8) and (11), it can be derived that

$$\begin{aligned} \dot{V} \leq\; & -\boldsymbol{\sigma}^T K \boldsymbol{\sigma} \\ & - \frac{\left\| B^T \left[\frac{\partial \boldsymbol{\sigma}(\mathbf{x}, t)}{\partial \mathbf{x}}\right]^T \boldsymbol{\sigma}(\mathbf{x}, t) \right\|^2 \rho_{tdc}^2(\mathbf{x}, t)}{\left\| B^T(\mathbf{x}, t) \left[\frac{\partial \boldsymbol{\sigma}(\mathbf{x}, t)}{\partial \mathbf{x}}\right]^T \boldsymbol{\sigma}(\mathbf{x}, t) \right\| \rho_{tdc}(\mathbf{x}, t) + \epsilon} \\ & + \left\| B^T(\mathbf{x}, t) \left[\frac{\partial \boldsymbol{\sigma}(\mathbf{x}, t)}{\partial \mathbf{x}}\right]^T \boldsymbol{\sigma}(\mathbf{x}, t) \right\| \rho_{tdc}(\mathbf{x}, t) \\ \leq\; & -\boldsymbol{\sigma}^T K \boldsymbol{\sigma} + \epsilon. \end{aligned}$$

Then the system is globally uniformly ultimately bounded under control (9) [12]. The ultimate bound over infinite time horizon is

$$\lim_{t \to \infty} \sup_{w \geq t} \|\boldsymbol{\sigma}(w)\| \leq \sqrt{\frac{\epsilon}{\lambda_{min}(K)}}$$

where $\lambda_{min}(K)$ denotes the minimum eigenvalue of K. ■

Remark 1: (10) adds a form of integral action into the controller (9). Integral action has been used in adaptive variable structure control such as [5] which estimates the switching gain and in integral variable structure control such as [13] which uses integration in the design of sliding surface. In this note, integral action is directly added into the control input.

Remark 2: Since usually $\alpha T_s \ll d_{max}(\mathbf{x},t)$ for a sufficiently small T_s, then it can be inferred from (6) that $\alpha T_s \ll \rho_{con}(\mathbf{x},t)$. If $\mathbf{d}$ is relatively slow varying with respect to T_s, say, $l_d(\mathbf{x},t) \le \alpha T_s$, then $l_d(\mathbf{x},t) \ll \rho_{con}(\mathbf{x},t)$. If $\dot{\mathbf{x}}(t-lT_s)$ is estimated accurately enough, i.e., $\gamma(\mathbf{x},t) \ll d_{max}(\mathbf{x},t)$, then according to (6) and (11), $\rho_{tdc}(\mathbf{x},t) \ll \rho_{con}(\mathbf{x},t)$, i.e., the switching gain is greatly reduced. This makes it feasible to choose a much smaller ϵ than that of the typical SMC to eliminate chattering motion and to improve tracking accuracy. The Time-Delay Control based disturbance estimation reduces chattering which is valid for the situations where $l_d(\mathbf{x},t) \le \alpha T_s \ll d_{max}(\mathbf{x},t)$ holds, i.e., system dynamics is slow varying.

4 Illustrative Example

To show the effectiveness of the proposed method, the control law was used to position a DC servo motor described by

$$J\ddot{\theta} + b_s\dot{\theta} + k_s\theta = \tau + d \tag{12}$$

where θ is the motor rotation angle, τ is the control torque, J is the total inertia, b_s is the viscous friction coefficient and k_s is the spring constant, d is an exogenous disturbance. The nominal values are $J_n = 4.0 \times 10^{-2}\ Kg \cdot m$, $k_{sn} = 0.17\ N \cdot m/rad$ and $b_{sn} = 0.05\ N \cdot m \cdot sec/rad$. Multiplicative perturbations for J, k_s, b_s are δ_J, δ_k, δ_b respectively and $|\delta_J| \le 50\%$, $|\delta_k| \le 50\%$, $|\delta_b| \le 50\%$. In this simulation, the real plant takes the extremal perturbations, i.e., $J = 6.0 \times 10^{-2}\ Kg \cdot m$, $k_s = 0.255\ N \cdot m/rad$ and $b_s = 0.075\ N \cdot m \cdot sec/rad$. The desired trajectory is $\theta_d = sin(\pi t)$, sliding surface is defined by $\sigma = (\dot{\theta} - \dot{\theta}_d) + 6(\theta - \theta_d)$ and a large disturbance $d = 10sin(2\pi t)$ is applied. Sampling interval is $T_s = 1\ msec$.

A typical SMC is implemented as

$$\begin{aligned}
\tau(t) &= \tau_c(t) + \tau_n(t), \\
\tau_c(t) &= k_{sn}\theta + (b_{sn} - 6J_n)\dot{\theta} + J_n(\ddot{\theta}_d + 6\dot{\theta}_d), \\
\tau_n(t) &= -k_1\sigma - \rho_{con}\frac{\sigma}{|\sigma| + \epsilon}, k_1 = 0.2, \\
\rho_{con} &\ge d_{max} = 0.5\sqrt{(6J_n)^2 + b_{sn}^2}|\dot{\theta}| + 0.5k_{sn}|\theta| \\
&\quad +0.5J_n|\ddot{\theta}_d + 6\dot{\theta}_d| + 10.
\end{aligned}$$

k_1 is used to speed up the reaching phase. $\epsilon = 4 \times 10^{-4}$ is chosen. Figure 1 shows the severe chattering due to the high switching gain and small ϵ. Figure 2 (dashed line) shows the steady-state tracking error is bounded by $1.25 \times 10^{-2}\ rad$.

To improve tracking accuracy while to reduce chattering motion, the following control law is derived according to the proposed method (9)

$$\begin{aligned}
\tau(t) &= \tau_c(t) + \tau_n(t) + \tau_{cmp}(t), \\
\tau_n(t) &= -k_1\sigma - \rho_{tdc}\frac{\sigma}{|\sigma| + \epsilon}, \\
\tau_{cmp}(t) &= \tau(t - lT_s) + J_n\hat{\ddot{\theta}}(t - lT_s) - b_{sn}\dot{\theta}(t - lT_s),
\end{aligned}$$

where $l = 1$. As $\ddot{\theta}$ may not be available, the following approximation was used

$$\hat{\ddot{\theta}}(t - lT_s) = \frac{\dot{\theta}(t) - \dot{\theta}(t - lT_s)}{lT_s}. \tag{13}$$

The lumped perturbation h can be derived from (12) as

$$h = \delta_J J_n\ddot{\theta} + \delta_b b_{sn}\dot{\theta} + \delta_k k_{sn}\theta + d. \tag{14}$$

It can be estimated that $l_d = 0.065$ according to (14). Assuming the approximation error in (13) is bounded by $\gamma = 0.005$, a low switching gain $\rho_{tdc} = l_d + \gamma = 0.07$ is used. For comparison with the typical SMC, $\epsilon = 4 \times 10^{-4}$ are selected again. Figure 2 (solid line) shows that the tracking accuracy is greatly improved and the steady-state error is bounded by $3 \times 10^{-4}\ rad$. Figure 3 shows the control profile is quite smooth.

For a fast varying disturbance, the switching gain ρ_{tdc} tends to be greater. Consider the case $d = 0.1sin(300\pi t)$, $l_d = 9.4$ is estimated according to (14) and note that l_d is of the same magnitude as d_{max}. $\rho_{tdc} = 9.5$ and $\epsilon_2 = 0.002$ are used. Figure 4 shows the chatter motion caused by the increased l_d. Figure 5 shows the bound of the steady-state tracking error is enlarged to be $1.5 \times 10^{-3}\ rad$.

In many practical situations, the estimated acceleration signal could be very noisy. Such a situation is simulated by applying white noise which is uniformly distributed, with zero mean and variance 0.005 to $\dot{\theta}(t)$. The estimated acceleration signal from (13) is shown in Figure 6. $\rho_{tdc} = 0.07$ and $\epsilon = 4 \times 10^{-4}$ are used. The steady-state tracking error in Figure 7 is still fairly acceptable. This is because the plant acts as a low-pass filter and considerably reduces the high frequency components of the white noise [6]. If the measurement noise is too severe, some signal processing methods are necessary to eliminate measurement noise. Observer design method can also be applied to stably reconstruct state variables and their derivatives and to ensure good tracking performance.

5 Conclusions

Typical SMC uses high switching gain to maintain robustness to system uncertainties and disturbance. To eliminate chattering, boundary layer is introduced in the cost of tracking accuracy. In this note, SMC design is incorporated with Time-Delay Control during sliding phase to reduce the switching gain. It facilitates a much smaller size of boundary layer to smoothen control profile and to improve tracking accuracy. System disturbance and uncertainty have to be slow varying to ensure the effectiveness of the proposed method, it also requires a reasonably good estimate of the derivative of system state.

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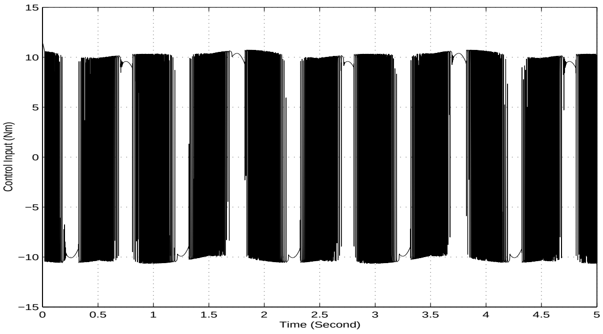

Figure 1: Control profile for typical SMC, $d = 10sin(2\pi t)$, $\epsilon = 4 \times 10^{-4}$

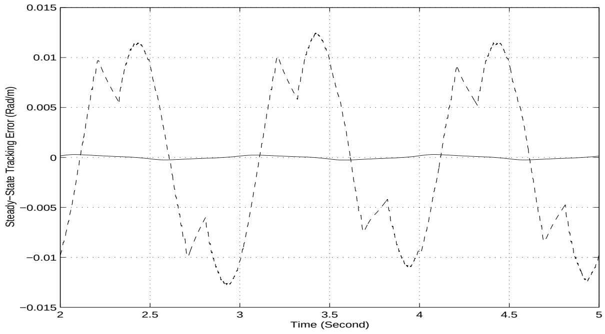

Figure 2: Steady-state tracking error for typical SMC and SMC with TDC, $d = 10sin(2\pi t)$, $\epsilon = 4 \times 10^{-4}$. Solid line – SMC with TDC; Dashed line – SMC

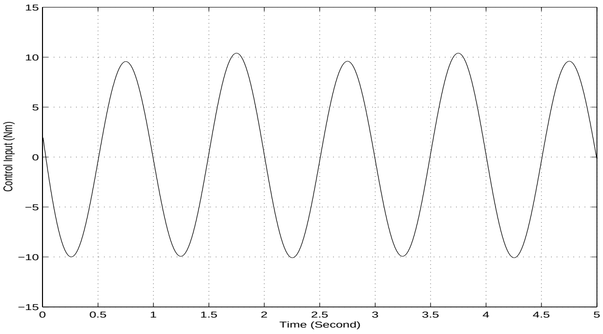

Figure 3: Control profile for SMC with TDC, $d = 10sin(2\pi t)$, $\rho_{tdc} = 0.07$, $\epsilon = 4 \times 10^{-4}$

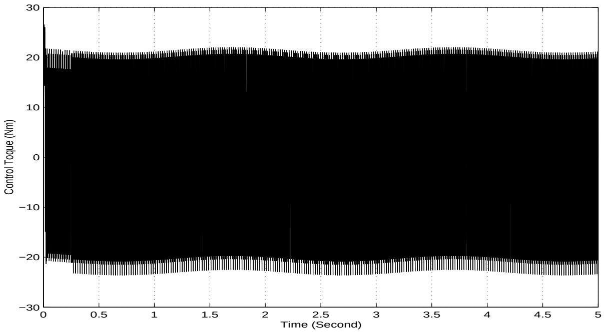

Figure 4: Control profile for SMC with TDC, $d = 10sin(300\pi t)$, $\rho_{tdc} = 9.5$, $\epsilon_2 = 0.002$

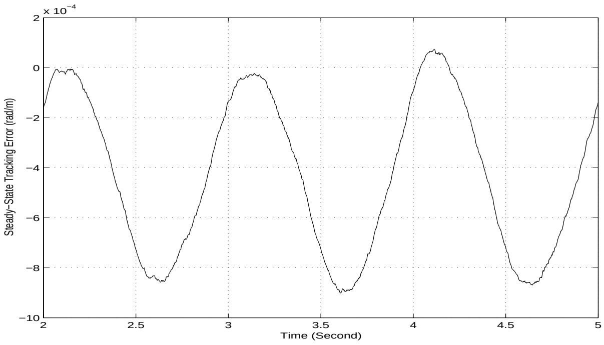

Figure 5: Steady-state tracking error for SMC with TDC, $d = 10sin(300\pi t)$, $\rho_{tdc} = 9.5$, $\epsilon_2 = 0.002$.

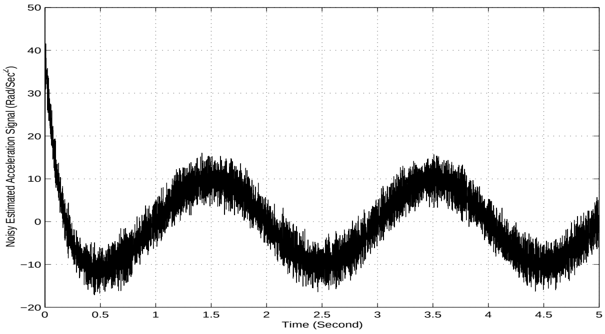

Figure 6: The noisy estimated acceleration signal

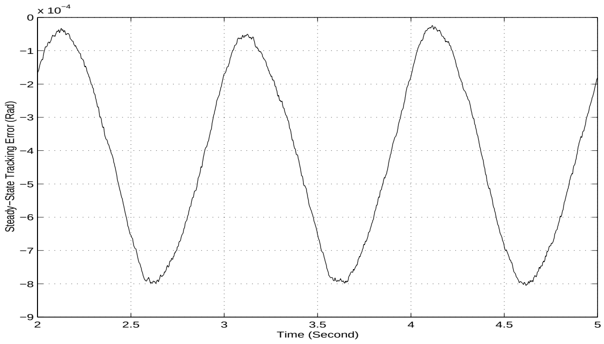

Figure 7: Steady-state tracking error for SMC with TDC, $d = 10sin(2\pi t)$, $\rho_{tdc} = 0.07$, $\epsilon = 4 \times 10^{-4}$ with measurement noise.