A Learning Variable Structure Controller of a Flexible One-Link Manipulator

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Abstract

In this paper, tip regulation of a flexible one-link manipulator by Learning Variable Structure Control (LVSC) is investigated. Sliding surface is designed according to a selected reference model which relocates system poles to be negative real ones, hence link vibration is eliminated. The proposed LVSC incorporates a learning mechanism to improve regulation accuracy. Rigorous proof shows: the state's tracking error sequence converges *uniformly* to zero; the uniformly bounded learning control sequence converges to the *equivalent control almost everywhere*. A more detailed journal version of this paper is presented in [1].

1 Dynamic Formulation and Problem Statement

Denote l, m_l, EI, τ, θ, μ and J_h as the length, mass, flexural rigidity, control torque, rigid angular displacement, elastic damping coefficient and actuator's moment of inertia of the flexible link. The flexible manipulator is only operated in the horizontal plane and consequently the gravity is not considered. The first m flexible modes are considered. The deflection is represented as $n(x,t) = \boldsymbol{\phi}^T(x)\mathbf{q}(t), \quad 0 \le x \le l$ where $\boldsymbol{\phi}(x) = [\ \phi_1(x) \ \cdots \ \phi_m(x)\ ]^T$ and $\mathbf{q}(t) = [\ q_1(t) \ \cdots \ q_m(t)\ ]^T$.

A. Dynamic Model

The derivation of the dynamic equations given below follows then as in [2]

$$\left\{ \begin{array}{l} m_{RR}\ddot{\theta} + \mathbf{m}_{RF}^T\ddot{\mathbf{q}} = u - 2\mathbf{q}^T M_{FF}\dot{\mathbf{q}}\dot{\theta} + d(t) \\ \mathbf{m}_{RF}\ddot{\theta} + M_{FF}\ddot{\mathbf{q}} = M_{FF}\mathbf{q}\dot{\theta}^2 - C_F\mathbf{q} - D_F\dot{\mathbf{q}} \end{array} \right. , \qquad (1)$$

where

$$m_{RR} = J_h + \frac{1}{3}ml^2 + m_t l^2 + \mathbf{q}^T M_{FF}\mathbf{q},$$

$$\mathbf{m}_{RF} = \frac{m}{l}\int_0^l x\boldsymbol{\phi}(x)dx + m_t l\boldsymbol{\phi}_e,$$

$$M_{FF} = m_t\boldsymbol{\phi}_e\boldsymbol{\phi}_e^T + \frac{m_l}{l}\int_0^l \boldsymbol{\phi}(x)\boldsymbol{\phi}^T(x)dx,$$

$$C_F = EI\int_0^l \operatorname{diag}\left(\left[\frac{\partial^2\phi_1(x)}{\partial x^2}\right]^2 \ \cdots \ \left[\frac{\partial^2\phi_m(x)}{\partial x^2}\right]^2 \right) dx,$$

$$D_F = \int_0^l \mu\boldsymbol{\phi}(x)\boldsymbol{\phi}^T(x)dx.$$

Subscript e is used to denote the value of the variable occurring at the tip, i.e., the value at $x = l$. $d(t)$ stands for bounded exogenous disturbance. The dynamic model (1) can be re-written into

$$M(\boldsymbol{\xi})\ddot{\boldsymbol{\xi}} + D(\boldsymbol{\xi},\dot{\boldsymbol{\xi}})\dot{\boldsymbol{\xi}} + C\boldsymbol{\xi} = \mathbf{b}_1 u + \mathbf{d}, \qquad (2)$$

$$\boldsymbol{\xi} = \begin{bmatrix} \theta \\ \mathbf{q} \end{bmatrix},\ \mathbf{b}_1 = \begin{bmatrix} 1 \\ \mathbf{0}_{m\times 1} \end{bmatrix},\ M = \begin{bmatrix} m_{RR} & \mathbf{m}_{RF}^T \\ \mathbf{m}_{RF} & M_{FF} \end{bmatrix},$$
$$C = \operatorname{diag}\begin{bmatrix} 0 & C_F \end{bmatrix},\ D = \begin{bmatrix} \mathbf{q}^T M_{FF}\dot{\mathbf{q}} & \mathbf{q}^T M_{FF}\dot{\theta} \\ -M_{FF}\mathbf{q}\dot{\theta} & D_F \end{bmatrix},$$
$$\mathbf{d} = \begin{bmatrix} d \\ \mathbf{0}_{m\times 1} \end{bmatrix},\ D_F = \textstyle\int_0^l \mu\boldsymbol{\phi}(x)\boldsymbol{\phi}^T(x)dx.$$

Defining state vector $\mathbf{x} \triangleq \left[\boldsymbol{\xi}^T \ \dot{\boldsymbol{\xi}}^T\right]^T$ yields

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}u + \mathbf{v} \qquad (3)$$

where $\mathbf{x}$ is measurable and

$$A = \begin{bmatrix} \mathbf{0}_{(m+1)\times(m+1)} & I_{(m+1)\times(m+1)} \\ -M^{-1}C & -M^{-1}D \end{bmatrix},$$

$\mathbf{b} = \begin{bmatrix} \mathbf{0}_{(m+1)\times 1} \\ M^{-1}\mathbf{b}_1 \end{bmatrix}$, $\mathbf{v} = \left[\mathbf{0}_{1\times(m+1)},\ M^{-1}\mathbf{d}\right]^T$. A and $\mathbf{b}$ can be decomposed into the linear part A_l and $\mathbf{b}_l$ which are constant matrices, and the nonlinear part ΔA and $\Delta\mathbf{b}$ as below

$$A = A_l + \Delta A, \quad \mathbf{b} = \mathbf{b}_l + \Delta\mathbf{b} \qquad (4)$$

where

$$A_l = \begin{bmatrix} \mathbf{0}_{(m+1)\times(m+1)} & I_{(m+1)\times(m+1)} \\ -M_l^{-1}C & -M_l^{-1}D_l \end{bmatrix},$$
$$\mathbf{b}_l = \begin{bmatrix} \mathbf{0}_{(m+1)\times 1} \\ M_l^{-1}\mathbf{b}_1 \end{bmatrix},\ M_l = \begin{bmatrix} m_{RRl} & \mathbf{m}_{RF}^T \\ \mathbf{m}_{RF} & M_{FF} \end{bmatrix},$$
$$m_{RRl} = J_h + \frac{1}{3}m_l l^2 + m_t l^2,\ D_l = \operatorname{diag}\begin{bmatrix} 0 & D_F \end{bmatrix}.$$

The tip angular displacement can be obtained as below by the assumed mode method

$$y = \mathbf{c}^T\mathbf{x},\ \mathbf{c} = \begin{bmatrix} 1 & \boldsymbol{\phi}_e^T\big/ l & 0 & \mathbf{0}_{1\times m} \end{bmatrix}^T. \qquad (5)$$

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The general form of tip transfer function considering the linear plant is

$$g(s)=k_g\frac{1+b_1s+b_2s^2+\cdots+b_{2m-1}s^{2m-1}+b_{2m}s^{2m}}{s^{2m+2}+a_{2m-1}s^{2m+1}+\cdots+a_1s^3+a_0s^2}$$

The conjugate poles result in sinusoidal oscillatory behavior of tip trajectory.

B. Repeatable Control Tasks

Learning control has been considered for repeatable tasks [3], [4]. The system (1) is deterministic and satisfies the resetting condition, i.e., $\mathbf{x}_i(0)=\mathbf{0}_{(2m+2)\times 1}$ where i denotes the iteration sequence $\forall i\in Z_+$ where Z_+ denotes the set of positive integers. Given a finite time interval $[0,\ T_f]$, the control objective is to design a VSC combined with iterative learning such that, as $i\to\infty$, $\mathbf{x}_i$ is regulated to the desired state $\mathbf{x}^*\triangleq\begin{bmatrix} r & \mathbf{0}_{1\times m} & 0 & \mathbf{0}_{1\times m}\end{bmatrix}^T$ where r is a constant reference input, meanwhile the tip vibration should be eliminated.

C. Sliding Surface Design

We design the sliding surface according to the linear plant which takes the *nominal* values, i.e., $\dot{\mathbf{x}}=A_{l,n}\mathbf{x}+\mathbf{b}_{l,n}u$ where the subscript n denotes the *nominal* value. A reference model is selected as

$$\hat{h}_1(s)=\frac{\alpha_{2m+2}}{s^{2m+2}+\alpha_1s^{2m+1}+\cdots+\alpha_{2m+1}s+\alpha_{2m+2}}$$

which contains no finite zero but all negative real poles. The sliding surface is designed as below [5],

$$\sigma=\mathbf{s}^T\mathbf{x}+\alpha_{2m+2}\int_0^t[\eta-r]\,d\tau+\mathbf{c}_4^T\int_0^t\mathbf{q}\dot{\theta}^2d\tau=0 \quad (6)$$

where $\eta=\mathbf{m}^T\mathbf{x}$,

$$\begin{aligned}\mathbf{m}^T&=[0\ 0\ \cdots\ 0\ k_{g,n}]\\&\quad\times\left[\mathbf{b}_{l,n}\ A_{l,n}\mathbf{b}_{l,n}\ \cdots\ A_{l,n}^{2m}\mathbf{b}_{l,n}\ A_{l,n}^{2m+1}\mathbf{b}_{l,n}\right]^{-1},\\ \mathbf{s}^T&=\left[c_1\ \mathbf{c}_2^T\ c_3\ \mathbf{c}_4^T\right]=\mathbf{m}^T\left[A_{l,n}^{2m+1}+\alpha_1A_{l,n}^{2m}\right.\\&\qquad\left.+\cdots+\alpha_{2m}A_{l,n}+\alpha_{2m+1}I_{(2m+2)\times(2m+2)}\right].\end{aligned}$$

Lemma 1: [6]. The following Linear Matrix Inequality

$$\begin{bmatrix} Q(\mathbf{x}) & S(\mathbf{x})\\ S^T(\mathbf{x}) & R(\mathbf{x})\end{bmatrix}>0$$

where $Q(\mathbf{x})=Q^T(\mathbf{x})$, $R(\mathbf{x})=R^T(\mathbf{x})$ and $S(\mathbf{x})$ depends affinely on $\mathbf{x}$, is equivalent to

$$R(\mathbf{x})>0,\quad Q(\mathbf{x})-S(\mathbf{x})R^{-1}(\mathbf{x})S^T(\mathbf{x})>0.$$

Since $M(\boldsymbol{\xi})$ is the inertia matrix of the flexible manipulator dynamics (2), hence according to *Lemma 1*, $M(\boldsymbol{\xi})>0$ ensures that

$$\Delta=m_{RR}-\mathbf{m}_{RF}^TM_{FF}^{-1}\mathbf{m}_{RF}>0. \quad (7)$$

Lemma 2: For the system (1) and the proposed sliding surface (6), the following stands

$$\gamma^{-1}\Delta\dot{\sigma}=u+\mathbf{k}^T\boldsymbol{\chi} \quad (8)$$

where $\lambda=c_{3,n}-\mathbf{c}_{4,n}^TM_{FF}^{-1}\mathbf{m}_{RF}$, $\Delta_l=m_{RRl}-\mathbf{m}_{RF}^TM_{FF}^{-1}\mathbf{m}_{RF}$, $\mathbf{1}\triangleq\begin{bmatrix}1 & \cdots & 1\end{bmatrix}^T$

$$\mathbf{k}=\begin{bmatrix}\lambda^{-1}\Delta_lc_{1,n}\\ M_{FF}^{-1}C_F\left(\mathbf{m}_{RF}-\lambda^{-1}\Delta_l\mathbf{c}_{4,n}\right)\\ M_{FF}^{-1}D_F\mathbf{m}_{RF}+\lambda^{-1}\Delta_l\left(\mathbf{c}_{2,n}-M_{FF}^{-1}D_F\mathbf{c}_{4,n}\right)\\ \lambda^{-1}\Delta_l\alpha_{2m+2}\\ -\mathbf{m}_{RF}\\ M_{FF}\mathbf{1}_{m\times1}\\ \lambda^{-1}c_1M_{FF}\mathbf{1}_{m\times1}\\ -\left(\lambda^{-1}M_{FF}^{-1}C_F\mathbf{c}_{4,n}\right)\otimes\left(M_{FF}\mathbf{1}_{m\times1}\right)\\ \lambda^{-1}\left(\mathbf{c}_{2,n}-M_{FF}^{-1}D_F\mathbf{c}_{4,n}\right)\otimes\left(M_{FF}\mathbf{1}_{m\times1}\right)\\ d\end{bmatrix},$$

$$\boldsymbol{\chi}=\begin{bmatrix}\dot{\theta} & \mathbf{q}^T & \dot{\mathbf{q}}^T & \eta-r & \mathbf{q}^T\dot{\theta}^2 & (\dot{\mathbf{q}}\circ\mathbf{q})^T & \dot{\theta}\,(\mathbf{q}\circ\mathbf{q})^T\\ \mathbf{q}^T\otimes(\mathbf{q}\circ\mathbf{q})^T & \dot{\mathbf{q}}^T\otimes(\mathbf{q}\circ\mathbf{q})^T & 1\end{bmatrix}^T.$$

Here, $\mathbf{a}\circ\mathbf{b}\triangleq[a_1b_1\ \cdots a_nb_n]^T$, $\mathbf{a}\otimes\mathbf{b}\triangleq\left[a_1\mathbf{b}^T\ \cdots\ a_n\mathbf{b}^T\right]^T$ where $\mathbf{a}=\{a_j\}\in\mathcal{R}^n$, $\mathbf{b}=\{b_j\}\in\mathcal{R}^n$.

Proof of Lemma 2: It can be derived from (1) that

$$\begin{aligned}\Delta\ddot{\theta}&=u-\mathbf{m}_{RF}^T\mathbf{q}\dot{\theta}^2+\mathbf{m}_{RF}^TM_{FF}^{-1}C_F\mathbf{q}\\&\quad+\mathbf{m}_{RF}^TM_{FF}^{-1}D_F\dot{\mathbf{q}}-2\mathbf{q}^TM_{FF}\dot{\mathbf{q}}+d, &(9)\\ \Delta\ddot{\mathbf{q}}&=-M_{FF}^{-1}\mathbf{m}_{RF}\Big(u-\mathbf{m}_{RF}^T\mathbf{q}\dot{\theta}^2+\mathbf{m}_{RF}^TM_{FF}^{-1}C_F\mathbf{q}\\&\quad+\mathbf{m}_{RF}^TM_{FF}^{-1}D_F\dot{\mathbf{q}}-2\mathbf{q}^TM_{FF}\dot{\mathbf{q}}+d\Big)\\&\quad+\Delta\mathbf{q}\dot{\theta}^2-\Delta M_{FF}^{-1}C_F\mathbf{q}-\Delta M_{FF}^{-1}D_F\dot{\mathbf{q}}. &(10)\end{aligned}$$

Differentiating (6) with respect to time once one obtains

$$\dot{\sigma}=c_1\dot{\theta}+\mathbf{c}_2^T\dot{\mathbf{q}}+c_3\ddot{\theta}+\mathbf{c}_4^T\ddot{\mathbf{q}}+\alpha_{2m+2}(\eta-r)+\mathbf{c}_4^T\mathbf{q}\dot{\theta}^2.$$

Combining (9), (10) and the above yields (8). ■

Lemma 3: For the system (1) and the proposed sliding surface (6), define that $\bar{\mathbf{x}}\triangleq\mathbf{x}-\begin{bmatrix} r & \mathbf{0}_{1\times m} & 0 & \mathbf{0}_{1\times m}\end{bmatrix}^T$, the closed-loop system dynamics is

$$\dot{\bar{\mathbf{x}}}=A_{cl}\bar{\mathbf{x}}+\mathbf{f}_{h.o.t.}(\bar{\mathbf{x}})+\mathbf{b}_{cl}\dot{\sigma} \quad (11)$$

where

$$A_{cl}=\begin{bmatrix}\mathbf{0}_{(m+1)\times(m+1)} & I_{(m+1)\times(m+1)}\\ \multicolumn{2}{c}{\boldsymbol{\xi}_1^T+\gamma^{-1}\alpha_{2m+2}\mathbf{m}_n^T}\\ \multicolumn{2}{c}{\boldsymbol{\xi}_2^T+\gamma^{-1}\alpha_{2m+2}M_{FF}^{-1}\mathbf{m}_{RF}\mathbf{m}_n^T}\end{bmatrix},$$

$$\mathbf{f}_{h.o.t.}=\begin{bmatrix}0\\ \mathbf{0}_{m\times1}\\ 0\\ \mathbf{q}\dot{\theta}^2\end{bmatrix},\quad \mathbf{b}_{cl}=\begin{bmatrix}0\\ \mathbf{0}_{m\times1}\\ \gamma^{-1}\\ \gamma^{-1}M_{FF}^{-1}\mathbf{m}_{RF}\end{bmatrix},$$

$$\boldsymbol{\xi}_1 = \begin{bmatrix} 0 \\ \gamma^{-1} C_F M_{FF}^{-1} \mathbf{c}_{4,n} \\ -\gamma^{-1} c_{1,n} \\ \gamma^{-1} D_F M_{FF}^{-1} \mathbf{c}_{4,n} - \gamma^{-1} \mathbf{c}_{2,n} \end{bmatrix}, \qquad \boldsymbol{\xi}_2 =$$

$$\begin{bmatrix} \mathbf{0}_{1\times m} \\ -\gamma^{-1} C_F M_{FF}^{-1} \mathbf{c}_{4,n} \mathbf{m}_{RF}^T M_{FF}^{-1} - C_F M_{FF}^{-1} \\ \gamma^{-1} c_{1,n} \mathbf{m}_{RF}^T M_{FF}^{-1} \\ \gamma^{-1} \left(\mathbf{c}_{2,n} - D_F M_{FF}^{-1} \mathbf{c}_{4,n} \right) \mathbf{m}_{RF}^T M_{FF}^{-1} - D_F M_{FF}^{-1} \end{bmatrix}.$$

Proof of Lemma 3: From (8), substitution of $u = -\mathbf{k}^T \boldsymbol{\chi} + \gamma^{-1} \Delta \dot{\sigma}$ into (9) and (10) obtains

$$\begin{aligned} \ddot{\theta} &= \boldsymbol{\xi}_1^T \mathbf{x} + \gamma^{-1} \alpha_{2m+2} (\eta - r) + \gamma^{-1} \dot{\sigma}, & (12) \\ \ddot{\mathbf{q}} &= \boldsymbol{\xi}_2^T \mathbf{x} + \gamma^{-1} M_{FF}^{-1} \mathbf{m}_{RF} \alpha_{2m+2} (\eta - r) + \mathbf{q} \dot{\theta}^2 \\ &\quad + \gamma^{-1} M_{FF}^{-1} \mathbf{m}_{RF} \dot{\sigma} & (13) \end{aligned}$$

Using $\eta - r = \mathbf{m}^T \bar{\mathbf{x}}$, (12) and (13) we have (11). ■

Setting $\dot{\sigma} = 0$ for (11), we have the system dynamics during ideal sliding mode as

$$\dot{\bar{\mathbf{x}}}_d = A_{cl} \bar{\mathbf{x}}_d + \mathbf{f}_{h.o.t.}(\bar{\mathbf{x}}_d) \tag{14}$$

where the subscript d denotes the system states or variables under ideal sliding mode. Since $\mathbf{f}_{h.o.t.}(\bar{\mathbf{x}}_d)$ is of higher order in $\bar{\mathbf{x}}_d$, i.e., $\sup \| \mathbf{f}_{h.o.t}(\bar{\mathbf{x}}_d) \| / \| \bar{\mathbf{x}}_d \| \to 0$, as $\| \bar{\mathbf{x}}_d \| \to 0$, then the system (14) is uniformly asymptotically stable around the equilibrium point $\bar{\mathbf{x}}_d = \mathbf{0}$ provided that A_{cl} is *Hurwitz*, (*i.e., has all its eigenvalues strictly in the left half-plane*).

Remark 1: According to (5), as $\bar{\mathbf{x}}_d \to \mathbf{0}_{(2m+2)\times 1}$, the tip angular displacement under *ideal* sliding mode $y_d \to r$ and produces *zero* steady-state output error.

2 Learning Variable Structure Controller

Decompose $\mathbf{k}$ of (8) into

$$\mathbf{k} = \mathbf{k}_n + \delta \mathbf{k} \tag{15}$$

where $\mathbf{k}_n$ can be calculated by setting $d = 0$, $\mathbf{q} = \mathbf{0}$, $\dot{\mathbf{q}} = \mathbf{0}$, m_t, J_h, μ, m_l and l to their nominal values, and $\delta \mathbf{k}$ is the residual. Denoting $|\mathbf{w}|_1 = [|w_1| \ \cdots \ |w_n|]^T$ where $\mathbf{w} = \{w_j\}$, $j = 1, \cdots, n$, a typical VSC is outlined as below

$$\begin{aligned} u &= -\mathbf{k}_n^T \boldsymbol{\chi} + u_v, \\ u_v &= -\left(\hat{\mathbf{k}}^T |\boldsymbol{\chi}|_1 + \epsilon \right) \mathrm{sign}(\gamma) \mathrm{sat}(\sigma, \ \varepsilon) & (16) \end{aligned}$$

where $\mathrm{sat}(\star, \ \varepsilon) \triangleq \left\{ \begin{array}{ll} \star / \varepsilon & |\star| \leq \varepsilon \\ \star / |\star| & |\star| > \varepsilon \end{array} \right.$,

$$\hat{k}_j = \sup \ |\delta k_j|, \ \hat{\mathbf{k}} = \left\{ \hat{k}_j \right\}, \ \delta \mathbf{k} = \{\delta k_j\}, \tag{17}$$

$j = 1, \ \cdots, \ \left(2m^2 + 5m + 3\right)$, $\epsilon > 0$. The term $-\mathbf{k}_n^T \boldsymbol{\chi}$ cancels the nominal part, and u_v is a switching control. If $|\sigma| > \varepsilon$, differentiating a Lyapunov function $V \triangleq \sigma^2/2$ with respect to t using (8) of *Lemma 2* and VSC law (16), one obtains

$$\begin{aligned} \dot{V} &= \sigma \Delta^{-1} \gamma \left(\gamma^{-1} \Delta \dot{\sigma} \right) = \sigma \Delta^{-1} \gamma \left(u + \mathbf{k}^T \boldsymbol{\chi} \right) \\ &< -\Delta^{-1} |\gamma| \hat{\mathbf{k}}^T |\boldsymbol{\chi}|_1 |\sigma| + \Delta^{-1} \gamma \delta \mathbf{k}^T \boldsymbol{\chi} \sigma - \Delta^{-1} |\gamma| \epsilon \varepsilon. \end{aligned}$$

$\hat{\mathbf{k}}$ in (17) ensures that $\dot{V} < -\Delta^{-1} |\gamma| \epsilon \varepsilon < 0$. Since $\sigma(0) = 0$, VSC law (16) achieves the ultimate bound $|\sigma| \leq \varepsilon$. $\mathrm{sat}(\sigma, \ \varepsilon)$ eliminates chattering at the cost of regulation accuracy. The larger the ε, the smoother the control input while the larger the steady-state error.

A. LVSC Configuration and Preliminaries

In the subsequent discussions, $\mathcal{B}(\mathcal{D})$ denotes a space of bounded functions on $\mathcal{D}$; $\mathcal{C}(\mathcal{D})$ denotes a space of continuous functions on $\mathcal{D}$; $\mathcal{UC}(\mathcal{D})$ denotes a space of uniformly continuous functions on $\mathcal{D}$.

According to (8), the *equivalent control* of u_v in (16) is

$$u_{eq} = -\delta \mathbf{k}^T \boldsymbol{\chi}_d. \tag{18}$$

Since $\bar{\mathbf{x}}_d \in \mathcal{B}([0, \ T_f])$ and r is a finite constant, $\mathbf{x}_d \in \mathcal{B}([0, \ T_f])$. The system parametric uncertainties and disturbance are also bounded, then $u_{eq} \in \mathcal{B}([0, \ T_f])$. Obtaining u_{eq} is our ultimate objective which realizes the perfect regulation.

To obtain u_{eq} and improve the regulation accuracy, a Learning Variable Structure Controller (LVSC) is proposed as below where i denotes the iteration number

$$\begin{aligned} u_i &= \beta_1 \left(-\mathbf{k}_n^T \boldsymbol{\chi}_i + u_{v,i} \right) + \beta_2 \left(u_{l,i-1} \right), & (19) \\ u_{v,i} &= -\left(\hat{\mathbf{k}}^T |\boldsymbol{\chi}_i|_1 + \epsilon \right) \mathrm{sign}(\gamma) \mathrm{sat}(\sigma_i, \varepsilon), & (20) \\ u_{l,i} &= \beta_2 \left(u_{l,i-1} \right) + u_{v,i}, \ u_{l,0} \triangleq 0, & (21) \end{aligned}$$

and $\beta_j(\star)$, $\forall j \in \{1, 2\}$ is a saturator defined as below

$$\beta_j(\star) \triangleq \left\{ \begin{array}{ll} \star & |\star| \leq M_j \\ (\star / |\star|) M_j & |\star| > M_j \end{array} \right. \tag{22}$$

where M_j is a positive-definite saturation bound.

Lemma 4: For the flexible manipulator system (1) under the control input (19) saturated by (22), i.e., $u_i \in \mathcal{B}([0, \ T_f])$, we have $\theta_i, \dot{\theta}_i, \mathbf{q}_i, \dot{\mathbf{q}}_i \in \mathcal{B}([0, \ T_f])$, $\forall i \in Z_+$.

Proof of Lemma 4: Since $u_i \in \mathcal{B}([0, \ T_f])$, $\forall i \in Z_+$ and the control duration T_f is finite, then the input energy is finite, hence the system kinetic energy $E_{k,i}$ and potential energy $E_{p,i}$ are all bounded. It has been derived in [7] that $E_{k,i} = \frac{1}{2} \dot{\boldsymbol{\xi}}_i^T M(\boldsymbol{\xi}_i) \dot{\boldsymbol{\xi}}_i$, $E_{p,i} = \frac{1}{2} \mathbf{q}_i^T C_F \mathbf{q}_i$. Both $M(\boldsymbol{\xi}_i)$ and C_F are positive-definite matrices, hence the boundedness of $E_{k,i}$ and $E_{p,i}$ concludes that $\dot{\theta}_i$, $\mathbf{q}_i$, $\dot{\mathbf{q}}_i \in \mathcal{B}([0, \ T_f])$. The bounded $\dot{\theta}_i$ and finite T_f also lead to that $\theta_i \in \mathcal{B}([0, \ T_f])$, $\forall i \in Z_+$. ■

According to (7) and $\mathbf{x}_i \in \mathcal{B}([0,\ T_f])$, we have

$$\underline{\Delta} = \inf_{i\in Z_+, t\in[0,\ T_f]} \Delta_i,\ \bar{\Delta} = \sup_{i\in Z_+, t\in[0,\ T_f]} \Delta_i$$

for some positive-definite constants $\underline{\Delta}$ and $\bar{\Delta}$. From *Lemma 4*, we have $\boldsymbol{\chi}_i \in \mathcal{B}([0,\ T_f])$, hence $\mathbf{k}_n^T\boldsymbol{\chi}_i \in \mathcal{B}([0,\ T_f])$ and $u_{v,i} \in \mathcal{B}([0,\ T_f])$, $\forall i \in Z_+$ according to (20). M_1 is a virtual saturation bound which can be sufficiently large such that

$$M_1 \geq \sup_{i\in Z_+, t\in[0,\ T_f]} \left| -\mathbf{k}_n^T\boldsymbol{\chi}_i + u_{v,i} \right|.$$

Hence the control law (19) is equivalent to below under the above relation

$$u_i = -\mathbf{k}_n^T\boldsymbol{\chi}_i + u_{v,i} + \beta_2\left(u_{l,i-1}\right). \tag{23}$$

Since $u_{v,i} \in \mathcal{B}([0,\ T_f])$, according to (21), $\beta_2(\star)$ ensures the uniform boundedness of $u_{l,i}$, $\forall t \in [0,\ T_f]$, $\forall i \in Z_+$. To guarantee the learnability of the learning system, M_2 should be sufficiently large such that

$$M_2 \geq \sup_{t\in[0,\ T_f]} |u_{eq}(t)|. \tag{24}$$

To evaluate the learning scheme (21), the following Performance Index is defined

$$J_i(t) \triangleq \int_0^t e^{-\lambda\tau}\left[u_{eq}(\tau) - u_{l,i}(\tau)\right]^2 d\tau \tag{25}$$

where λ is a positive constant. To facilitate LVSC analysis, we develop another three Lemmas which reveal the boundedness relationship among σ_i, $\mathbf{x}_i$, $\boldsymbol{\chi}_d - \boldsymbol{\chi}_i$ and J_i.

Lemma 5: For the system (1) under the saturator (22) and the sliding surface (6), the followings hold

$$\|\bar{\mathbf{x}}_d - \bar{\mathbf{x}}_i\| \leq l_1 \int_0^t \|\bar{\mathbf{x}}_d - \bar{\mathbf{x}}_i\| d\tau + \|\mathbf{b}_{cl}\| \cdot |\sigma_i|, \tag{26}$$

$$\|\bar{\mathbf{x}}_d - \bar{\mathbf{x}}_i\| \leq l_1\|\mathbf{b}_{cl}\| \int_0^t |\sigma_i| e^{l_1(t-\tau)} d\tau + \|\mathbf{b}_{cl}\| \cdot |\sigma_i| \tag{27}$$

where

$$l_1 \triangleq \|A_{cl}\| + \sup_{i\in Z_+, t\in[0,\ T_f]} \left(\dot{\theta}_i^2 + \left|\dot{\theta}_d + \dot{\theta}_i\right| \cdot \|\mathbf{q}_d\|\right).$$

Since $\dot{\theta}_i$, $\mathbf{q}_i$, $\dot{\mathbf{q}}_i$ are all bounded, we have from (8) that

$$\|\boldsymbol{\chi}_d - \boldsymbol{\chi}_i\| \leq l_2\|\bar{\mathbf{x}}_d - \bar{\mathbf{x}}_i\| = l_2\|\mathbf{x}_d - \mathbf{x}_i\| \tag{28}$$

for a positive-definite constant l_2.

Lemma 6: Under the same conditions as those in *Lemma 5* the following holds

$$\int_0^t e^{-\lambda\tau}|\sigma_i| \cdot \|\boldsymbol{\chi}_d - \boldsymbol{\chi}_i\| d\tau \leq l_3 \int_0^t e^{-\lambda\tau}\sigma_i^2 d\tau \tag{29}$$

for a positive-definite constant $l_3 \triangleq l_1 l_2 \|\mathbf{b}_{cl}\| e^{l_1 T_f} T_f + l_2\|\mathbf{b}_{cl}\|$.

Lemma 7: Under the same conditions as those in *Lemma 5* the following stands

$$\|\bar{\mathbf{x}}_d - \bar{\mathbf{x}}_i\| \leq l_4 J_i^{\frac{1}{2}}(T_f), \tag{30}$$

$$|\sigma_i| \leq \underline{\Delta}^{-1}|\gamma| \left(T_f^{\frac{1}{2}} e^{\lambda T_f} + l_2 l_4 \|\delta\mathbf{k}\| T_f\right) J_i^{\frac{1}{2}}(T_f) \tag{31}$$

where $l_4 = \underline{\Delta}^{-1}|\gamma| \cdot \|\mathbf{b}_{cl}\| e^{lT_f} T_f^{\frac{1}{2}}$, $l \triangleq \max(\lambda,\ l_1 + \underline{\Delta}^{-1} l_2 |\gamma| \cdot \|\mathbf{b}_{cl}\| \cdot \|\delta\mathbf{k}\|)$.

For the proofs of *Lemma 5, 6* and *7*, see **Appendix**.

B. Convergence Analysis

Theorem: For the system (1) under the control laws (19)-(22), as $i \to \infty$, $\mathbf{x}_i(t)$ converges *uniformally* to $\mathbf{x}_d(t)$, $\sigma_i(t)$ converges *uniformally* to 0, and $u_{l,i}(t)$ converges to $u_{eq}(t)$ *almost everywhere*, $\forall t \in [0,\ T_f]$.

Proof of Theorem: The relation (24) which ensures learnability warrants the following inequality

$$\left[u_{l,i} - u_{eq}\right]^2 \geq \left[\beta_2\left(u_{l,i}\right) - u_{eq}\right]^2,\ \forall i \in Z_+.$$

Under the learning law (21) and the above inequality, $J_i - J_{i-1}$, $\forall i \geq 2 \cap (i \in Z_+)$ can be derived as

$$\begin{aligned} & \int_0^t e^{-\lambda\tau}\left\{\left[u_{l,i} - u_{eq}\right]^2 - \left[u_{l,i-1} - u_{eq}\right]^2\right\} d\tau \\ \leq & \int_0^t e^{-\lambda\tau}\left\{\left[u_{l,i} - u_{eq}\right]^2 - \left[\beta_2(u_{l,i-1}) - u_{eq}\right]^2\right\} d\tau \\ = & \int_0^t e^{-\lambda\tau}\left\{u_{v,i}^2 - 2u_{v,i}\left[u_{eq} - \beta_2(u_{l,i-1})\right]\right\} d\tau. \end{aligned} \tag{32}$$

Substitution of (23), the equivalent of the control law (19), into (8) using relation (15) obtains

$$\gamma^{-1}\Delta_i\dot{\sigma}_i = u_{v,i} + \beta_2(u_{l,i-1}) + \delta\mathbf{k}^T\boldsymbol{\chi}_i. \tag{33}$$

From (18) and the above we have

$$\begin{aligned} u_{eq} &- \beta_2(u_{l,i-1}) \\ &= -\delta\mathbf{k}^T\left(\boldsymbol{\chi}_d - \boldsymbol{\chi}_i\right) + u_{v,i} - \gamma^{-1}\Delta_i\dot{\sigma}_i. \end{aligned} \tag{34}$$

Substitution of (34) into (32) obtains

$$\begin{aligned} & J_i(t) - J_{i-1}(t) \\ \leq & \int_0^t e^{-\lambda\tau}\left[-u_{v,i}^2 + 2\delta\mathbf{k}^T\left(\boldsymbol{\chi}_d - \boldsymbol{\chi}_i\right)u_{v,i} \right. \\ & \left. + 2\gamma^{-1}\Delta_i u_{v,i}\dot{\sigma}_i\right] d\tau \\ \leq & \int_0^t e^{-\lambda\tau}\left[2\delta\mathbf{k}^T\left(\boldsymbol{\chi}_d - \boldsymbol{\chi}_i\right)u_{v,i} + 2\gamma^{-1}\Delta_i u_{v,i}\dot{\sigma}_i\right] d\tau \\ \leq & 2\|\delta\mathbf{k}\| \int_0^t e^{-\lambda\tau}\|\boldsymbol{\chi}_d - \boldsymbol{\chi}_i\| \cdot |u_{v,i}| d\tau \\ & + 2\gamma^{-1}\int_0^t e^{-\lambda\tau}\Delta_i u_{v,i}\dot{\sigma}_i d\tau \end{aligned} \tag{35}$$

The control law (20) can be rewritten into

$$\begin{aligned} u_{v,i} &= -\rho_i sign(\gamma)\sigma_i, \\ \rho_i &= \begin{cases} \left(\hat{\mathbf{k}}^T|\boldsymbol{\chi}_i|_1+\epsilon\right)|\sigma_i|^{-1} & |\sigma_i|>\varepsilon \\ \left(\hat{\mathbf{k}}^T|\boldsymbol{\chi}_i|_1+\epsilon\right)\varepsilon^{-1} & |\sigma_i|\le\varepsilon \end{cases}. \end{aligned} \quad (36)$$

According to *Lemma 4*, $\mathbf{x}_i \in \mathcal{B}([0,\ T_f])$ under the saturated control input (19). Hence $\sigma_i \in \mathcal{B}([0,\ T_f])$ from (6) and $\boldsymbol{\chi}_i \in \mathcal{B}([0,\ T_f])$ from (8), and we have

$$\begin{aligned} \underline{\rho} &= \epsilon\left[\sup_{i\in Z_+,\ t\in[0,\ T_f]}|\sigma_i|\right]^{-1}, \\ \bar{\rho} &= \varepsilon^{-1}\sup_{i\in Z_+,\ t\in[0,\ T_f]}\left(\hat{\mathbf{k}}^T|\boldsymbol{\chi}_i|_1+\epsilon\right) \end{aligned}$$

for some positive-definite constants $\underline{\rho}$ and $\bar{\rho}$. Since $\sigma_i(0)=0$, substituting of (36) into (35) using *Lemma 6* and taking the integration by parts one obtains

$$\begin{aligned} & J_i(t)-J_{i-1}(t) \le 2\|\delta\mathbf{k}\|\bar{\rho}l_3\int_0^t e^{-\lambda\tau}\sigma_i^2 d\tau \\ & -2|\gamma|^{-1}\int_0^t e^{-\lambda\tau}\Delta_i\rho_i\sigma_i\dot{\sigma}_i d\tau \\ \le\ & 2\|\delta\mathbf{k}\|\bar{\rho}l_3\int_0^t e^{-\lambda\tau}\sigma_i^2 d\tau - |\gamma|^{-1}\underline{\Delta\rho}\int_0^t e^{-\lambda\tau}d\sigma_i^2 \\ =\ & -|\gamma|^{-1}\underline{\Delta\rho}e^{-\lambda t}\sigma_i^2 \\ & -|\gamma|^{-1}\underline{\Delta\rho}\left(\lambda-2|\gamma|\underline{\Delta}^{-1}\|\delta\mathbf{k}\|\underline{\rho}^{-1}\bar{\rho}l_3\right)\int_0^t e^{-\lambda\tau}\sigma_i^2 d\tau. \end{aligned}$$

There exists a sufficiently large λ such that $\lambda \ge 2|\gamma|\underline{\Delta}^{-1}\|\delta\mathbf{k}\|\underline{\rho}^{-1}\bar{\rho}l_3$ to ensure that

$$J_i(t)-J_{i-1}(t) \le -|\gamma|^{-1}\underline{\Delta\rho}e^{-\lambda t}\sigma_i^2(t). \quad (37)$$

From (37), taking the summation over $j=1$ to i yields

$$J_i(t)-J_1(t) \le -|\gamma|^{-1}\underline{\Delta\rho}e^{-\lambda t}\sum_{j=1}^{i}\sigma_j^2(t).$$

Since $J_i(t)\ge 0$, as $i\to\infty$ we have

$$\sum_{j=1}^{\infty}\sigma_j^2(t) \le |\gamma|\underline{\Delta}^{-1}\underline{\rho}^{-1}e^{\lambda T_f}J_1(T_f).$$

Since the bounded $u_{l,i}$ ensures that $J_i(T_f)$ is bounded according to (25), then $\sum_{j=1}^{\infty}\sigma_j^2(t)$ is convergent, consequently $\lim_{j\to\infty}\sigma_j(t)=0,\ \forall t\in[0,\ T_f]$. According to (27) and (28) we have

$$\lim_{i\to\infty}\bar{\mathbf{x}}_i(t)=\bar{\mathbf{x}}_d(t),\ \lim_{i\to\infty}\boldsymbol{\chi}_i(t)=\boldsymbol{\chi}_d(t),\ \forall t\in[0,\ T_f]. \quad (38)$$

Rearranging (34) using (21) gives $u_{eq}-u_{l,i} = -\delta\mathbf{k}^T(\boldsymbol{\chi}_d-\boldsymbol{\chi}_i)-\gamma^{-1}\Delta_i\dot{\sigma}_i$. From (25) and (38),

$$\lim_{i\to\infty}J_i(T_f)=\lim_{i\to\infty}\int_0^{T_f}e^{-\lambda\tau}\gamma^{-2}\Delta_i^2\dot{\sigma}_i^2 d\tau.$$

Since $u_{v,i}\in\mathcal{B}([0,\ T_f])$ and $\boldsymbol{\chi}_i\in\mathcal{B}([0,\ T_f])$, from (33) we have $\dot{\sigma}_i\in\mathcal{B}([0,\ T_f])$, which as well as $\lim_{i\to\infty}\sigma_i(t)=0$ conclude that

$$\lim_{i\to\infty}J_i(T_f)\le\gamma^{-2}\bar{\Delta}^2\lim_{i\to\infty}\int_0^{T_f}\dot{\sigma}_i(\tau)d\sigma_i(\tau)=0. \quad (39)$$

According to (30), (31) and the above, we have

$$\begin{aligned} & \lim_{i\to\infty}\sup_{t\in[0,\ T_f]}\|\bar{\mathbf{x}}_d(t)-\bar{\mathbf{x}}_i(t)\|=0, \\ & \lim_{i\to\infty}\sup_{t\in[0,\ T_f]}|\sigma_i(t)|=0,\ \forall t\in[0,\ T_f], \end{aligned}$$

i.e., $\mathbf{x}_i(t)$ converges *uniformally* to $\mathbf{x}_d(t)$, $\sigma_i(t)$ converges *uniformally* to 0, $\forall t\in[0,\ T_f]$. From (25) and (39), $u_{l,i}(t)$ converges to $u_{eq}(t)$ *almost everywhere*, $\forall t\in[0,\ T_f]$ which ends the proof. ■

Remark 2: The only direct addressing for obtaining *equivalent control* appears in [8] which uses a first-order low-pass filter. It requires $\Omega/\tau\to 0$ and $\tau\to 0$ where $|\sigma|\le\Omega$ and τ is the time constant of the filter. This shows the difficulty in achieving *equivalent control* in general: it demands infinite switching frequency to produce an infinitesimal bound Ω of σ and an infinite filter bandwidth due to the worst case control environment. These two stringent requirements can be relaxed when repeatable control tasks are concerned.

3 Conclusion

In this paper, the proposed LVSC scheme incorporates a simple learning mechanism to approximate the equivalent control of VSC under a repeatable regulation task of flexible one-link manipulators. Uniformly bounded control ultimately nullifies the state's tracking error. For practical considerations, the learning mechanism is further conducted in frequency domain by means of *Fourier* series expansion which achieves better regulation performance in [1], where the numerical simulations attached confirm the effectiveness and robustness of the proposed approach.

Appendix: Proofs of *Lemma 5, 6* and *7*

Proof of Lemma 5: From (11) and (14) we have

$$\begin{aligned} & \dot{\bar{\mathbf{x}}}_d-\dot{\bar{\mathbf{x}}}_i = A_{cl}(\bar{\mathbf{x}}_d-\bar{\mathbf{x}}_i) \\ & +\begin{bmatrix} 0 & \mathbf{0}_{1\times m} & 0 & \mathbf{q}_d^T\dot{\theta}_d^2-\mathbf{q}_i^T\dot{\theta}_i^2 \end{bmatrix}^T-\mathbf{b}_{cl}\dot{\sigma}_i. \end{aligned}$$

According to *Lemma 4*, it can be obtained that

$$\begin{aligned} & \|\mathbf{q}_d\dot{\theta}_d^2-\mathbf{q}_i\dot{\theta}_i^2\| = \|\mathbf{q}_d\dot{\theta}_d^2-\mathbf{q}_d\dot{\theta}_i^2+\mathbf{q}_d\dot{\theta}_i^2-\mathbf{q}_i\dot{\theta}_i^2\| \\ \le\ & \sup_{i\in Z_+,t\in[0,\ T_f]}\left(\|\mathbf{q}_d\|\cdot|\dot{\theta}_d+\dot{\theta}_i|\right)\left|\dot{\theta}_d-\dot{\theta}_i\right| \\ & +\sup_{i\in Z_+,t\in[0,\ T_f]}\dot{\theta}_i^2\|\mathbf{q}_d-\mathbf{q}_i\|. \end{aligned} \quad (40)$$

Under the *resetting condition*, $\bar{\mathbf{x}}_d(0) = \bar{\mathbf{x}}_i(0)$ and $\sigma_i(0) = 0$, Estimating the difference $\mathbf{x}_d - \mathbf{x}_i$ on the norm with regard to (40) one obtains (26). Applying *Bellman-Gronwall Lemma I* [9] to (26), we obtain (27). ■

Proof of Lemma 6: Since $0 \le \nu \le \tau \le t \le T_f$, then $0 \le \tau - \nu \le \tau \le T_f$ and $-\frac{\lambda}{2}\tau \le -\frac{\lambda}{2}\nu$. Using *Hölder inequality* [9], (27) and (28) we have that

$$\begin{aligned}
&\int_0^t e^{-\lambda\tau}|\sigma_i| \cdot \|\boldsymbol{\chi}_d - \boldsymbol{\chi}_i\| d\tau - l_2\|\mathbf{b}_{cl}\| \int_0^t e^{-\lambda\tau}\sigma_i^2 d\tau \\
\le\ & l_2 \int_0^t e^{-\lambda\tau}|\sigma_i| \cdot \|\mathbf{x}_d - \mathbf{x}_i\| d\tau - l_2\|\mathbf{b}_{cl}\| \int_0^t e^{-\lambda\tau}\sigma_i^2 d\tau \\
\le\ & l_1 l_2 \|\mathbf{b}_{cl}\| \int_0^t e^{-\lambda\tau}|\sigma_i(\tau)| \left[\int_0^\tau e^{l_1(\tau-\nu)}|\sigma_i(\nu)| d\nu\right] d\tau \\
\le\ & l_1 l_2 \|\mathbf{b}_{cl}\| e^{l_1 T_f} \int_0^t e^{-\lambda\tau}|\sigma_i(\tau)| \left[\int_0^\tau |\sigma_i(\nu)| d\nu\right] d\tau \\
=\ & l_1 l_2 \|\mathbf{b}_{cl}\| e^{l_1 T_f} \left[\int_0^t e^{-\frac{\lambda}{2}\tau}|\sigma_i(\tau)| d\tau\right]^2 \\
\le\ & l_1 l_2 \|\mathbf{b}_{cl}\| e^{l_1 T_f} T_f \int_0^t e^{-\lambda\tau}\sigma_i^2(\tau) d\tau.
\end{aligned}$$

Rearranging the above yields (29). ■

Proof of Lemma 7: From (8), (15), (19) and (21) it can be obtained that

$$\dot{\sigma}_i = \Delta_i^{-1}\gamma\left(u_i + \mathbf{k}^T\boldsymbol{\chi}_i\right) = \Delta_i^{-1}\gamma\left(u_{l,i} + \delta\mathbf{k}^T\boldsymbol{\chi}_i\right).$$

Substituting (18) into the above yields

$$\dot{\sigma}_i = -\Delta_i^{-1}\gamma\left[u_{eq} - u_{l,i} + \delta\mathbf{k}^T\left(\boldsymbol{\chi}_d - \boldsymbol{\chi}_i\right)\right]$$

Since $\sigma_i(0) = 0$, integrating the above obtains

$$\begin{aligned}
\sigma_i &= -\int_0^t \Delta_i^{-1}\gamma\left(u_{eq} - u_{l,i}\right) d\tau \\
&\quad - \int_0^t \Delta_i^{-1}\gamma\delta\mathbf{k}^T\left(\boldsymbol{\chi}_d - \boldsymbol{\chi}_i\right) d\tau.
\end{aligned}$$

From the above and (28), we have

$$\begin{aligned}
|\sigma_i| &\le \underline{\Delta}^{-1}|\gamma| \int_0^t |u_{eq} - u_{l,i}| d\tau \\
&\quad + \underline{\Delta}^{-1} l_2 |\gamma| \cdot \|\delta\mathbf{k}\| \int_0^t \|\bar{\mathbf{x}}_d - \bar{\mathbf{x}}_i\| d\tau. \qquad (41)
\end{aligned}$$

Substitution of the above into (26) using the *Hölder inequality* [9] and *Bellman-Gronwall Lemma II* [9] yields

$$\begin{aligned}
&\|\bar{\mathbf{x}}_d(t) - \bar{\mathbf{x}}_i(t)\| \\
\le\ & \left(l_1 + \underline{\Delta}^{-1} l_2 |\gamma| \cdot \|\delta\mathbf{k}\| \cdot \|\mathbf{b}_{cl}\|\right) \int_0^t \|\bar{\mathbf{x}}_d - \bar{\mathbf{x}}_i\| d\tau \\
& + \underline{\Delta}^{-1}|\gamma| \cdot \|\mathbf{b}_{cl}\| \int_0^t |u_{eq} - u_{l,i}| d\tau \\
\le\ & \underline{\Delta}^{-1}|\gamma| \cdot \|\mathbf{b}_{cl}\| e^{lT_f} \int_0^t e^{-l\tau}|u_{eq} - u_{l,i}| d\tau \\
\le\ & \underline{\Delta}^{-1}|\gamma| \cdot \|\mathbf{b}_{cl}\| e^{lT_f} T_f^{\frac{1}{2}} \left[\int_0^t e^{-\lambda\tau}(u_{eq} - u_{l,i})^2 d\tau\right]^{\frac{1}{2}} \\
=\ & l_4 J_i^{\frac{1}{2}}(T_f). \qquad (42)
\end{aligned}$$

Since $0 \le \frac{\lambda}{2}\tau \le \lambda\tau \le \lambda T_f$, using the *Hölder inequality* [9] we have

$$\begin{aligned}
\int_0^t |u_{eq} - u_{l,i}| d\tau &\le e^{\lambda T_f} \int_0^{T_f} e^{-\frac{\lambda}{2}\tau} |u_{eq} - u_{l,i}| d\tau \\
&\le e^{\lambda T_f} T_f^{\frac{1}{2}} J_i^{\frac{1}{2}}(T_f) \qquad (43)
\end{aligned}$$

Substitution of (42) and (43) into (41) obtains (31). ■

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