Robust Global Terminal Sliding Mode Control of SISO Nonlinear Uncertain Systems

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Abstract: A global terminal sliding mode controller is proposed for higher order SISO nonlinear dynamic systems by employing the fast terminal sliding mode concept in both the reaching phase and the sliding phase. The inherent dynamical properties of the fast terminal sliding modes and the recursive mechanism for application in higher order systems are explored. A control design procedure is developed. It is shown that, by suitably choosing the parameters of the fast terminal sliding modes, the system state variables will reach the fast terminal sliding manifolds within a desired finite time, and stay in the sliding modes thereafter, resulting in the convergence to the equilibrium in a finite time which can also be prespecified. The control law designed, which is called "the global terminal sliding mode control", is nonlinear and continuous. It does not cause chattering in the reaching phase and sliding phase. The robustness analysis shows that the proposed global terminal sliding mode controller has superior robustness in system uncertainties and external disturbances. Simulation studies are presented to validate the proposed scheme.

1. Introduction

Sliding mode control systems are well known for their invariant properties to certain internal parameter variations and external disturbances, and have been successfully applied in many fields [1,2]. The sliding mode is attained by designing the control laws, which drive the system to reach and remain on the intersection of a set of prescribed sliding manifolds. When in the sliding mode, the system exhibits invariance properties, such as robustness to certain internal parameter variations and external disturbances.

The dynamics performance of a sliding mode control system is determined by the prescribed sliding manifolds upon which the control structure is switched. The most commonly used sliding manifolds are linear hyperplanes. Such hyperplanes guarantee the asymptotic stability of the sliding mode. That is, the system states reach the equilibrium in infinite time. Nonlinear switching manifolds may give rise to a better control performance, provided the nonlinearity is added purposefully. The terminal sliding mode concept has been proposed to address the finite time control issue [3,4,5]. In particular, the fast terminal sliding mode concept [5], is particularly effective in terms of delivering fast and finite time control performance as well as high-precision.

The sliding mode control process involves two phases – the reaching phase and the sliding phase. Discontinuous control is usually employed in order to deliver a finite time reachability of the switching manifolds. The discontinuity in control is a major contributor to the "chattering" behaviors well known to exist in sliding mode control systems. There has been an interest in developing chattering free sliding mode control, e.g. [8].

In this paper, we propose a fast terminal sliding mode controller that can force the system to reach the nonlinear switching manifolds, characterized by the fast terminal sliding modes, in the reaching phase. Then the fast terminal sliding modes will take the system to reach its equilibrium in finite time. The control designed is continuous so the chattering behaviors will not occur. We then study the robustness issue of the controller with respect to bounded uncertainties and disturbances. Conditions to ensure the robustness are given. Simulation results are provided to demonstrate the effectiveness of the controller proposed.

This paper is organized as follows. Section 2 presents the main results. Section 3 depicts the simulation results and Section 4 concludes the paper.

2. Main Results

2.1 The Fast Terminal Sliding Mode Concept and the SISO Nonlinear Systems

The fast terminal sliding mode concept [5], or fast terminal attractor, is depicted as

$$s = \dot{x} + \alpha x + \beta x^{q/p} = 0 \tag{1}$$

where x is a scalar variable, $\alpha, \beta > 0$ are constants and $q, p \ (q < p)$ are odd positive integers. It can be derived that the time to reach the equilibrium $x = 0$ is [5]

$$t_s = \frac{p}{\alpha(p-q)} \ln \frac{\alpha x(0)^{(p-q)/p} + \beta}{\beta} \tag{2}$$

The fast convergence performance in comparison with the conventional linear sliding mode can be demonstrated by the following example using Matlab. Consider $\alpha = 1, \beta = 1$ and initial condition $x(0) = 1$. First let us assume $p = 3,\ q = 1$. From (2) one can easily find that the time to reach zero is

$$\mathrm{t_s} = 1.03972077083992\,.$$

We now compare the above with the situation where *p* and *q* are set to 1. The simulation suggests that at approximate $\mathrm{t_s} = 1.03969999999990$ for the situation $p = 3, q = 1$

$$x(\mathrm{t_s}) = 0.00000009178540$$

while the system with *p=1* and *q=1*

$$x(\mathrm{t_s}) = 0.12500519281775$$

It is evident that the convergence rate of the fast terminal sliding mode is far better than its linear counter part. The obvious reason is when close to the equilibrium, the convergence rate of the linear sliding mode exponentially slows down while the convergence rate of the fast sliding mode accelerates exponentially.

The recursive structure based on the fast sliding mode concept for higher order systems can be derived as [6]

$$\begin{aligned} s_1 &= \dot{s}_0 + \alpha_0 s_0 + \beta_0 s_0^{q_0/p_0} \\ s_2 &= \dot{s}_1 + \alpha_1 s_1 + \beta_1 s_1^{q_1/p_1} \\ &\vdots \qquad\quad \vdots \\ s_{n-1} &= \dot{s}_{n-2} + \alpha_{n-1} s_{n-2} + \beta_{n-2} s_{n-2}^{q_{n-2}/p_{n-2}} \end{aligned} \tag{3}$$

Here $\alpha_i, \beta_i > 0$ and q_i, p_i are positive odd integers $(i = 1,2,...,n-2)$. One can easily see that if s_{n-1} reaches zero, $s_{n-2}, s_{n-3},..., s_0$ will reach zero subsequently according to the dynamical structure of the terminal attractor (1).

In this paper, without loss of generality, we consider the higher order SISO nonlinear system described by

$$\begin{aligned} \dot{x}_i &= x_{i+1} \qquad i = 1,2,\cdots,n-1 \\ \dot{x}_n &= f(x) + g(x)u \end{aligned} \tag{4}$$

where $f(x)$ and $g(x)$ are smooth scalar fields on R^n, $g(x) \neq 0$ and $u \in R^1$. Note that any relative degree *n* nonlinear SISO system can be formulated as (3) using the Lie brackets (See [7] for details).

2.2 Fast Terminal Sliding Mode Control Design

In this section, we propose a controller design based on the fast sliding mode concept. Instead of using the conventional control design principle $s_{n-1}\dot{s}_{n-1} < -k|s_{n-1}|$ [3], which usually results in a discontinuous controller, we use the fast terminal sliding mode concept in the reaching phase as well. The main result is presented below.

Theorem 1 For system (4), if we choose the following control law

$$\begin{aligned} u(t) = -\frac{1}{g(x)}\Big(& f(x) + \sum_{k=0}^{n-2} \alpha_k s_k^{(n-k-1)} \\ &+ \sum_{k=0}^{n-2} \beta_k \frac{d^{n-k-1}}{dt^{n-k-1}} s_k^{(q_k/p_k)} + \phi\, s_{n-1} + \gamma\, s_{n-1}^{q/p}\Big) \end{aligned} \tag{5}$$

with $s_0 = x_1$, then the system states will reach the sliding manifold $s_{n-1} = 0$ according to the terminal attractor $\dot{s}_{n-1} = -\phi\, s_{n-1} - \gamma\, s_{n-1}^{q/p}$ in finite time $t_{s_{n-1}}$, where

$$t_{s_{n-1}} = \frac{p}{\phi(p-q)} \ln \frac{\phi\, x_1(0)^{(p-q)/p} + \gamma}{\gamma} \tag{6}$$

with $\phi, \gamma > 0$, and *p*, *q* being odd positive integer (*q*<*p*). The system will follow the recursive structure (3) to converge to the system equilibrium in finite time.

Proof. Taking the first order derivative of s_{n-1}, we can get

$$\dot{s}_{n-1} = \ddot{s}_{n-2} + \alpha_{n-2}\dot{s}_{n-2} + \beta_{n-2}\frac{d}{dt} s_{n-2}^{q_{n-2}/p_{n-2}}$$

Since $s_i = \dot{s}_{i-1} + \alpha_{i-1}s_{i-1} + \beta_{i-1}s_{i-1}^{q_{i-1}/p_{i-1}}$, $i = n-1, n-2, \cdots, 1$, and the l th order derivative of s_i is

$$s_i^{(l)} = s_{i-1}^{(l+1)} + \alpha_{i-1}s_{i-1}^{(l)} + \beta_{i-1}\frac{d^l}{dt^l} s_{i-1}^{q_{i-1}/p_{i-1}}$$

Then it can be easily induced step by step till the following equation is arrived

$$\begin{aligned} \dot{s}_{n-1} &= s_0^{(n)} + \sum_{k=0}^{n-2} \alpha_k s_k^{(n-k-1)} + \sum_{k=0}^{n-2} \beta_k \frac{d^{n-k-1}}{dt^{n-k-1}} s_k^{(q_k/p_k)} \\ &= \dot{x}_n + \sum_{k=0}^{n-2} \alpha_k s_k^{(n-k-1)} + \sum_{k=0}^{n-2} \beta_k \frac{d^{n-k-1}}{dt^{n-k-1}} s_k^{(q_k/p_k)} \\ &= f(x) + g(x)u(t) + \sum_{k=0}^{n-2} \alpha_k s_k^{(n-k-1)} \\ &\quad + \sum_{k=0}^{n-2} \beta_k \frac{d^{n-k-1}}{dt^{n-k-1}} s_k^{(q_k/p_k)} \end{aligned} \tag{7}$$

Substituting the control law (5) into (7) yields

$$\dot{s}_{n-1} = -\phi\, s_{n-1} - \gamma\, s_{n-1}^{q/p} \tag{8}$$

Given the conditions in Theorem 1, (8) is a fast terminal attractor discussed in Section 2.1. Therefore $s_{n-1}=0$ will be reached in time defined by (6). According to the recursive structure (3) [6], the system states will reach the system equilibrium in finite time. Q.E.D.

Note that in order to avoid the singularity problem in the terminal sliding modes, the conditions

$$\frac{q_k}{p_k} > \frac{n-k-1}{n-k}\,,\ k = n-2,\dots 0$$

must be satisfied [6].

2.3 Robustness Analysis

The strength of the sliding mode control lies in its robustness in parameter uncertainties and external disturbances. We now investigate the robustness of the proposed global sliding mode control scheme.

Let us consider the following SISO nonlinear uncertain system described

$$\begin{aligned} \dot{x}_i &= x_{i+1} \qquad i = 1,2,\cdots,n-1 \\ \dot{x}_n &= f_0(x) + \Delta f(x) + g(x)u + d(x) \end{aligned} \tag{9}$$

where $f_0(x)$ and $g(x)$ are the smooth nominal scalar fields on R^n, $g(x)\neq 0$ and $u\in R^1$. $\Delta f(x)$ and $d(x)$ represent the system uncertainties and external disturbance respectively. Assume that $\left|\Delta f(x)+d(x)\right|\le L$, we have the following theorem.

Theorem 2. For system (9), if we choose the following control law

$$\begin{aligned} u(t) = -\frac{1}{g(x)}\Big(& f_0(x) + \sum_{k=0}^{n-2}\alpha_k s_k^{(n-k-1)} \\ & + \sum_{k=0}^{n-2}\beta_k \frac{d^{\,n-k-1}}{dt^{\,n-k-1}} s_k^{(q_k/p_k)} + \phi\, s_{n-1} + \gamma\, s_{n-1}^{v/w}\Big) \end{aligned} \tag{10}$$

then the system will reach the neighborhood Δ of the sliding manifold $s_{n-1}=0$ according to the terminal attractor $\dot{s}_{n-1} = -\phi\, s_{n-1} - \gamma' s_{n-1}^{v/w}$ at least in finite time

$$t'_{s_{n-1}} = \frac{p}{\phi(p-q)} \ln\frac{\phi\, x_1(0)^{(p-q)/p} + \eta}{\eta} \tag{11}$$

where

$$\begin{aligned} \gamma' &= \gamma - \frac{\Delta f(x) + d(x)}{s_{n-1}^{q/p}} \\ \gamma &= \frac{L}{\left|s_{n-1}^{q/p}\right|} + \eta,\ \eta > 0 \end{aligned} \tag{12}$$

$$\Delta = \left\{ x\!:\ \left|s_{n-1}\right| \le \left(\frac{L}{\gamma}\right)^{p/q} \right\}$$

with $\phi,\gamma>0$, and *p*, *q* being odd positive integer (*q*<*p*).

Proof. Because $f(x)$ includes the uncertainties, replacing $f(x)$ with its nominal part $f_0(x)$ in control law (5) and substituting the control (10) into (9) yields

$$\begin{aligned} \dot{s}_{n-1} &= f(x) - f_0(x) - \phi\, s_{n-1} - \gamma\, s_{n-1}^{q/p} + d(x) \\ &= \Delta f(x) + d(x) - \phi\, s_{n-1} - \gamma\, s_{n-1}^{q/p} \\ &= -\phi\, s_{n-1} - \left(\gamma - \frac{\Delta f(x)+d(x)}{s_{n-1}^{q/p}}\right) s_{n-1}^{q/p} \\ &= -\phi\, s_{n-1} - \gamma' s_{n-1}^{q/p} \end{aligned}$$

To be able to have the fast terminal convergence, we must make the following inequality hold:

$$\gamma' = \gamma - \frac{\Delta f(x)+d(x)}{s_{n-1}^{q/p}} > 0$$

According to the conditions in (12), we have that

$$\begin{aligned} \gamma' &= \eta + \frac{L}{\left|s_{n-1}^{q/p}\right|} - \frac{\Delta f(x)+d(x)}{s_{n-1}^{q/p}} \\ &> \eta + \frac{L}{\left|s_{n-1}^{q/p}\right|} - \frac{\left|\Delta f(x)+d(x)\right|}{\left|s_{n-1}^{q/p}\right|} \ge \eta > 0 \end{aligned}$$

It means that conditions in (12) guarantees $\gamma' \ge \eta > 0$. Thus it can be seen that in the region Δ constrained by

$$\left|s_{n-1}\right| > \left(\frac{L}{\gamma}\right)^{p/q}$$

The condition $\gamma' > 0$ holds hence the fast terminal convergence is guaranteed as shown in Section 2.1. Therefore, it can be said that the system state will reach the neighborhood of the equilibrium Δ constrained by $\left|s_{n-1}\right| < \left(\frac{L}{\gamma}\right)^{p/q}$ at least in finite time (11) (see also Proof of Theorem 1). Hence the proof of the theorem is completed. Q.E.D.

Remark. As a matter of fact, if we choose large enough $\gamma > L$ and p/q, we can always make the neighborhood of the sliding manifold s_{n-1} Δ small enough since the neighborhood is defined by $\left|s_{n-1}\right| < \left(\frac{L}{\gamma}\right)^{p/pq}$. For example, if $L=1, q/p=\frac{1}{9}, \gamma=2$, then δ will be $\left|s_{n-1}\right| \le \left(\frac{L}{\gamma}\right)^{p/q} = 1.95\times10^{-3}$. The strength of our method lies in the choice of values of p, q.

3. Simulations

Now we present some simulations for validating the proposed scheme. Let us consider the following second order SISO nonlinear system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \cos x_1 + (x_1^2+1)u \end{aligned}$$

The terminal sliding manifold is designed as

$$s_1 = \dot{s}_0 + \alpha s_0 + \beta s_0^{q_0/p_0} = 0$$

where $s_0 = x_1$. In the reaching mode, we design a global terminal sliding mode controller to push the system state to the sliding manifold by the following fast terminal attractor

$$\dot{s} = -\phi s - \gamma s^{q/p}$$

According to Section 2, the global terminal sliding mode controller was given as

$$\begin{aligned} u &= \frac{1}{x_1^2+1}\left(-\cos x_1 - \alpha\dot{x}_1 - \beta\frac{q}{p}x_1^{(q_0-p_0)/p_0}\dot{x}_1 - \phi s - \gamma s^{q/p}\right) \\ &= \frac{1}{x_1^2+1}\left(-\cos x_1 - \alpha x_2 - \beta\frac{q}{p}x_1^{(q_0-p_0)/p_0}x_2 - \phi s - \gamma s^{q/p}\right) \end{aligned}$$

The designed control parameters are chosen as $\alpha=2, \beta=1, p_0=9, q_0=5, \phi=10, \gamma=10, p=3, q=1$. The simulation results are shown in Figure 1. To compare with the corresponding linear sliding manifold, we used the fast terminal attractor only in the reaching mode, but the sliding manifold is linear with $p=q$, the other parameters are the same as Figure 1. The results of the simulations are shown in Figure 2. From the figures it appears that the dynamical performances of both controls are similar. However, Figure 1(c) and Figure 2(c), which depict the transient performances of the system under the two controls between 1.98sec-2.03sec, tell the difference. The global terminal sliding mode controller much improved the convergence rate towards zero.

To verify the robustness of global terminal sliding mode controller, let $f_0(x)=\cos(x), \Delta f(x)+d(x)=\sin(x)$. The results of the simulations under the global terminal sliding mode controller are shown in Figure 3. One can easily see that the uncertainty/disturbance were well suppressed.

4. Conclusions

From the above analysis, design and simulations we can conclude that, first, the global sliding mode controller guarantees that the system reach the sliding manifolds in finite time and then the terminal sliding modes will further take the system states to the origin in finite time. Furthermore, the finite time to reach the sliding manifolds can be adjusted to the desired values in the design phase by choosing adequate p, q. In addition, the proposed controller is a continuous one, hence the troublesome chattering modes in the conventional sliding mode control systems are avoided. Second, the global sliding mode controller is robust to the bounded system uncertainties and disturbances, and can push the system states into a small enough neighborhood of the sliding manifolds fairly easily with small enough ratio q/p.

Acknowledgements

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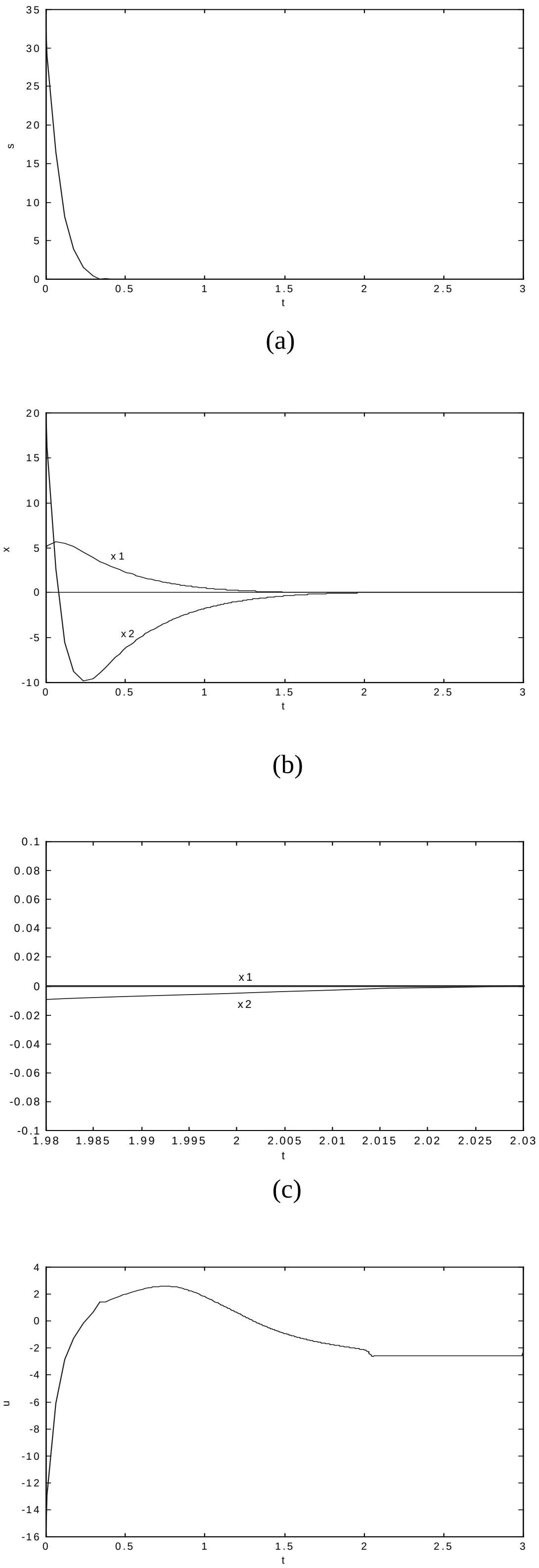

(a)

(b)

(c)

(d)

Figure 1. Global terminal sliding mode control with $p_0 = 9, q_0 = 5$.

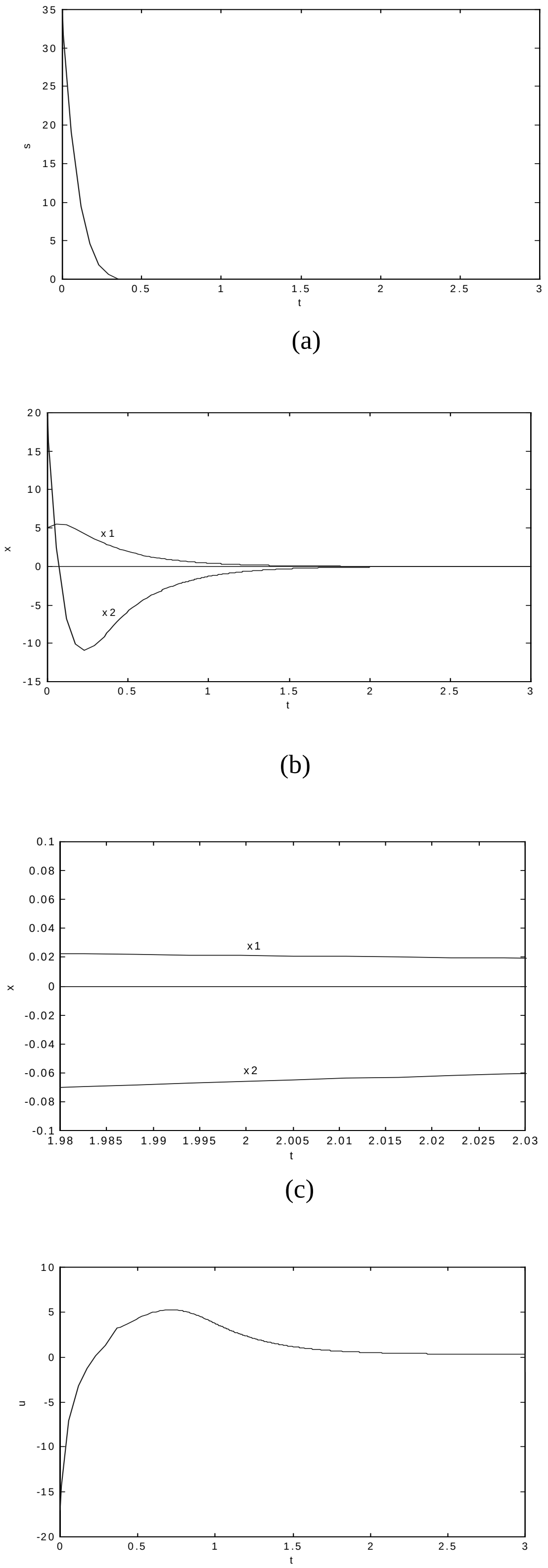

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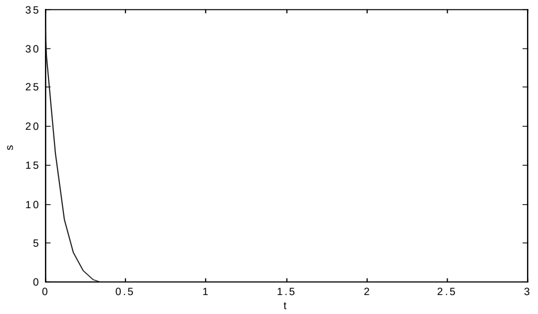
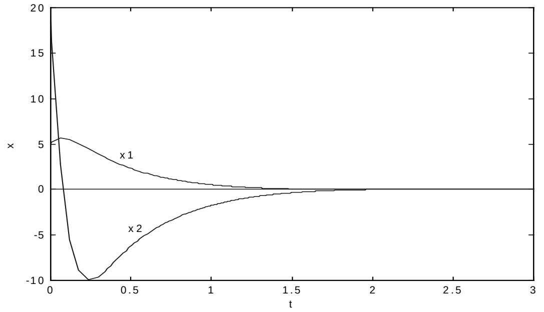
Figure 2. Global terminal sliding mode control with. $p_0 = q_0$



(a)



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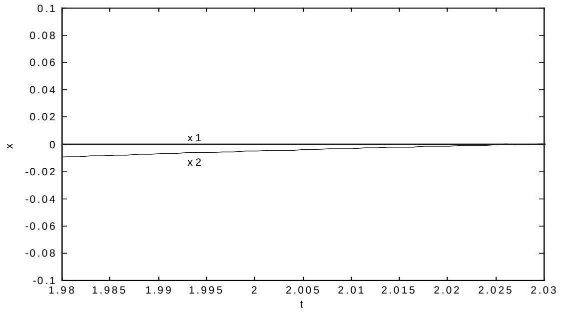

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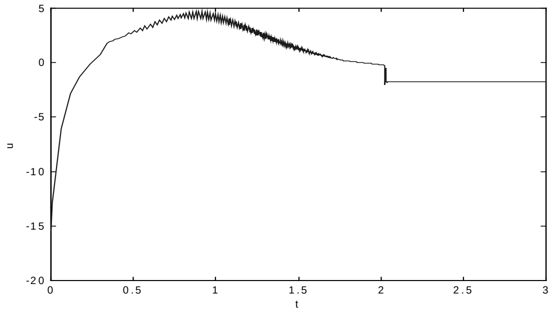

(d)

Figure 3. Global terminal sliding mode control for an uncertain system.

For all above figures: (a) Sliding function.(b) System states. (c) A segment of system states. (d) Control