New State-Dependent Scaling for Uncertain Dynamics: Nonlinear Global Stabilization and Performance

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Abstract

This paper proposes a simple and systematic design approach to global robust stabilization of nonlinear systems in the presence of dynamic and static uncertainties. An extended concept of state-dependent scaling is newly introduced for robustification of feedback control against dynamic uncertainties. This paper presents a recursive design procedure which provides a global stabilizing state-feedback controller whenever the system belongs to a new class of strict-feedback systems allowing both dynamic and static uncertainties. The state-dependent scaling method reduces problems of robust $\mathcal{L}_2$ disturbance attenuation and robust almost disturbance decoupling to a special case of the robust stabilization design.

1 Introduction

Global robust stabilization of nonlinear systems with unknown parameters and uncertain static nonlinearities with known functional bounds has been extensively studied for a special class of nonlinear systems called the strict-feedback form (possibly including stable zero dynamics). For example, backstepping with parameter uncertainty was pursued in the works of [2, 3, 8] and references therein. However, backstepping design with unknown unmodeled dynamics has less attention than backstepping with the parameter uncertainty. A gain margin for a fixed linear unmodeled dynamics at the system input was investigated in [1], where dynamic nonlinear damping was introduced in order to robustify the state-feedback control law. In contrast to static uncertainty, inadequate high gain domination may lead to the loss of robustness to unmodeled dynamics[14]. Dynamic uncertainties may cause dramatic shrinking of the region of attraction or finite escape time[1]. Optimal control design posses a certain margin of stability[14]. However, in general, an inverse optimal control are not sufficient to guarantee global robustness to dynamic uncertainties although it may establish robustness to static uncertainties. The concept of robust control Lyapunov functions defined in [8] is not applicable to dynamic uncertainties either. Usually, redesign of control Lyapunov functions constructed by backstepping are required to robustify control against dynamic uncertainties[13, 15, 16]. These references consider input unmodeled dynamics as strict passive systems and they do not address the robustness problem for $\mathcal{L}_2$-gain bounded uncertainties which have become a popular and useful model recently in linear robust control. Both types of modeling are practically important as they complement each other in diverse situations. This paper initiates unified investigation of robust backstepping which directly ensures global asymptotic stability for $\mathcal{L}_2$-gain bounded dynamic and static uncertainty. For this purpose, the design method in [8] is extended in this paper. The development in this paper can be also considered as a 'global robustification' of the previous result[10] against dynamic uncertainties. This paper successfully extends the state-dependent scaling design[10] by allowing the scaling to depend on the state variable even for

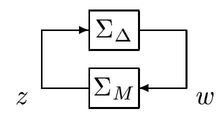

Figure 1: Uncertain system Σ for robust stability

dynamic uncertainties.

This paper encompasses $\mathcal{L}_2$ disturbance attenuation and $\mathcal{L}_2$ almost disturbance decoupling of strict-feedback nonlinear systems subject to static and dynamic uncertainties. This paper refers to them as robust performance problem. Recently, Su et al.[4] has extended the results of [5] and [6] by allowing for parameter uncertainty in the system with the triangular structure. The controlled output can be corrupted by the disturbance input in [4]. It, however, cannot involve the control input. In contrast, this paper is aimed at solving the robust disturbance attenuation problem where the controlled output can involve the control input, and the uncertainty is allow to be dynamic as well. Due to this extension, input unmodeled dynamics can be incorporated in the robust disturbance attenuation design. This paper demonstrates that, with the extended concept of state-dependent scaling, we can regard the robust performance problem as a special case of the robust stabilization problem.

2 Systems with dynamic and static uncertainty

Consider the uncertain nonlinear control system Σ shown in Fig.1. The part denoted by Σ_M represents a nominal system which is described by

$$\Sigma_M : \begin{cases} \dot{x} = A(x)x + B(x)w \\ z = C(x)x + D(x)w \end{cases} , \begin{array}{l} x(t) \in \mathcal{R}^n, \ w(t) \in \mathcal{R}^p \\ z(t) \in \mathcal{R}^p \end{array} \quad (1)$$

$$w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix}, \quad z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_m \end{bmatrix}, \quad \begin{array}{c} w_i(t), z_i(t) \in \mathcal{R}^{p_i} \\ p_i \geq 0 \\ p = \sum_{i=1}^m p_i \end{array} \quad (2)$$

The system Σ_Δ represents an uncertainty. In general, the representation (1) of a nonlinear plant Σ_M is not unique. Without loss of generality, it is assumed that w_i and z_i have the same dimension. The matrices A, B, C, and D are assumed to be $\mathcal{C}^0$ functions of x. The uncertain part Σ_Δ is a nonlinear mapping $\Delta : z \mapsto w$ which has the following structure.

$$w = \Delta z = \text{block-diag}[\Delta_1, \Delta_2, \cdots, \Delta_m] z \ . \quad (3)$$

The dimension of input and output vectors of Δ_i can be zero. Each mapping $\Delta_i : z_i \mapsto w_i$ is allowed to have two types of components[1]:

$$\Delta_i : z_i = \begin{bmatrix} z_{id} \\ z_{is} \end{bmatrix} \mapsto w_i = \begin{bmatrix} w_{id} \\ w_{is} \end{bmatrix}, \quad w_i = \begin{bmatrix} \Delta_{id} & 0 \\ 0 & \Delta_{is} \end{bmatrix} z_i \ . \quad (4)$$

[1]Repeated static uncertainties can be also incorporated in Δ_i without any difficulties as shown in [10]. This paper does not distinguish repeated static uncertainties from the set of static uncertainties to avoid notational complexity.

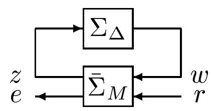

Figure 2: Uncertain system $\bar{\Sigma}$ for robust performance

Here, Δ_{id} represents a dynamic system. Δ_{is} denotes a static system. It is unnecessary for Δ_i to have the both types. The dynamic uncertainty Δ_{id} is defined by

$$\Delta_{id}\ :\left\{\begin{array}{l}\dot{x}_{\Delta_i}=f_{\Delta_{id}}(x_{\Delta_i},z_{id},t)\\ w_{id}=h_{\Delta_{id}}(x_{\Delta_i},z_{id},t)\end{array}\right. , \tag{5}$$

where $f_{\Delta_{id}}(0,0,t)=0$ and $h_{\Delta_{id}}(0,0,t)=0$ for all $t\geq 0$. The state of Σ is $x_{cl}=[x^T,x_\Delta^T]^T=[x^T,x_{\Delta_1}^T,x_{\Delta_2}^T,\cdots,x_{\Delta_m}^T]^T\in\mathcal{R}^n\times\mathcal{R}^{n_\Delta}$. The static part Δ_{is} is defined by

$$\Delta_{is}\ : w_{is}=h_{\Delta_{is}}(z_{is},t), \tag{6}$$

where $h_{\Delta_{is}}(0,t)=0$ for all $t\geq 0$. This paper concentrates on the following class of uncertainties.

Definition 1 *The uncertainty Σ_Δ is said to be admissible if* **(i)**-**(ii)** *are satisfied for $i=1,2,\ldots,m$:* **(i)** *The equilibrium $x_{\Delta_i}=0$ of Δ_{id} is globally uniformly asymptotically stable and Δ_{id} has $\mathcal{L}_2$-gain less than or equal to one globally with a $\mathcal{C}^1$ storage function $V_{\Delta i}(x_{\Delta_i})$ which is positive definite and radially unbounded.* **(ii)** *Δ_{is} satisfies $\|z_{is}\|^2\geq\|w_{is}\|^2$ for all $t\in[0,\infty)$.*

In the condition (i) we imply

$$V_{\Delta i}(x_{\Delta_i}(0))+\int_0^T(z_{id}^Tz_{id}-w_{id}^Tw_{id})dt\geq V_{\Delta i}(x_{\Delta_i}(T)),\ \forall T>0$$

The system Σ is said to be robustly globally uniformly asymptotically stable if the equilibrium $x_{cl}=0$ is globally uniformly asymptotically stable for all admissible uncertainties Σ_Δ. The system Σ_M not only describes a nominal plant, but also includes information about location, magnitude, structure and nonlinearity(in x) of uncertainties arising from the system Σ.

This paper addresses robust disturbance attenuation as well as the robust stabilization. For the robust disturbance attenuation problem, we consider another uncertain system $\bar{\Sigma}$ shown in Fig.2. The signal w and z are defined as in (2). The uncertain system Σ_Δ defined in (3-6) is connected to the nominal system $\bar{\Sigma}_M$ through z and w. The exogenous signal r and the controlled output e are

$$r=\begin{bmatrix}r_1\\ r_2\\ \vdots\\ r_m\end{bmatrix},\quad e=\begin{bmatrix}e_1\\ e_2\\ \vdots\\ e_m\end{bmatrix},\quad \begin{array}{c}r_i(t),e_i(t)\in\mathcal{R}^{q_i}\\ q_i\geq 0\\ q=\sum_{i=1}^m q_i\end{array} \tag{7}$$

It is assumed that the system $\bar{\Sigma}_M$ is described by

$$\bar{\Sigma}_M\ :\left\{\begin{array}{l}\dot{x}=A(x)x+\bar{B}(x)\bar{w}\\ \bar{z}=\bar{C}(x)x+\bar{D}(x)\bar{w}\end{array}\right. ,\ \begin{array}{c}x(t)\in\mathcal{R}^n\\ \bar{w}(t),\bar{z}(t)\in\mathcal{R}^{p+q}\end{array} \tag{8}$$

$$\bar{w}=\begin{bmatrix}w_1\\ r_1\\ w_2\\ r_2\\ \vdots\\ w_m\\ r_m\end{bmatrix},\quad \bar{z}=\begin{bmatrix}z_1\\ e_1\\ z_2\\ e_2\\ \vdots\\ z_m\\ e_m\end{bmatrix}\in\mathcal{R}^{p+q} \tag{9}$$

Each definition of $\bar{w}$ and $\bar{z}$ interlaces two signals in order to fit for a recursive method to be introduced. The system $\bar{\Sigma}$ is said to have $\mathcal{L}_2$-gain less than or equal to λ if the mapping from r to e has such gain. The system $\bar{\Sigma}$ is said to achieve robust global performance if $\bar{\Sigma}$ is robustly globally uniformly asymptotically stable and it has $\mathcal{L}_2$-gain less than or equal to one globally for all admissible uncertainties Σ_Δ.

3 SD scaling for dynamic uncertainty

In order to evaluate robustness of the system Σ with respect to the dynamic uncertainty Δ_{id}, this paper proposes a new pair of scaling matrices as follows:

$$\begin{aligned}\mathbf{\Phi}_{id}&=\Big\{\Phi_{id}(x)=\phi_d(V_0(x))\hat{\phi}_iI:\\ &\qquad\left.\begin{array}{l}\phi_d(\cdot)\in\mathcal{C}^0,\quad\phi_d(s)>0,\ \forall s\in[0,\infty),\quad\hat{\phi}_i>0\\ \exists\mu_\phi(\cdot)\in\mathcal{K}\text{ s.t. }\dfrac{s}{\phi_d(s)}\geq\mu_\phi(s),\ \forall s\in[0,\infty)\end{array}\right\}\end{aligned} \tag{10}$$

$$\mathbf{\Theta}_{id}=\{\Theta_{id}(x)=\theta_{id}(x)I:\theta_{id}(\cdot)\in\mathcal{C}^0,\ \theta_{id}(x)>0\ \forall x\in\mathcal{R}^n\}\qquad i=1,2,\ldots,m \tag{11}$$

where $\hat{\phi}_i$, $i=1,2,\ldots,m$ are real scalars and I is an identity matrix which is compatible in size with z_{id}. All sets $\mathbf{\Phi}_{id}$, $i=1,2,\ldots,m$ are defined with a common function $\phi_d(s)$. The scalar $V_0(\cdot)$ is a $\mathcal{C}^0$ function $\mathcal{R}^n\to[0,\infty)$ of the state x. Both $\mathbf{\Phi}_{id}$ and $\mathbf{\Theta}_{id}$ are 'state-dependent'. For the static uncertainty Δ_{is}, a pair of scaling matrices are defined by

$$\mathbf{\Phi}_{is}=\{\Phi_{is}(x)=\phi_{is}(x)I:\phi_{is}(\cdot)\in\mathcal{C}^0,\ \phi_{is}(x)>0\ \forall x\in\mathcal{R}^n\} \tag{12}$$

$$\mathbf{\Theta}_{is}=\{\Theta_{is}(x)=\theta_{is}(x)I:\theta_{is}(\cdot)\in\mathcal{C}^0,\ \theta_{is}(x)>0\ \forall x\in\mathcal{R}^n\} \tag{13}$$

The identity matrices are compatible in size with z_{is}. Note that $\mathbf{\Phi}_{id}$ is identical with $\mathbf{\Theta}_{id}$ when $\phi_d\hat{\phi}_i=\theta_{id}$ holds. In the same manner, $\mathbf{\Phi}_{is}$ is identical with $\mathbf{\Theta}_{is}$ when $\phi_{is}=\theta_{is}$. The definition of $\mathbf{\Theta}_{id}$, $\mathbf{\Phi}_{is}$ and $\mathbf{\Theta}_{is}$ is the same as the state-dependent scaling employed in [10, 9]. The new scaling function $\mathbf{\Phi}_{id}$ is different from those state-dependent scaling matrices in that the growth of the state dependence of $\mathbf{\Phi}_{id}$ is constrained. Constant scaling matrices are included in $\mathbf{\Phi}_{id}$. The following lemma will play an important role.

Lemma 1 *Suppose that $\eta(x)$ is a $\mathcal{C}^1$ function of $x\in\mathcal{R}^n$ which is positive definite and radially unbounded. Let $\phi(\cdot)$ be a $\mathcal{C}^0$ function which fulfills the following properties.*

$$\begin{array}{l}\phi(s)>0,\ \forall s\in[0,\infty)\\ \exists\mu(\cdot)\in\mathcal{K}\text{ s.t. }\dfrac{s}{\phi(s)}\geq\mu(s),\ \forall s\in[0,\infty)\end{array}$$

Then, the function

$$\zeta(x)=\int_0^{\eta(x)}\frac{1}{\phi(s)}ds$$

is $\mathcal{C}^1$, positive definite and radially unbounded.

For $i=1,2,\ldots,m$, define $\Phi_i(x)$ and $\Theta_i(x)$ as

$$\mathbf{\Phi}_i=\left\{\Phi_i(x)=\begin{bmatrix}\Phi_{id}(x)&0\\ 0&\Phi_{is}(x)\end{bmatrix}:\begin{array}{l}\Phi_{id}\in\mathbf{\Phi}_{id}\\ \Phi_{is}\in\mathbf{\Phi}_{is}\end{array}\right\}$$

$$\mathbf{\Theta}_i=\left\{\Theta_i(x)=\begin{bmatrix}\Theta_{id}(x)&0\\ 0&\Theta_{is}(x)\end{bmatrix}:\begin{array}{l}\Theta_{id}\in\mathbf{\Theta}_{id}\\ \Theta_{is}\in\mathbf{\Theta}_{is}\end{array}\right\}$$

Two sets of scaling matrices for the whole system Σ_Δ are

$$\mathbf{\Phi}=\left\{\Phi(x)=\underset{i=1}{\overset{m}{\text{block-diag}}}\,\Phi_i(x),\ \Phi_i\in\mathbf{\Phi}_i\right\} \tag{14}$$

$$\mathbf{\Theta}=\left\{\Theta(x)=\underset{i=1}{\overset{m}{\text{block-diag}}}\,\Theta_i(x),\ \Theta_i\in\mathbf{\Theta}_i\right\} \tag{15}$$

Now, using the scaling matrices, we characterize robust stability of Σ shown in Fig.1. We consider the diffeomorphism between $x\in\mathcal{R}^n$ and $\chi\in\mathcal{R}^n$ as follows:

$$\chi=S(x)x\ . \tag{16}$$

The time-derivative of χ is obtained as

$$\dot{\chi} = \left[\frac{\partial S}{\partial x_1}x, \frac{\partial S}{\partial x_2}x, \cdots, \frac{\partial S}{\partial x_n}x\right]\dot{x} + S(x)\dot{x} = T(x)\dot{x}\ .$$

Let $\chi_{[\kappa]}$ and $\chi_{\langle\kappa\rangle}$ denote

$$\chi_{[\kappa]} = [\chi_1, \chi_2, \cdots, \chi_\kappa]^T\ , \qquad \chi_{\langle\kappa+1\rangle} = [\chi_{\kappa+1}, \chi_{\kappa+2}, \cdots, \chi_n]^T$$

respectively. Note that $x = x_{[n]} = x_{\langle 1\rangle}$. The following is the main result of this section.

Theorem 1 *Suppose that there exist an integer* $\kappa \in [0, n]$, *constant symmetric matrices* $P_{[\kappa]} \in \mathcal{R}^{\kappa\times\kappa}$, $P_{\langle\kappa+1\rangle} \in \mathcal{R}^{(n-\kappa)\times(n-\kappa)}$, *and scaling functions* $\Phi \in \mathbf{\Phi}$, $\Theta \in \mathbf{\Theta}$ *such that*

$$\begin{bmatrix} S^{-T}A^TT^T\Xi+\Xi TAS^{-1} & \Xi TB & S^{-T}C^T\Phi \\ B^TT^T\Xi & -\Theta & D^T\Phi \\ \Phi CS^{-1} & \Phi D & -\Phi \end{bmatrix} < 0 \qquad (17)$$

$$P = \begin{bmatrix} P_{[\kappa]} & 0 \\ 0 & P_{\langle\kappa+1\rangle} \end{bmatrix} > 0 \qquad (18)$$

$$\Theta \leq \Phi \qquad (19)$$

are satisfied for all $x \in \mathcal{R}^n$ *with*

$$V_0(x) = \chi_{[\kappa]}^T P_{[\kappa]} \chi_{[\kappa]} \qquad (20)$$

$$\Xi(x) = \begin{bmatrix} I_\kappa & 0 \\ 0 & \phi_d(V_0(x))I_{n-\kappa} \end{bmatrix} P \qquad (21)$$

Then, the system Σ *is robustly globally uniformly asymptotically stable.*

Note that $\Xi = P$ holds in either case of $\kappa = 0$ and $\kappa = n$. When $\kappa = 0$ is chosen, we use the definition $\phi_d(V_0) = 1$, and the function $V_0(x)$ is not involved in Theorem 1. In the case of $\kappa = 0$ and $\Theta = \Phi$, Theorem 1 reduces to a theorem proposed in [10]. In Section 5, it will be shown that such a choice $\kappa = 0$ is not sufficient for achieving global stabilization against dynamic uncertainties. The first term of the Lyapunov function

$$V(x_{cl}) = \int_0^{V_{0[\kappa]}(\chi_{[\kappa]})} \frac{1}{\phi_d(s)} ds + \chi_{\langle\kappa+1\rangle}^T P_{\langle\kappa+1\rangle} \chi_{\langle\kappa+1\rangle} + \sum_{i=1}^{m} \hat{\phi}_i V_{\Delta i}(x_{\Delta i}) \qquad (22)$$

used in the above theorem is similar to Lyapunov functions appearing frequently in recent Lyapunov techniques(e.g. [17, 11, 13, 14, 16]) for various purposes. However, the idea and purpose for which this paper employs the function are distinct from those of them. In this paper, the integrand $1/\phi_d(s)$ in (22) is given a character of scaling and the function is determined by the matrix inequality (17). Theorem 1 demonstrates how to determine the integrand for guaranteeing robustness of stability with respect to dynamic uncertainties whose locations and structures are prescribed *a priori*. This new guideline for constructing a Lyapunov function for robustifying a nonlinear system against dynamic uncertainties is an important contribution of this paper. An explicit procedure for selecting $\phi_d(s)$ for a class of nonlinear systems will be presented in this paper.

The inequality (17) implies that the 'scaled' system

$$\dot{x} = A(x)x + B(x)\Theta^{-1/2}w$$
$$z = \Phi^{1/2}C(x)x + \Phi^{1/2}D(x)\Theta^{-1/2}w$$

has $\mathcal{L}_2$-gain less than one with a storage function x^TS^TPSx in the case of $\kappa \in \{0, n\}$. However, the original system (1) unnecessarily has $\mathcal{L}_2$-gain less than one since Φ and Θ are functions of the state x. The mapping Φ and Θ do not necessarily commute with the nonlinear functions of $\Delta : z \mapsto w$. In this sense, Theorem 1 is much more general and less conservative than the classical input-output small gain theorem for dissipative systems. Indeed, introducing the state-dependence which the classical small gain theorem lacks is crucial to guarantee global properties.

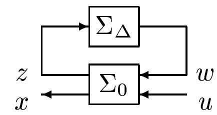

Figure 3: Uncertain plant Σ_P for robust stabilization

For evaluating the robust performance of $\bar{\Sigma}$, this paper introduces the following sets.

$$\bar{\mathbf{\Phi}}_{id} = \left\{ \bar{\Phi}_{id}(x) = \phi_d(V_0(x)) \begin{bmatrix} \hat{\phi}_i I & 0 \\ 0 & I \end{bmatrix} : \begin{array}{l} \phi_d(\cdot) \in \mathcal{C}^0, \quad \phi_d(s) > 0,\ \forall s \in [0,\infty), \quad \hat{\phi}_i > 0 \\ \exists \mu_\phi(\cdot) \in \mathcal{K} \text{ s.t. } \dfrac{s}{\phi_d(s)} \geq \mu_\phi(s),\ \forall s \in [0,\infty) \end{array} \right\} \qquad (23)$$

$$\bar{\mathbf{\Theta}}_{id} = \left\{ \bar{\Theta}_{id}(x) = \begin{bmatrix} \theta_{id}(x)I & 0 \\ 0 & \bar{\theta}_{id}(x)I \end{bmatrix} : \begin{array}{l} \theta_{id}(\cdot) \in \mathcal{C}^0, \quad \theta_{id}(x) > 0\ \forall x \in \mathcal{R}^n \\ \bar{\theta}_{id}(\cdot) \in \mathcal{C}^0, \quad \bar{\theta}_{id}(x) > 0\ \forall x \in \mathcal{R}^n \end{array} \right\} \qquad (24)$$

$$\bar{\mathbf{\Phi}}_i = \left\{ \bar{\Phi}_i(x) = \begin{bmatrix} \bar{\Phi}_{id}(x) & 0 \\ 0 & \Phi_{is}(x) \end{bmatrix} : \begin{array}{l} \bar{\Phi}_{id} \in \bar{\mathbf{\Phi}}_{id} \\ \Phi_{is} \in \mathbf{\Phi}_{is} \end{array} \right\} \qquad (25)$$

$$\bar{\mathbf{\Theta}}_i = \left\{ \bar{\Theta}_i(x) = \begin{bmatrix} \bar{\Theta}_{id}(x) & 0 \\ 0 & \Theta_{is}(x) \end{bmatrix} : \begin{array}{l} \bar{\Theta}_{id} \in \bar{\mathbf{\Theta}}_{id} \\ \Theta_{is} \in \mathbf{\Theta}_{is} \end{array} \right\} \qquad (26)$$

$$\bar{\mathbf{\Phi}} = \left\{ \bar{\Phi}(x) = \operatorname*{block-diag}_{i=1}^{m} \bar{\Phi}_i(x),\ \bar{\Phi}_i \in \bar{\mathbf{\Phi}}_i \right\} \qquad (27)$$

$$\bar{\mathbf{\Theta}} = \left\{ \bar{\Theta}(x) = \operatorname*{block-diag}_{i=1}^{m} \bar{\Theta}_i(x),\ \bar{\Theta}_i \in \bar{\mathbf{\Theta}}_i \right\} \qquad (28)$$

where $i = 1, 2, \ldots, m$. The block partition of $\bar{\Phi}_{id}$ and $\bar{\Theta}_{id}$ is compatible in size with $[z_{id}^T, e_i^T]^T$. The following theorem reduces the performance robustness problem to a 'scaling problem' which is very similar to Theorem 1.

Theorem 2 *Suppose that there exist an integer* $\kappa \in [0, n]$, *constant symmetric matrices* $P_{[\kappa]} \in \mathcal{R}^{\kappa\times\kappa}$, $P_{\langle\kappa+1\rangle} \in \mathcal{R}^{(n-\kappa)\times(n-\kappa)}$, *and scaling functions* $\bar{\Phi} \in \bar{\mathbf{\Phi}}$, $\bar{\Theta} \in \bar{\mathbf{\Theta}}$ *such that (18) and*

$$\begin{bmatrix} S^{-T}A^TT^T\Xi+\Xi TAS^{-1} & \Xi T\bar{B} & S^{-T}\bar{C}^T\bar{\Phi} \\ \bar{B}^TT^T\Xi & -\bar{\Theta} & \bar{D}^T\bar{\Phi} \\ \bar{\Phi}\bar{C}S^{-1} & \bar{\Phi}\bar{D} & -\bar{\Phi} \end{bmatrix} < 0 \qquad (29)$$

$$\bar{\Theta} \leq \bar{\Phi} \qquad (30)$$

are satisfied for all $x \in \mathcal{R}^n$ *with (20) and (21). Then, the system* $\bar{\Sigma}$ *achieves the robust global performance.*

If $\hat{\phi}_i = 1$ and $\theta_{id} = \bar{\theta}_{id}$ are set for all i satisfying $q_i > 0$, $\bar{\mathbf{\Phi}}_{id}$ and $\bar{\mathbf{\Theta}}_{id}$ are identical with $\mathbf{\Phi}_{id}$ and $\mathbf{\Theta}_{id}$, respectively. Thus, the robust performance problem can be recasted as a problem of robust stability in the presence of dynamic uncertainties. The scaling approach gives a unified framework for not only robust stability, but also robust performance problems.

4 State feedback control problem

From this section, we tackle state-feedback control design. Our primary problem is to find a state-feedback controller

$$\Sigma_K \ :\ u = K(x)x \qquad (31)$$

which globally uniformly asymptotically stabilize the uncertain nonlinear plant Σ_P shown in Fig.3. The feedback loop consisting of Σ_0 and Σ_K corresponds to the nominal system Σ_M in Fig.1. We focuses on a class of uncertain nonlinear systems Σ_P in a special form. The system Σ_0 in Fig.3 is a nominal part of the plant, which is described by

$$\Sigma_0 : \begin{cases} \dot{x} = A(x)x + B(x)w + G(x)u & x(t) \in \mathcal{R}^n, u(t) \in \mathcal{R} \\ z = C(x)x + D(x)w + H(x)u' & w(t), z(t) \in \mathcal{R}^{p+\pi} \end{cases} \quad (32)$$

$$w = \begin{bmatrix} w_1 \\ w_{U1} \\ \vdots \\ w_n \\ w_{Un} \end{bmatrix}, \; z = \begin{bmatrix} z_1 \\ z_{U1} \\ \vdots \\ z_n \\ z_{Un} \end{bmatrix}, \; \begin{matrix} w_i(t) \in \mathcal{R}^{p_i} \\ w_{Ui}(t) \in \mathcal{R}^{\pi_i} \\ z_i(t) \in \mathcal{R}^{p_i} \\ z_{Ui}(t) \in \mathcal{R}^{\pi_i} \\ p_i \geq 0, \; \pi_i \geq 0 \\ i = 1, 2, ..., n \end{matrix} \; \begin{matrix} p = \sum_{i=1}^n p_i \\ \\ \\ \pi = \sum_{i=1}^n \pi_i \end{matrix} \quad (33)$$

The system Σ_Δ represents the uncertain part of Σ_P and the mapping $\Delta : z \mapsto w$ has the following structure.

$$w = \Delta z = \text{block-diag}[\Delta_1, \Delta_{U1}, \Delta_2, \Delta_{U2}, \cdots, \Delta_n, \Delta_{Un}]z, \quad (34)$$

where some of $\Delta_i : z_i \mapsto w_i$ and $\Delta_{Ui} : z_{Ui} \mapsto w_{Ui}$, $i = 1, 2, \ldots, n$ can be zero in vector dimension. The vectors z_{Ui} and w_{Ui} are input and output, respectively, of the mapping Δ_{Ui}. The blocks Δ_{Ui} represent uncertainty in the virtual control coefficients which appear in backstepping(See [10]). Each of Δ_i and Δ_{Ui}, $i = 1, \ldots, n$ is allowed to have two types of components defined in (4-6). All uncertainties Δ_i, Δ_{Ui}, $i = 1, 2, \ldots, n$ are assumed to be admissible in the sense of Definition 1. We assume that Σ_0 has the following structure.

$$A(x) = \begin{bmatrix} a_{11} & a_{12} & 0 & \cdots & \cdots & 0 \\ a_{21} & a_{22} & a_{23} & 0 & & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ & & & & \ddots & 0 \\ a_{n-1,1} & a_{n-1,2} & \cdots & \cdots & & a_{n-1,n} \\ a_{n1} & a_{n2} & \cdots & \cdots & & a_{nn} \end{bmatrix}, \; G(x) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ a_{n,n+1} \end{bmatrix} \quad (35)$$

$$a_{ij}(x) = a_{ij}(x_1, x_2, \cdots, x_i), \quad 1 \leq i \leq n, \; 1 \leq j \leq i+1 \quad (36)$$

$$B(x) = \begin{bmatrix} B_{11} & U_{L1} & 0 & 0 & \cdots & 0 & 0 \\ B_{21} & U_{21} & B_{22} & U_{L2} & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ B_{n1} & U_{n1} & B_{n2} & U_{n2} & \cdots & B_{nn} & U_{Ln} \end{bmatrix} \quad (37)$$

$$D(x) = \begin{bmatrix} D_1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & D_2 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & D_n & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}, \; H(x) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ U_{Rn} \end{bmatrix} \quad (38)$$

$$C(x) = \begin{bmatrix} C_{11} & 0 & 0 & \cdots & 0 & 0 \\ 0 & U_{R1} & 0 & \cdots & 0 & 0 \\ C_{21} & C_{22} & 0 & \ddots & 0 & 0 \\ 0 & 0 & U_{R2} & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ C_{n-1,1} & C_{n-1,2} & \cdots & \cdots & C_{n-1,n-1} & 0 \\ 0 & 0 & \cdots & \cdots & 0 & U_{R,n-1} \\ C_{n1} & C_{n2} & \cdots & \cdots & C_{n,n-1} & C_{nn} \\ 0 & 0 & \cdots & \cdots & 0 & 0 \end{bmatrix}, \quad (39)$$

where $B_{ij} \in \mathcal{R}^{1 \times p_i}$, $C_{ij} \in \mathcal{R}^{p_i \times 1}$, $D_i \in \mathcal{R}^{p_i \times p_i}$, $U_{L,i} \in \mathcal{R}^{1 \times \pi_i}$, $U_{R,i} \in \mathcal{R}^{\pi_i \times 1}$ and $U_{li} \in \mathcal{R}^{1 \times \pi_i}$ are consistent with

$$B_{ij}(x) = B_{ij}(x_1, x_2, \cdots, x_i), \; C_{ij}(x) = C_{ij}(x_1, x_2, \cdots, x_i) \quad (40)$$

$$D_i(x) = D_i(x_1, x_2, \cdots, x_i) \quad (41)$$

$$U_{Li}(x) = U_{Li}(x_1, x_2, \cdots, x_i), \; U_{Ri}(x) = U_{Ri}(x_1, x_2, \cdots, x_i) \quad (42)$$

$$U_{li}(x) = U_{li}(x_1, x_2, \cdots, x_l), \quad i+1 \leq l \leq n \quad (43)$$

for $1 \leq i \leq n$ and $1 \leq j \leq i$. We also need an assumption

$$I - D_i(x)D_i^T(x) > 0, \quad \forall x \in \mathcal{R}^n, \forall i =, 1, 2, \ldots, n \quad (44)$$

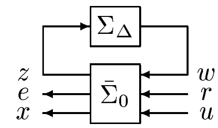

Figure 4: Uncertain plant $\bar{\Sigma}_P$ for robust performance

for guaranteeing the existence of a unique solution to the interconnection $(\Sigma_0, \Sigma_\Delta)$ in Fig. 3 for all admissible uncertainties. Finally, we use a standard assumption

$$a_{i,i+1}^2(x) > U_{Ri}(x)^T U_{Ri}(x) U_{Li}(x) U_{Li}^T(x), \quad \forall x \in \mathcal{R}^n \quad (45)$$

for $i = 1, 2, \ldots, n$ so that coefficients of virtual and actual inputs of Σ_P cannot be made zero by uncertainties[8].

An uncertain system Σ_P which admits a nominal part Σ_0 satisfying these structural conditions (35-45) is said to be in the extended robust strict-feedback form[10]. This class of systems extends the robust strict-feedback form[8] in that dynamic uncertainties are allowed.

According to Theorem 1, the uncertain system Σ_P is globally uniformly asymptotically stabilized by a state-feedback controller Σ_K for all admissible uncertainties Σ_Δ if there exist P $\Phi \in \mathbf{\Phi}$, $\Theta \in \mathbf{\Theta}$ such that (18) and

$$M = \begin{bmatrix} \hat{S}^T \hat{A}^T T^T \Xi + \Xi T \hat{A} \hat{S} & \Xi T B & \hat{S}^T \hat{C}^T \Phi \\ B^T T^T \Xi & -\Theta & D^T \Phi \\ \Phi \hat{C} \hat{S} & \Phi D & -\Phi \end{bmatrix} < 0 \quad (46)$$

$$\Theta \leq \Phi \quad (47)$$

are satisfied for all $x \in \mathcal{R}^n$ with (20-21), where $\hat{A}$ and $\hat{S}$ are

$$\hat{S} = \begin{bmatrix} S^{-1} \\ KS^{-1} \end{bmatrix}, \quad \hat{A} = [\, A \;\; G \,], \quad \hat{C} = [\, C \;\; H \,]$$

Our next design problem is to find a state-feedback controller Σ_K in (31) which globally uniformly asymptotically stabilize $\bar{\Sigma}_P$ shown in Fig.4 and achieves $\mathcal{L}_2$-gain less than or equal to one. Assume that $\bar{\Sigma}_0$ in Fig.4 is described by

$$\bar{\Sigma}_0 : \begin{cases} \dot{x} = A(x)x + \bar{B}(x)\bar{w} + G(x)u & x(t) \in \mathcal{R}^n, u(t) \in \mathcal{R} \\ \bar{z} = \bar{C}(x)x + \bar{D}(x)\bar{w} + \bar{H}(x)u' & \bar{w}(t), \bar{z}(t) \in \mathcal{R}^{p+\pi+q+\rho} \end{cases} \quad (48)$$

$$\bar{w} = \begin{bmatrix} w_1 \\ r_1 \\ w_{U1} \\ r_{U1} \\ \vdots \\ w_n \\ r_n \\ w_{Un} \\ r_{Un} \end{bmatrix}, \; \bar{z} = \begin{bmatrix} z_1 \\ e_1 \\ z_{U1} \\ e_{U1} \\ \vdots \\ z_n \\ e_n \\ z_{Un} \\ e_{Un} \end{bmatrix}, \; \begin{matrix} w_i(t) \in \mathcal{R}^{p_i}, \quad z_i(t) \in \mathcal{R}^{p_i} \\ r_i(t) \in \mathcal{R}^{q_i}, \quad e_i(t) \in \mathcal{R}^{q_i} \\ w_{Ui}(t) \in \mathcal{R}^{\pi_i}, \; z_{Ui}(t) \in \mathcal{R}^{\pi_i} \\ r_{Ui}(t) \in \mathcal{R}^{\rho_i}, \; e_{Ui}(t) \in \mathcal{R}^{\rho_i} \\ p_i \geq 0, \; q_i \geq 0, \; \pi_i \geq 0, \; \rho_i \geq 0 \\ i = 1, 2, \cdots, n \\ p = \sum_{i=1}^n p_i, \; \pi = \sum_{i=1}^n \pi_i \\ q = \sum_{i=1}^n q_i, \; \rho = \sum_{i=1}^n \rho_i \end{matrix} \quad (49)$$

$$w = \begin{bmatrix} w_1 \\ w_{U1} \\ w_2 \\ w_{U2} \\ \vdots \\ w_n \\ w_{Un} \end{bmatrix}, \; z = \begin{bmatrix} z_1 \\ z_{U1} \\ z_2 \\ z_{U2} \\ \vdots \\ z_n \\ z_{Un} \end{bmatrix}, \; r = \begin{bmatrix} r_1 \\ r_{U1} \\ r_2 \\ r_{U2} \\ \vdots \\ r_n \\ r_{Un} \end{bmatrix}, \; e = \begin{bmatrix} e_1 \\ e_{U1} \\ e_2 \\ e_{U2} \\ \vdots \\ e_n \\ e_{Un} \end{bmatrix} \quad (50)$$

The system Σ_Δ is the same as that for the robust stabilization problem. We define the extended robust strict-feedback form for $\bar{\Sigma}_P$ of the robust performance problem in the same way that we did for Σ_P except the following replacement.

$$\begin{matrix} B_{ij} & \to & \bar{B}_{ij} \in \mathcal{R}^{1 \times (p_i + q_i)} & , & C_{ij} & \to & \bar{C}_{ij} \in \mathcal{R}^{(p_i + q_i) \times 1} \\ D_i & \to & \bar{D}_i \in \mathcal{R}^{(p_i + q_i) \times (p_i + q_i)} & , & U_{Li} & \to & \bar{U}_{Li} \in \mathcal{R}^{1 \times (\pi_i + \rho_i)} \\ U_{ij} & \to & \bar{U}_{ij} \in \mathcal{R}^{1 \times (\pi_j + \rho_j)} & , & U_{Ri} & \to & \bar{U}_{Ri} \in \mathcal{R}^{(\pi_i + \rho_i) \times 1} \end{matrix}$$

Matrices $\hat{A}$, B, $\hat{C}$ and D are defined according to the the above replacement of their components. According to Theorem 2, by replacing $\{\Phi, \Theta\}$ with $\{\bar{\Phi}, \bar{\Theta}\}$, the inequality (46) again gives a sufficient condition for a state-feedback controller Σ_K achieving the robust performance for the uncertain system $\bar{\Sigma}_P$.

5 Recursive design with scaling

Let $x_{[k]}$ denote the first k components of the state x:

$$x_{[k]} = [x_1, x_2, \cdots, x_k]^T .$$

The nonsingular matrix $S(x)$ of smooth functions, the state-feedback law and the symmetric matrix P are chosen as

$$S(x) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -s_1 & 1 & 0 & \cdots & 0 \\ s_1 s_2 & -s_2 & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ (-1)^{n-1} s_1 \cdots s_{n-1} & \cdots & s_{n-2}s_{n-1} & -s_{n-1} & 1 \end{bmatrix} \quad (51)$$

$$u = s_n(x)\chi_n \quad (52)$$

$$P = \mathrm{diag}[P_1, P_2, \cdots, P_n] \quad (53)$$

where functions $s_1(x_{[1]})$, $s_2(x_{[2]})$, $\cdots$, $s_n(x_{[n]})$ are smooth and P_i, $i = 1, \ldots, n$ are positive real numbers. Candidates of state-dependent scaling are chosen in the form of

$$\Phi = \underset{j=1}{\overset{2n}{\text{block-diag}}}\, \Phi_j, \quad \Phi_j = \begin{cases} \phi_j(x_{[(j+1)/2]})I & \text{for odd } j \\ \phi_j(x_{[j/2]})I & \text{for even } j \end{cases} \quad (54)$$

$$\Theta = \underset{j=1}{\overset{2n}{\text{block-diag}}}\, \Theta_j, \quad \Theta_j = \begin{cases} \theta_j(x_{[(j+1)/2]})I & \text{for odd } j \\ \theta_j(x_{[j/2]})I & \text{for even } j \end{cases} \quad (55)$$

$$\phi_j(x) > 0, \quad \theta_j(x) > 0, \quad \forall x \in \mathcal{R}^n$$

The block partition of Φ and Θ is compatible in size with the uncertain mapping (34) of the form

$$\Delta = \underset{j=1}{\overset{2n}{\text{block-diag}}}\, \hat{\Delta}_j, \quad \hat{\Delta}_j = \begin{cases} \Delta_{(j+1)/2} & \text{for odd } j \\ \Delta_{U(j/2)} & \text{for even } j \end{cases}$$

Let $M_{[k]}(x_k)$ be defined by

$$M_{[k]} = \begin{bmatrix} \left\{ \begin{array}{c} \hat{S}_{[k]}^T \hat{A}_{[k]}^T T_{[k]}^T \Xi_{[k]} + \\ \Xi_{[k]} T_{[k]} \hat{A}_{[k]} \hat{S}_{[k]} \end{array} \right\} & \Xi_{[k]} T_{[k]} B_{[k]} & \hat{S}_{[k]}^T \hat{C}_{[k]}^T \Phi_{[k]} \\ B_{[k]}^T T_{[k]}^T \Xi_{[k]} & -\Theta_{[k]} & D_{[k]}^T \Phi_{[k]} \\ \Phi_{[k]} \hat{C}_{[k]} \hat{S}_{[k]} & \Phi_{[k]} D_{[k]} & -\Phi_{[k]} \end{bmatrix}$$

for $k = 1, 2, \ldots, n$, where individual matrices are given by

$$\hat{A}_{[k]} = \begin{bmatrix} a_{11} & a_{12} & 0 & \cdots & \cdots & 0 \\ a_{21} & a_{22} & a_{23} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{k-1,1} & a_{k-1,2} & \cdots & \cdots & a_{k-1,k} & 0 \\ a_{k1} & a_{k2} & \cdots & \cdots & a_{kk} & a_{k,k+1} \end{bmatrix}$$

$$B_{[k]} = \begin{bmatrix} B_{11} & U_{L1} & 0 & 0 & \cdots & 0 & 0 \\ B_{21} & U_{21} & B_{22} & U_{L2} & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ B_{k1} & U_{k1} & B_{k2} & U_{k2} & \cdots & B_{kk} & U_{Lk} \end{bmatrix}$$

$$\hat{C}_{[k]} = \begin{bmatrix} C_{11} & 0 & 0 & \cdots & 0 & 0 \\ 0 & U_{R1} & 0 & \cdots & 0 & 0 \\ C_{21} & C_{22} & 0 & \ddots & 0 & 0 \\ 0 & 0 & U_{R2} & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ C_{k,1} & C_{k,2} & \cdots & \cdots & C_{k,k} & 0 \\ 0 & 0 & \cdots & \cdots & 0 & U_{R,k} \end{bmatrix} \quad \begin{array}{l} \Xi_{[k]} = \underset{j=1}{\overset{k}{\mathrm{diag}}}\, \Xi_j \\ \Phi_{[k]} = \underset{j=1}{\overset{2k}{\text{block-diag}}}\, \Phi_j \\ \Theta_{[k]} = \underset{j=1}{\overset{2k}{\text{block-diag}}}\, \Theta_j \end{array}$$

$$D_{[k]} = \begin{bmatrix} D_1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & D_2 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & D_k & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}, \quad \hat{S}_{[k]} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ s_1 & 1 & 0 & \cdots & 0 \\ 0 & s_2 & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & s_{k-1} & 1 \\ 0 & \cdots & 0 & 0 & s_k \end{bmatrix}$$

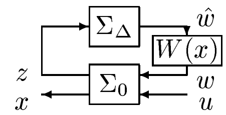

Figure 5: Modified plant $\hat{\Sigma}_P$

In a similar manner, $T_{[k]}$ is the $k \times k$ upper left part of T. The next properties can be verified easily.

Proposition 1 *For $k = 1, 2 \ldots, n$, $M_{[k]}$ satisfies following.*

(i) *$M_{[k]}$ is independent of $\{x_{k+1}, x_{k+2}, \cdots, x_n\}$.*

(ii) *$M_{[k]}$ does not include $\{s_{k+1}, \cdots, s_{n-1}, s_n\}$, $\{\Phi_{2k+1}, \Phi_{2k+2}, \cdots, \Phi_{2n}\}$ and $\{\Theta_{2k+1}, \Theta_{2k+2}, \cdots, \Theta_{2n}\}$.*

(iii) *$M_{[k]}(x_{[k]}) < 0$ implies $M_{[k-1]}(x_{[k-1]}) < 0$.*

(iv) $M_{[n]}(x_{[n]}) = M(x)$

On the basis of this proposition, a new recursive procedure for robust backstepping is now proposed.

Robust backstepping via state-dependent scaling *Solve $M_{[k]}(x_{[k]}) < 0$ for $\{s_k(x_{[k]})$, $\phi_{2k-1}(x_{[k]})$, $\phi_{2k}(x_{[k]})$, $\theta_{2k-1}(x_{[k]})$, $\theta_{2k}(x_{[k]})\}$ from $k = 1$ through $k = n$.*

Now, how to construct a solution in the k-th step of the robust backstepping and the existence of the solution are addressed.

Theorem 3 *Consider the system Σ_P in the extended robust strict-feedback form. Suppose that Σ_P satisfies either:*

(i) only Δ_j is allowed to involve dynamic uncertainties,

(ii) only Δ_{Uj} is allowed to involve dynamic uncertainties,

(iii) only $\{\Delta_i,\ \Delta_{Ui} : i = 1, 2, \ldots, j\}$ are allowed to involve dynamic uncertainties, and matrices $\hat{A}_{[j]}$, $B_{[j]}$, $\hat{C}_{[j]}$ and $D_{[j]}$ are independent of x.

for an integer $j \in [1, n]$. Then, the system Σ_P can be rendered robustly globally uniformly asymptotically stable by the smooth state-feedback control law (52).

The proof is based on Theorem 1 with (20) and $\kappa = j$. The new class of scaling (10) enables us to solve the global robustification of stability against 'dynamic' uncertainties belongings to the above three cases. The success relies heavily on the introduction of the new scaling (10). Indeed, in general, the robust stabilization is globally solvable by constant scaling only in the $j = 1$ case of (i)[10]. The proof of Theorem 3 demonstrates how to construct the globally stabilizing state-feedback law explicitly. The robust backstepping is reduced to simple scalar inequalities which are affine in the new state-dependent scaling. Although admissible uncertainties in Definition 1 have unit gain, Theorem 3 implies the global stabilizability in the presence of uncertainties having any finite gain. As a matter of fact, if the uncertainties are bounded from above by γ_i, we can apply Theorem 3 to a new system Σ_0 whose corresponding input channel in w is multiplied by γ_i. Note that the multiplication should not violate (44) and (45). It is verified that no restriction of locations of dynamic uncertainties is necessary if the nominal system Σ_0 is linear. Linear systems in the extended robust strict-feedback form are stabilizable for all admissible dynamic nonlinear uncertainties by a linear control $u = s_n \chi$ since all scaling matrices can be chosen constants.

When dynamic uncertainties are involved in all Δ_i and Δ_{Ui}, $i = 1, 2, \ldots, n$, Theorem 3 does not guarantee the robustness of global stability. It is, however, possible to obtain a sort of stability margin.

Theorem 4 *Suppose that Σ_0 and Σ_Δ define an uncertain system Σ_P in the extended robust strict-feedback form. Then, there exists $\bar{W} \in \mathcal{R}^{(p+\pi)\times(p+\pi)}$ consisting of $\mathcal{C}^0$ functions*

$$\bar{W}(x) = \text{block-diag}[\gamma_1(x)I, \gamma_2(x)I, \cdots, \gamma_{2n-1}(x)I, I], \ \ \bar{W}(0) = I$$

such that the system $\hat{\Sigma}_P$ shown in Fig.5 can be rendered robustly globally uniformly asymptotically stable by the smooth state-feedback control law (52) for all diagonal matrices $W(x)$ satisfying $0 < W(x) \le \bar{W}(x)$ for all $x \in \mathcal{R}^n$. The block partition of $\bar{W}$ is consistent with the that of Δ.

The claim and the design procedure described by Theorem 3 are also applicable to the performance robustness problem.

Corollary 1 *Consider the system $\bar{\Sigma}_P$ in the extended robust strict-feedback form. Suppose that $\bar{\Sigma}_P$ satisfies either:*

(a) $r = r_j$ and Theorem 3(i)
(b) $r = r_{U,j}$ and Theorem 3(ii)
(c) $r = [r_1^T, r_{U,1}^T, \cdots, r_j^T, r_{U,j}^T]^T$ and Theorem 3(iii).

for an integer $j \in [1, n]$. Then, the system Σ_P can be made to achieve the robust global performance by the smooth state-feedback control law (52).

Although Corollary 1 by itself shows unit $\mathcal{L}_2$-gain disturbance attenuation, the state-feedback law can be designed for arbitrarily small $\mathcal{L}_2$-gain λ unless (44) and (45) are violated. Thus, the robust almost $\mathcal{L}_2$ disturbance decoupling problem can be solved by the smooth state-feedback (52) for arbitrarily large $\mathcal{L}_2$-gain class of dynamic and static uncertainties.

6 Examples of uncertain systems

Uncertain systems listed below are only a few from many examples for which the state-dependent scaling method solves the robust stabilization and performance *globally*.

Example 1 The first example is a nonlinear system involving input unmodeled dynamics which is described by

$$\begin{aligned} \dot{x} &= f(x) + g(x)u + l(x)d(t,\xi) + m(x)s(t,x) + b(x)r \\ \dot{\xi} &= p(t,\xi,h(x)u) \\ e &= c(x)u \end{aligned} \tag{56}$$

where the dynamic uncertain components $\{p(\cdot,\cdot,\cdot), d(\cdot,\cdot)\}$ and the static uncertainty $s(\cdot,\cdot)$ belong to the following sets.

$$\left\{ \begin{array}{l} \dot{\xi} = p(t,\xi,v) \\ w = d(t,\xi) \end{array} \right. \quad \mathcal{L}_2\text{-gain of } v \mapsto w \ \le \gamma \tag{57}$$

$$\begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} = s(t,x) = \begin{bmatrix} \vdots \\ s_i(t, x_{[i]}) \\ \vdots \end{bmatrix}, \quad \|z_i\| \le \gamma \|x_{[i]}\| \tag{58}$$

The ξ-subsystem represents the input unmodeled dynamics. Suppose that $f(\cdot)$, $g(\cdot)$, $l(\cdot)$, $m(\cdot)$ and $b(\cdot)$ are in the form of

$$f(x) = \begin{bmatrix} a_{11}(x_1) & a_{12}(x_1) & 0 & \cdots & 0 \\ a_{21}(x_{[2]}) & a_{22}(x_{[2]}) & a_{23}(x_{[2]}) & 0 & \cdots \\ \vdots & \ddots & & \ddots & \ddots \\ a_{n1}(x) & \cdots & \cdots & \cdots & a_{nn}(x) \end{bmatrix} x, \quad g(x) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ g_n(x) \end{bmatrix} \tag{59}$$

$$l(x) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ l_n(x) \end{bmatrix}, \ m(x) = \operatorname*{block-diag}_{i=1}^{n} m_i(x_{[i]}), \ b(x) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_n(x) \end{bmatrix} \tag{60}$$

$$a_{i,i+1}^2 > 0, \ i = 1, 2, \dots, n-1, \quad g_n^2 > 0, \ x \in \mathcal{R}^n \tag{61}$$

$$g_n^2 > (h^T h + c^T c)(\gamma^2 l_n^2 + \lambda^{-2} b_n^2), \ x \in \mathcal{R}^n \tag{62}$$

The above requirement is a sort of controllability property of the system (56) for all admissible uncertainties. Then, according to Corollary 1, for each $\gamma, \lambda > 0$, there exist a smooth feedback $u = K(x)x$ such that the system (56) is globally uniformly asymptotically stabilized and the mapping from r to e has $\mathcal{L}_2$-gain less than or equal to λ for all admissible uncertainties. In contrast to redesign[13, 15, 16], we are able to directly construct a control Lyapunov function for the input unmodeled dynamics and a robustifying controller at the same time.

Example 2 The second example is

$$\begin{aligned} \dot{x} &= f(x) + g(x)u + l(x)d(t,\xi) + m(x)s(t,x) + b(x)r \\ \dot{\xi} &= p(t,\xi,\bar{h}(x)x) \\ e &= \bar{c}(x)x + e(x)r \end{aligned} \tag{63}$$

The uncertainties are defined as (57) and (58). We assume

$$I > \lambda^{-2} e(x) e^T(x), \quad x \in \mathcal{R}^n \tag{64}$$

which is obviously necessary for the $\mathcal{L}_2$-gain performance. The uncertain system is supposed to be controllable in the sense of (59) and (61) in addition to (60).

Example 3 The next example is a nonlinear system with an unknown stable zero dynamics.

$$\begin{aligned} \dot{x} &= f(x) + g(x)u + l(x)d(t,\xi) + m(x)s(t,x) + b(x)r \\ \dot{\xi} &= p(t,\xi,\bar{h}(x_1)x_1) \\ e &= \bar{c}(x_1)x_1 + e(x_1)r \end{aligned} \tag{65}$$

The uncertainties are the same as (57) and (58). Thus, the ξ-subsystem can be regarded as an asymptotically stable zero dynamics. The condition (64) is necessary again. We suppose that the uncertain system meets (59) and (61). This time, functions l, m and b are

$$l(x) = \begin{bmatrix} l_1(x_1) \\ l_2(x_{[2]}) \\ \vdots \\ l_n(x) \end{bmatrix}, m(x) = \operatorname*{block-diag}_{i=1}^{n} m_i(x_{[i]}), \ b(x) = \begin{bmatrix} b_1(x_1) \\ b_2(x_{[2]}) \\ \vdots \\ b_n(x) \end{bmatrix}$$

Example 4 Consider the following cascade connection of a linear system and a nonlinear system.

$$\begin{aligned} \dot{\eta} &= A_\eta \eta + G_\eta \zeta_1 + L_\eta d(t,\xi) + M_\eta s_\eta(t,\eta) + B_\eta r \\ \dot{\zeta} &= F_\zeta(\eta,\zeta) + G_\zeta(\eta,\zeta)u + L_\zeta(\eta,\zeta)d(t,\xi) \\ &\qquad + M_\zeta(\eta,\zeta)s_\zeta(t,\eta,\zeta) + B_\zeta(\eta,\zeta)r \\ \dot{\xi} &= p(t,\xi,H_\eta \zeta_1) \\ e &= \begin{bmatrix} C_\eta \eta + E_\eta r \\ U_\eta \zeta_1 \end{bmatrix} \end{aligned} \tag{66}$$

The linear part is the η-subsystem defined by constant matrices $\{A_\eta, G_\eta, L_\eta, M_\eta, B_\eta, C_\eta, E_\eta, U_\eta, H_\eta\}$. The input ζ_1 of the η-subsystem is the output of the nonlinear ζ-subsystem. The functions $s_\eta(\cdot,\cdot)$ and $s_\zeta(\cdot,\cdot)$ are static uncertain systems. The linear system has the input unmodeled uncertainty described by the nonlinear ξ-system. Without loss of generality, (A_η, G_η) and (A_η, L_η) are in the controlablity canonical form. The uncertainties affect the nonlinear ζ-subsystem as well as the linear η-subsystem. Definitions of the uncertain components are the same as (57) and (58). For a compact notation, we define the state variable x as

$$x = \begin{bmatrix} \eta \\ \zeta \end{bmatrix} \in \mathcal{R}^n, \quad x_{[k]} = \eta \in \mathcal{R}^k, \quad \zeta_1 = x_{k+1}$$

for some $k \in [1, n-1]$. Then, the overall system (66) is

$$\begin{aligned} \dot{x} &= f(x) + g(x)u + l(x)d(t,\xi) + m(x)s(t,x) + b(x)r \\ \dot{\xi} &= p(t,\xi,H_\eta x_{k+1}) \\ e &= \begin{bmatrix} C_\eta x_{[k]} + E_\eta r \\ U_\eta x_{k+1} \end{bmatrix}, \qquad s(t,x) = \begin{bmatrix} s_\eta(t,\eta) \\ s_\zeta(t,\eta,\zeta) \end{bmatrix} \end{aligned} \tag{67}$$

Suppose that F_ζ, G_ζ, L_ζ, M_ζ and B_ζ are consistent with

$$f(x) = \begin{bmatrix} A_\eta\eta + G_\eta\zeta_1 \\ F_\zeta(\eta,\zeta) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & 0 & \cdots & 0 \\ a_{21} & a_{22} & a_{23} & 0 & \cdots \\ \vdots & \ddots & & \ddots & \ddots \\ a_{n1} & \cdots & \cdots & \cdots & a_{nn} \end{bmatrix} x$$

$$\begin{array}{l} a_{ij} = a_{ij}(x_{[i]}) \\ k \le i \le n,\ 1 \le j \le i+1 \end{array}, \quad g(x) = \begin{bmatrix} 0 \\ G_\zeta(\eta,\zeta) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ g_n(x) \end{bmatrix}$$

$$l(x) = \begin{bmatrix} L_\eta \\ L_\zeta(\eta,\zeta) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ l_k \\ l_{k+1}(x_{[k+1]}) \\ \vdots \\ l_n(x) \end{bmatrix}$$

$$m(x) = \begin{bmatrix} M_\eta & 0 \\ 0 & M_\zeta(\eta,\zeta) \end{bmatrix}$$

$$b(x) = \begin{bmatrix} B_\eta \\ B_\zeta(\eta,\zeta) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_k \\ b_{k+1}(x_{[k+1]}) \\ \vdots \\ b_n(x) \end{bmatrix}$$

$$M_\eta = \text{block-diag}_{i=1}^{k}\, m_i$$
$$M_\zeta(\eta,\zeta) = \text{block-diag}_{i=k+1}^{n}\, m_i(x_{[i]})$$

The condition $I > \lambda^{-2} E_\eta E_\eta^T$ is necessary for achieving the $\mathcal{L}_2$-gain performance. Finally, we assume

$$a_{i,i+1}^2(x) > 0,\ i = 1,2,\ldots,n-1, \quad g_n^2(x) > 0, \quad x \in \mathcal{R}^n$$
$$a_{k,k+1}^2 > (H_\eta^T H_\eta + U_\eta^T U_\eta)(\gamma^2 l_k^2 + \lambda^{-2} b_k^2)$$

Then, we can apply Corollary 1 to the uncertain system (67). It is worth noting that the robust stabilization and performance problem can be solved globally even if the η-subsystem is nonlinear, namely, $\{A_\eta, G_\eta, L_\eta, M_\eta, B_\eta, C_\eta, E_\eta, U_\eta, H_\eta\}$ are allowed to be funcitons of η.

7 Concluding remarks

In this paper, the robust nonlinear control was addressed for a class of nonlinear systems subject to static nonlinear uncertainties and unmodeled nonlinear dynamics in various locations and structures. A new class of scaling matrices is invented for capturing the influence of dynamic uncertainties on global asymptotic stabilization and disturbance attenuation. The robust stabilization and the performance problem have been reduced to a state-dependent scaling problem in a unified way. The state-dependent scaling has led us to an interesting Lyapunov function which ensures global robustness with respect to dynamic uncertainties. This paper has proposed a recursive procedure for state-feedback control design which recursively selects the scaling matrices from the new class of state-dependent scaling.

This paper has derived solutions on the assumption that information of the dynamic system Σ_Δ is completely unknown except the $\mathcal{L}_2$-gain, or equivalently, the ratio between the average powers of output and input of Σ_Δ, which is the direct extension of the peak magnitude of the bode frequency plot to nonlinear systems. This formulation suits the situation where a less amount of information is available for the dynamics. Note that frequency weighting functions and nonlinear weighting functions of x can be included in Σ_M and Σ_0. On the other hand, if the dynamics we regard as Σ_Δ is not essentially unknown, we may have information about the nonlinear gain function of Σ_Δ together with its Lyapunov function. In such a case, the nonlinear small-gain theorem[11, 7] for input-to-state stable systems may be useful[12]. It is worth mentioning that the calculation of robust controllers for $\mathcal{L}_2$-gain bounded uncertainty considered in this paper does not involve any information about the Lyapunov function of Σ_Δ, its lower and upper functional bounds.

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