# COMMON QUADRATIC LYAPUNOV-LIKE FUNCTION WITH ASSOCIATED SWITCHING REGIONS FOR TWO UNSTABLE SECOND-ORDER LTI SYSTEMS

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**Abstract**

In the present paper we utilize Lyapunov-like functions in the qualitative analysis of switched systems. Specifically, for a class of second-order switched systems consisting of two unstable subsystems, we explore in detail some necessary and sufficient conditions for the existence of common quadratic Lyapunov-like functions. We find that the existence of quadratic Lyapunov-like functions is closely related to the recent work on conic switching laws.

**Keywords:** Second-order switched systems, quadratic Lyapunov-like functions, conic switching laws, stabilization

## 1. Introduction

In recent works in the qualitative study of switched systems, several basic problems have been addressed and studied in depth (see, e.g., [6] and the references cited therein), two of which stand out. One is to determine testable conditions that guarantee the asymptotic stability of a switched system under arbitrary switchings, while the other is to determine one (all) possible switching sequence(s) that render a switched system asymptotically stable (see, e.g., [6], [9], [13], [14], [12], [11], and the references cited therein). The endeavor of determining a common quadratic Lyapunov function for a given switched system (see, e.g, [7], [8], [10]) is closely related to the first problem.

The solutions to the above problems are still far from satisfactory, especially for high-dimensional switched systems. However, for second-order switched systems, some succinct solutions appear to be possible (see, e.g., [4], [10], [13] and [14]).

In the present paper, we address a problem that is related to both of the above problems. Our motivation for this problem can be illustrated by the following example.

**Example 1.1.** ([1], [5, 6]) Consider the harmonic oscillator with position measurements described by the following equations

$$\begin{aligned} \frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u, \\ y &= x_1. \end{aligned}$$

Although the above system is both controllable and observable, it cannot be stabilized by static (even discontinuous) state output feedback [1]; however, it is stabilizable by 2-state hybrid output feedback [1, 5]. By letting $u = -y$ and $u = \frac{1}{2}y$, we obtain the following systems, respectively,

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{1.1}$$

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{2} & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{1.2}$$

Define $V(x) \triangleq x_1^2 + x_2^2$. If the system (1.1) is active in the first and third quadrants, while the system (1.2) is active in the second and fourth quadrants, we will have $\dot{V} < 0$ whenever $x_1 x_2 \neq 0$, which implies that the entire switched system is asymptotically (and hence, for linear systems, exponentially) stable, by LaSalle's principle. In [4], we pointed out that the above system can be stabilized by 2-state hybrid output feedback with arbitrary small gains. We can repeat the above test by letting $u = \epsilon_1 y$ and $u = \epsilon_2 y$ ($\epsilon_2 < 0 < \epsilon_1$, $\epsilon_1, \epsilon_2$ arbitrarily small) and $V(x) \triangleq x_1^2 + x_2^2$, and the same switching strategy as before. □

We have seen from the above example that a hybrid output feedback problem can be solved if we can determine a common quadratic Lyapunov-like function $V(x)$ and a partition of the entire region $\Re^2$ associated with it. We thus pose the following question: under what conditions can such a common quadratic Lyapunov-like function be found?

In the present paper, we will study in detail, for the case of second-order switched systems consisting of two unstable subsystems (whose coefficient matrices have imaginary eigenvalues with nonnegative real parts), necessary and sufficient conditions for the existence of common quadratic Lyapunov-like functions, along with partitions of the entire region $\Re^2$ (if the above two subsystems are of the same direction), or a common conic region (if the above two subsystems are of opposite direction).

**Notation:** For a set $\Omega \subset \Re^2$, we will use $int(\Omega)$, $clo(\Omega)$ and $\Omega^c$ to denote the interior, closure and the complement of $\Omega$, respectively. For a real matrix $P$, $P > 0$ ($P \geq 0$) will denote that $P$ is a symmetric positive (nonnegative) definite matrix.

## 2. Terminology and Preliminary Results

In the present section, we introduce some terminology and preliminary results that will be used in the next section.

As in [14], we say that a subsystem is of **clockwise (counterclockwise) direction** in some region $\Omega$ if starting from any nonzero initial condition in $\Omega$, its trajectory moves in the clockwise (counterclockwise) direction. We will modify the definition of direction of a second-order LTI system in [14] in the following way to make it more easily understood from the vector field point of view.

**Definition 2.1.** We define the direction of a second-order LTI system $\dot{x} = Ax$ to be *clockwise* in the conic region $\{x : (x_2, -x_1)Ax \geq 0\}$ and to be *counterclockwise* in the conic region $\{x : (x_2, -x_1)Ax \leq 0\}$. □

In the present paper, we focus on second-order switched systems described by

$$\dot{x}(t) = A_i x(t), \quad i = 1, 2, \tag{2.1}$$

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where $A_i$ $(i = 1, 2)$ has a pair of purely imaginary eigenvalues or an unstable focus. Clearly, for the switched systems considered herein, the vector field of every subsystem has a unique direction.

For clarity, we present the following definition for the above switched systems, though it can be extended to other cases after slight modifications.

**Definition 2.2.** For switched system (2.1), if $A_1$ and $A_2$ are of the same direction, suppose that there exist two sets of regions $\{\Omega_{1p_1}, p_1 = 1, 2, \ldots, k_1\}$ (associated with $A_1$), $\{\Omega_{2p_2}, p_2 = 1, 2, \ldots, k_2\}$ (associated with $A_2$) satisfying (i) $0 \in clo(\Omega_{ip_i})$ and $int(\Omega_{ip_i}) \neq \emptyset$ for $p_i = 1, 2, \ldots, k_i$, $i = 1, 2$; (ii) the interior of the intersection for each pair of the above sets is empty; (iii) the union of the above sets is $\Re^2$. We call a $C^1$ function $V(x)$ satisfying $V(x) > 0, x \neq 0, V(0) = 0$ a *common weakly Lyapunov-like function* for switched system (2.1) associated with partition $\{\Omega_{ip_i}, p_i = 1, 2, \ldots, k_i\}$, $i = 1, 2$ if inequality for each $i$, $\dot{V} = \frac{dV}{dx} A_i x \leq 0$ holds in $int(\Omega_{ip_i})$ but equality does not hold in at least one of these regions. If $A_1$ and $A_2$ are of opposite directions, suppose that there exists a region $\Omega$ such that $0 \in clo(\Omega)$ and $int(\Omega) \neq \emptyset$. We call a $C^1$ function $V(x)$ satisfying $V(x) > 0, x \in int(\Omega), x \neq 0, V(0) = 0$ a *common weakly Lyapunov-like function for switched system (2.1) with region* $\Omega$ if the inequality $\dot{V} = \frac{dV}{dx} A_i x \leq 0$ holds for $x \in int(\Omega), i = 1, 2$ and equality does not hold for at least one of the subsystems.

If in the above definition, equality in the expression of $\dot{V}$ does not hold on every interior of the specified regions, then we drop the word **weakly**. □

In the sequel, we use the abbreviations **CQWLLF** and **CQLLF** to denote common quadratic weakly Lyapunov-like function and common quadratic Lyapunov-like function, respectively.

**Remark 2.1.** The above definition is different from the conventional Lyapunov function used in the literature for switched systems, especially in the case when $A_1$ and $A_2$ are of opposite directions. Also, we know that for switched system (2.1), the existence of a CQWLLF/CQLLF implies exponential stability. Furthermore, if switched system (2.1) consists of two subsystems of opposite directions, then we may conclude that the existence of a CQWLLF/CQLLF implies exponential stability under *arbitrary switchings inside the common region*. The last property is related to work in [10]. □

**Remark 2.2.** If $V(x) = x^T P x$ $(P > 0)$ is a CQWLLF such that $\dot{V}(x) = 0$ holds for a fixed subsystem $A_i$ with three linearly independent vectors $x_1$, $x_2$, $x_3$, then we have that $A_i^T P + P A_i = 0$, which implies that $A_i$ is neutrally stable (i.e., $A_i$ has a pair of purely imaginary eigenvalues). □

The common (quadratic) Lyapunov functions* proposed in [7], [10], [8], [11] and [12] are obviously common (quadratic) Lyapunov-like functions as defined above. However, the reverse is not always true. We cite the following results to demonstrate the difference.

**Proposition 2.1.** (Sufficiency [12], necessity [2]) For the switched system (of arbitrary dimension) described by

$$\dot{x}(t) = A_i x(t), \quad i = 1, 2, \tag{2.2}$$

where $A_i$ $(i = 1, 2)$ is unstable, there exists a switching strategy that makes (2.2) quadratically stable† if and only if there exists an $\alpha \in (0, 1)$ such that the convex combination system $\alpha A_1 + (1 - \alpha) A_2$ is stable.

For $A_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 1 & -2 \\ 1 & 0 \end{pmatrix}$, there exists no stable convex combination system, i.e., there exists no common quadratic Lyapunov function for the switched system consisting of the above two subsystems. However, common quadratic Lyapunov-like functions of the form $V(x) = x^T P x$ $(P > 0)$ do exist (see Section 3).

In the following, we summarize the conic switching laws proposed in [13] and [14] for switched systems consisting of two subsystems with unstable foci. These results along with Remark 2.2 will be used extensively in Section 3. Let $x = (x_1, x_2)^T$. As in [13], [14], we define the following regions as partitions of the entire plane,

$$\begin{aligned} E_{is} &= \{x | x^T f_i(x) = x^T A_i x \leq 0\}, \\ E_{iu} &= \{x | x^T f_i(x) = x^T A_i x \geq 0\}, \ i = 1, 2, \end{aligned} \tag{2.3}$$

and we denote

$$\Delta(x) = (A_1 x)^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} A_2 x. \tag{2.4}$$

To design stabilizing switching control laws, we must identify the following two different cases.

**Case 1. Two Subsystems of the Same Direction**

Without loss of generality, we assume that both subsystems are of clockwise direction. We define the following conic regions:

$$\begin{aligned} \Omega_1 &= E_{1s} \cap E_{2u}, \\ \Omega_2 &= E_{1u} \cap E_{2s}, \end{aligned} \tag{2.5}$$

$$\begin{aligned} \Omega_3 &= E_{1s} \cap E_{2s} \cap \{x | \Delta(x) \leq 0\}, \\ \Omega_4 &= E_{1s} \cap E_{2s} \cap \{x | \Delta(x) \geq 0\}, \end{aligned} \tag{2.6}$$

$$\begin{aligned} \Omega_5 &= E_{1u} \cap E_{2u} \cap \{x | \Delta(x) \leq 0\}, \\ \Omega_6 &= E_{1u} \cap E_{2u} \cap \{x | \Delta(x) \geq 0\}. \end{aligned} \tag{2.7}$$

By associating subsystem 1 with $\Omega_1$, $\Omega_3$, $\Omega_5$ and subsystem 2 with $\Omega_2$, $\Omega_4$, $\Omega_6$, we obtain the **conic switching law** proposed in [14].

The following result concerns the stabilizability of the switched system.

**Theorem 2.1.** Let $l_1$ be a ray that goes through the origin. Let $x_0 \neq 0$ be on $l_1$. Let $x^*$ be the point on $l_1$ where the trajectory intersects $l_1$ for the first time after leaving $x_0$, when the switched system evolves according to the conic switching law. The switched system (2.1) with subsystems of the same direction is asymptotically stabilizable if and only if $\|x^*\|_2 < \|x_0\|_2$ by the conic switching law. □

**Case 2. Two Subsystems of Opposite Directions**

Without loss of generality, we assume that subsystem $A_1$ is of clockwise direction and subsystem $A_2$ is of counterclockwise direction. We introduce the following conic regions:

$$\begin{aligned} \Omega_1 &= E_{1s} \cap E_{2u}, \\ \Omega_2 &= E_{1u} \cap E_{2s}, \end{aligned} \tag{2.8}$$

$$\begin{aligned} \Omega_3 &= E_{1s} \cap E_{2u} \cap \{x | \Delta(x) \geq 0\}, \\ \Omega_4 &= E_{1s} \cap E_{2u} \cap \{x | \Delta(x) \leq 0\}, \end{aligned} \tag{2.9}$$

*A common quadratic Lyapunov function for (2.1) is defined to be of the form $x^T P x$ where $P > 0$, satisfying the inequality $A_i^T P + P A_i \leq -Q$, $i = 1, 2$, for some matrix $Q > 0$ .

†Quadratic stability means that there exists an $\epsilon > 0$ such that $\dot{V} < -\epsilon x^T x$ for $x \neq 0$. Obviously, the existence of a common quadratic Lyapunov function implies quadratic stability.

$$\begin{aligned} \Omega_5 &= E_{1u} \cap E_{2s} \cap \{x | \Delta(x) \geq 0\}, \\ \Omega_6 &= E_{1u} \cap E_{2s} \cap \{x | \Delta(x) \leq 0\}. \end{aligned} \quad (2.10)$$

**Theorem 2.2.** The switched system (2.1) with two subsystems of opposite directions is asymptotically stabilizable if and only if $int(\Omega_1) \cup int(\Omega_3) \cup int(\Omega_5) \neq \emptyset$. □

If the switched system is asymptotically stabilizable, then the **conic switching law** can also be obtained as in [14] which makes the system asymptotically stable. The conic switching law first forces the trajectory into the interior of one of the conic regions $\Omega_1$, $\Omega_3$, $\Omega_5$ (there must be one available), and then switches to another subsystem upon intersecting the boundary of the region so as to keep the trajectory inside the region. It is also shown in [3] that a conic switching law is always robust not only in the sense that the control law is *flexible* (that is, the switchings do not have to happen exactly on the switching boundaries), but also in the sense that the Lyapunov stability (resp., Lagrange stability) properties of the switched system are preserved in the presence of certain kinds of vanishing perturbations (resp., non-vanishing perturbations).

**Remark 2.3.** The above conic switching law is also true if one subsystem is neutrally stable (i.e., it has a pair of purely imaginary eigenvalues) while the other is either neutrally stable or unstable with foci. □

Other cases of combinations of the subsystems (though not exclusively) have also been studied in [14].

## 3. Analysis of the Existence of Common Quadratic Lyapunov-like Functions

In the present section, the existence problem of the CQ(W)LLFs for second-order switched system (2.1) is addressed. Various necessary and sufficient conditions are presented and methodologies to construct these quadratic functions are discussed as well.

We divide the discussion into several cases.

**(A) $A_1$, $A_2$ both have a pair of purely imaginary eigenvalues**

**Theorem 3.1.** If $A_1 \neq kA_2$ for any real constant $k$, then switched system (2.1) under condition (A) is stabilizable and there exists a CQLLF associated with a partition of the plane (if $A_1, A_2$ are of the same direction) or a common conic region (if $A_1, A_2$ are of the opposite directions).

*Proof.* It is well known that there exists a nonsingular real matrix $W$ such that $W^{-1}A_1W = \begin{pmatrix} 0 & \\ - & 0 \end{pmatrix}$ with $\geq 0$. Note that any nonsingular linear transformation does not affect stability properties. Therefore, without loss of generality, we assume that (note that $t\ ace(A_2) = 0$)

$$A_1 = \begin{pmatrix} 0 & \\ - & 0 \end{pmatrix} ( \ > 0), \quad A_2 = \begin{pmatrix} c & b \\ a & -c \end{pmatrix}.$$

Since $det(A_2) = -c^2 - ab > 0$, we have by Definition 2.1 that if $b > 0$, $A_2$ is of clockwise direction (so is $A_1$), and if $b < 0$, $A_2$ is of counterclockwise direction. We consider the following two cases.

**(A1) $b > 0$ ($\Rightarrow a < 0$), and $A_1, A_2$ are of the same direction**

In the present case, direct calculations show that $P_1 = \begin{pmatrix} -a & c \\ c & b \end{pmatrix} > 0$ satisfies that $A_2^T P_1 + P_1 A_2 = 0$. We construct the following quadratic function

$$V_1(x) = x^T x + x^T P_1 x. \quad (3.1)$$

We have

$$\begin{aligned} \dot{V}_1(x)|_{A_1} &= \frac{dV_1(x)}{dx} A_1 x = x^T \begin{pmatrix} -2c & -(a+b) \\ -(a+b) & 2c \end{pmatrix} x, \\ \dot{V}_1(x)|_{A_2} &= \frac{dV_1(x)}{dx} A_2 x = x^T \begin{pmatrix} 2c & a+b \\ a+b & -2c \end{pmatrix} x. \end{aligned}$$

Clearly, $\dot{V}_1(x)|_{A_1} + \dot{V}_1(x)|_{A_2} = 0$. Let

$$\Omega_{11} = \{x : \dot{V}_1(x)|_{A_1} \leq 0\}, \quad \Omega_{12} = \{x : \dot{V}_1(x)|_{A_1} > 0\},$$
$$\Omega_{21} = \{x : \dot{V}_1(x)|_{A_2} \leq 0\}, \quad \Omega_{22} = \{x : \dot{V}_1(x)|_{A_2} > 0\}.$$

Since $det \begin{pmatrix} 2c & a+b \\ a+b & -2c \end{pmatrix} < 0$ (the case $c = 0$, $a + b = 0$ is excluded under the assumption that $A_1 \neq kA_2$ for any real constant $k$), we know that $int(\Omega_{12}) \neq \emptyset$ and $int(\Omega_{22}) \neq \emptyset$. Therefore, a partition of the plane associated with $V_1(x)$ can be chosen as $\Omega_{11}$ (for $A_1$) and $\Omega_{21} \cap \Omega_{12}$ (for $A_2$); or $\Omega_{11} \cap \Omega_{22}$ (for $A_1$) and $\Omega_{21}$ (for $A_2$). □

**(A2) $b < 0$ ($\Rightarrow a > 0$), $A_1, A_2$ are of opposite directions**

In the present case, direct calculations show that $P_2 = \begin{pmatrix} a & -c \\ -c & -b \end{pmatrix} > 0$ satisfies that $A_2^T P_2 + P_2 A_2 = 0$. We now choose $V_2(x) = x^T x + x^T P_2 x$. By direct computation, we obtain that $\dot{V}_1(x)|_{A_1} - \dot{V}_1(x)|_{A_2} = 0$. Therefore, $\Omega_{11} = \{x : \dot{V}_1(x)|_{A_1} \leq 0\}$ is the common region for $V_2(x)$. □

**Remark 3.1.** If $A_1 = kA_2$ for some real constant $k$, then there is no way to find a common (even) weakly Lyapunov-like function. Therefore, condition $A_1 \neq kA_2$ for any $k \in \Re$ serves as a necessary and sufficient condition for the existence of a CQLLF for switched system (2.1) under condition (A). □

Theorem 3.1 and Remark 3.1 imply that a switched system consisting of two subsystems that are neutrally stable is always stabilizable by some switching laws provided that one coefficient matrix is not a multiple of the other. This interesting result has frequently been observed in the literature, but as far as the authors know, has not been proved rigorously.

**(B) $A_1$ has a pair of purely imaginary eigenvalues, $A_2$ has unstable foci**

In the present case, we have the following result.

**Theorem 3.2.** For switched system (2.1) under condition (B), there exists a CQWLLF with associated switching regions if and only if switched system (2.1) is stabilizable by the conic switching law stated in Section 2.

*Proof.* (Necessity) By Remark 2.1, we know that the existence of a CQWLLF implies that switched system (2.1) is stabilizable. Therefore, it is also stabilizable by the conic switching law (refer to [14]).

(Sufficiency) If switched system (2.1) is stabilizable by the conic switching law, we can construct the following CQWLLF with appropriate switching regions. Let $P_1 > 0$ satisfy $A_1^T P_1 + P_1 A_1 = 0$. Let $V(x) = x^T P_1 x$ and $\Omega = \{x : x^T (A_2^T P_1 + P_1 A_2) x < 0\}$. It is obvious that $int(\Omega) = \Omega \neq \emptyset$. Otherwise, $x^T (A_2^T P_1 + P_1 A_2) x \geq 0$ for all $x \in \Re^2$ and thus $\dot{V}(x) \geq 0$, no matter what kind of switching strategy is employed, which is a contradiction to our assumption.

If $A_1, A_2$ are of the same direction, the partition of the plane can be chosen as follows: $\Re^2 \backslash \Omega$ (for $A_1$) and $\Omega$ (for $A_2$). If $A_1, A_2$ are of the same direction, we can choose $\Omega$ to be the common switching region for $V(x)$. □

One may ask whether it is possible to replace CQWLLF by

CQLLF in the above theorem. In general this is not true. We have the following result instead.

**Theorem 3.3.** For switched system (2.1) under condition (B), there exists a CQLLF with associated switching regions if $A_1, A_2$ are of opposite directions and switched system (2.1) is stabilizable by the conic switching law.

*Proof.* Without loss of generality, we assume that

$$A_1 = \begin{pmatrix} 0 & \\ - & 0 \end{pmatrix} (\ \ > 0), \quad A_2 = \begin{pmatrix} c_1 & b \\ a & c_2 \end{pmatrix}. \tag{3.2}$$

Under the assumption
**(B1) $A_1, A_2$ are of opposite directions**,
we must have that $c_1 + c_2 > 0, (c_1 + c_2)^2 - 4(c_1c_2 - ab) = (c_1 - c_2)^2 + 4ab < 0$ (unstable foci) and $a > 0, b < 0$ (opposite direction). Denote

$$P_\epsilon = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \epsilon \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We have

$$A_1^T P_\epsilon + P_\epsilon A_1 = 2 \begin{pmatrix} -\epsilon & 0 \\ 0 & \epsilon \end{pmatrix},$$

$$A_2^T P_\epsilon + P_\epsilon A_2 = \begin{pmatrix} -2c_1 & a+b \\ a+b & 2c_2 \end{pmatrix} + \epsilon \begin{pmatrix} 2a & c_1 + c_2 \\ c_1 + c_2 & 2b \end{pmatrix}.$$

Let

$$\begin{aligned} \Omega &= \{x : x^T(A_2^T + A_2)x < 0\}, \\ \Omega_\epsilon &= \{x : x^T(A_2^T P_\epsilon + P_\epsilon A_2)x < 0\}. \end{aligned}$$

(Sufficiency) Since switched system (2.1) is stabilizable by the conic switching law, we must have that $\Omega \neq \emptyset$. Notice that $x^T(A_1^T P_\epsilon + P_\epsilon A_1)x = 2\epsilon\ \ (x_2^2 - x_1^2)$. If $\{x : x_2^2 - x_1^2 < 0\} \cap \Omega \neq \emptyset$, then by continuity, for $\epsilon > 0$ sufficiently small, $P_\epsilon > 0$ and $\{x : x_2^2 - x_1^2 < 0\} \cap \Omega_\epsilon \neq \emptyset$ is the common conic region for $V(x) = x^T P_\epsilon x$. Otherwise, we must have that $\{x : x_2^2 - x_1^2 > 0\} \cap \Omega_\epsilon \neq \emptyset$. Then for $\epsilon < 0$ sufficiently small, $P_\epsilon > 0$ and $\{x : x_2^2 - x_1^2 > 0\} \cap \Omega_\epsilon \neq \emptyset$ is the common conic region for $V(x) = x^T P_\epsilon x$. □

Now consider the problem under the assumption
**(B2) $A_1, A_2$ are of the same direction**.

Without loss of generality, we assume that $A_1, A_2$ have the form (3.2), where in $A_2$, we have $a < 0, b > 0$ (same direction), $c_1 + c_2 > 0$ and $(c_1 - c_2)^2 + 4ab < 0$ (unstable foci) and $(a + b)^2 - 4c_1c_2 > 0$ (existence of conic law).

In the present case, it is more likely that there exists no CQLLF with associated partitions even though the switched system is stabilizable by the conic switching law. Let us first look at the following example.

**Example 3.1.** Consider the switched system consisting of subsystems $A_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}$. We show that there exists no CQLLF. Suppose there exist a CQLLF. By definition, the CQLLF should be of the form $V(x) = x^T Px$ with $P > 0$. Without loss of generality, we assume that $(P_{22} = 1)$

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} k & d \\ d & 0 \end{pmatrix}. \tag{3.3}$$

Obviously,

$$P > 0 \Longleftrightarrow 1 + k - d^2 > 0. \tag{3.4}$$

We now have that

$$\begin{aligned} \Omega_1 &\triangleq \{x : \dot{V}(x)|_{A_1} < 0\} = \{x : -dx_1^2 + kx_1x_2 + dx_2^2 < 0\}, \\ \Omega_2 &\triangleq \{x : \dot{V}(x)|_{A_2} < 0\} \\ &= \{x : (1 + k - d)x_1^2 + (1 + 2k + d)x_1x_2 + 2dx_2^2 < 0\}. \end{aligned}$$

Therefore, there exists a CQLLF $V(x) = x^T Px$ if and only if there exist $k$ and $d$ such that (3.4) is satisfied and $clo(\Omega_1 \cup \Omega_2) = \Re^2$.

We need the following lemma to proceed.

**Lemma 3.1.** The conic region represented by $\{x : ax_1^2 + bx_1x_2 + cx_2^2 < 0\}$ is bounded by two straight lines of angle less than (resp., equal to or larger than) $\frac{\pi}{2}$ if and only if $a + c > 0$ (resp., $a + c = 0$ or $a + c < 0$).

*Proof.* This can be computed directly. Alternatively, notice that $ax_1^2 + bx_1x_2 + cx_2^2 = (a + c)x_1^2 + (-cx_1^2 + bx_1x_2 + cx_2^2)$ and the conic region $\{x : -cx_1^2 + bx_1x_2 + cx_2^2 < 0\}$ is bounded by two straight lines of angle $\frac{\pi}{2}$. □

By the above lemma it is clear that if $1 + k + d > 0$ then $clo(\Omega_1 \cup \Omega_2) \neq \Re^2$, since $\Omega_1$ is bounded by two straight lines of angle $\frac{\pi}{2}$. If $d^2 + d \geq 0$, then by (3.4), we have $1 + k + d > d^2 + d \geq 0$. Therefore, we must have $d^2 + d < 0$, i.e., $-1 < d < 0$. We now find that the $x_1$-axis ($x_1 \neq 0$) is strictly outside of $\Omega_1$ and $\Omega_2$, which means that $clo(\Omega_1 \cup \Omega_2) = \Re^2$ will not be satisfied. □

Suppose there exists a CQLLF. Then by Remark 2.1, the switched system is stabilizable by the conic switching law. By definition, the CQLLF should be of the form $V(x) = x^T Px$ with $P > 0$ defined in (3.3). We now have that

$$\Omega_1 \triangleq \{x : \dot{V}(x)|_{A_1} < 0\} = \{x : -dx_1^2 + kx_1x_2 + dx_2^2 < 0\}, \tag{3.5}$$

$$\begin{aligned} \Omega_2 \triangleq \{x : \dot{V}(x)|_{A_2} < 0\} = \{x : ((1 + k)c_1 + da)x_1^2 \\ +(a + b + bk + (c_1 + c_2)d)x_1x_2 + (c_2 + db)x_2^2 < 0\}. \end{aligned} \tag{3.6}$$

Therefore, there exists a CQLLF $V(x) = x^T Px$ if and only if there exist $k$ and $d$ such that $clo(\Omega_1 \cup \Omega_2) = \Re^2$.

In the following, we present an equivalent condition for the existence of a CQLLF under (B2). Let

$$A \triangleq (1 + k)c_1 + da, \quad B \triangleq a + b + bk + (c_1 + c_2)d, \quad C \triangleq c_2 + db. \tag{3.7}$$

We have the following result.

**Proposition 3.1.** The existence of a CQLLF under condition (B2) is equivalent to the existence of $(d, k)$ such that condition (3.4) and the following conditions hold

$$\begin{aligned} &A + C < 0 \\ &((a + b)^2 - (c_1 + c_2)^2)d^2 + (a + b)(c_1 - c_2)dk - c_1c_2k^2 = 0 \\ &A + B + C = 0\,(k > 0) \quad \text{or} \quad A - B + C = 0\,(k < 0). \end{aligned}$$

*Proof:* Refer to the Appendix. □

From Proposition 3.1, we know that the problem whether there exists a CQLLF under condition (B2) is elementary but highly nontrivial. Thus far, we are not able to prove or disprove that for the present case there exists no CQLLF.

**(C) $A_1, A_2$ both have unstable foci**

Without loss of generality, we assume that

$$A_1 = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} (\alpha, \beta > 0), \quad A_2 = \begin{pmatrix} c_1 & b \\ a & c_2 \end{pmatrix}.$$

**(C1)** $A_1, A_2$ **are of the same direction**

Under the condition that switched system (2.1) is stabilizable by the conic switching law, we must have the following constraints (since $intE_{2u} \neq \emptyset$)

$$b > 0, \quad a < 0, \quad c_1 + c_2 > 0,$$
$$(a+b)^2 - 4c_1c_2 > 0 > (c_1 - c_2)^2 + 4ab.$$

Let $V(x) = x^T Px$ with $P$ defined as in (3.3). Then

$$\Omega_1 \triangleq \{x : \dot{V}(x)|_{A_1} < 0\} = \{x : (\alpha + k\alpha - d\beta)x_1^2 + (k\beta + 2d\alpha)x_1x_2 + (\alpha + d\beta)x_2^2 < 0\}, \quad (3.8)$$

$$\Omega_2 \triangleq \{x : \dot{V}(x)|_{A_2} < 0\} = \{x : ((1+k)c_1 + da)x_1^2 + (a + b + bk + (c_1 + c_2)d)x_1x_2 + (c_2 + db)x_2^2 < 0\}. \quad (3.9)$$

As in (B2) we know that the existence of CQLLF $V(x) = x^T Px$ is equivalent to the existence of $k, d$ such that $1 + k - d^2 > 0$ and $clo(\Omega_1 \cup \Omega_2) = \Re^2$.

Under the present case, the existence of CQLLF is less likely than under (B2). We conjecture that for the present case (i.e., (C1)), there exists no CQLLF.

It is not difficult to show that if the above conjecture is true then there exists no CQWLLF as well in the present case. This follows directly from Remark 2.2.

**(C2)** $A_1, A_2$ **are of the opposite directions**

We assume that switched system (2.1) is stabilizable by the conic switching law. In the present case, $A_2$ satisfies $b < 0, a > 0, c_1 + c_2 > 0$, $(a+b)^2 - 4c_1c_2 > 0 > (c_1 - c_2)^2 + 4ab$. Using the notation introduced in Section 2, we have $E_{1u} = \Re^2$, $E_{2s} = \{x : c_1x_1^2 + (a+b)x_1x_2 + c_2x_2^2 \leq 0\}$, $\{x : \Delta(x) \geq 0\} = \{x : (a\alpha + c_1\beta)x_1^2 + ((a+b)\beta + \alpha(c_2 - c_1))x_1x_2 + (\beta c_2 - b\alpha)x_2^2 \leq 0\}$, $\Omega_5 = E_{2s} \cap \{x : \Delta(x) \geq 0\}$. By Theorem 2.2, we have that $int(\Omega_5) \neq \emptyset$.

Let $V(x)$ be defined as in (3.3) and let $\Omega_1$ and $\Omega_2$ be the same as in (3.8) and (3.9), respectively. Then the necessary and sufficient condition for the existence of a CQLLF $V(x)$ with $P > 0$ is that there exist constants $k, d$ such that $1 + k - d^2 > 0$ and $\Omega_1 \cap \Omega_2 \neq \emptyset$. The last inequality is equivalent to the existence of some constant $s \in [-\infty, \infty]$ such that

$$\alpha + k\alpha - d\beta + (k\beta + 2d\alpha)s + (\alpha + d\beta)s^2 < 0,$$
$$(1+k)c_1 + da + (a + b + kb + d(c_1 + c_2))s + (c_2 + db)s^2 < 0.$$

The special case $s = \pm\infty$ is equivalent to the existence of $d$ such that $\alpha + d\beta < 0, c_2 + db < 0$ (i.e., $x_2$-axis is strictly inside $\Omega_1 \cap \Omega_2$), which is equivalent to the condition

$$c_2\beta - b\alpha < 0. \quad (3.10)$$

**Example 3.2.** Let $A_1 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, $A_2 = \begin{pmatrix} 3 & -1 \\ 7 & -2 \end{pmatrix}$. Then $A_1, A_2$ satisfy (C1) and condition (3.10). We pick $d = -\frac{3}{2}$ to satisfy that $\alpha + d\beta = 1 + d < 0$ and $c_2 + db = -2 - b < 0$. Now we have $P = \begin{pmatrix} 3 & -\frac{3}{2} \\ \frac{3}{2} & 1 \end{pmatrix}$ and

$$\Omega_1 \cap \Omega_2 = \{x : 9x_1^2 - 2x_1x_2 - x_2^2 < 0\} \cap \{x : -3x_1^2 + 5x_1x_2 - x_2^2 < 0\} \neq \emptyset.$$

Clearly $V(x) = x^T Px$ is a CQLLF with common region $\Omega_1 \cap \Omega_2$.

The above example shows that it is at least possible to construct a CQLLF for some special cases. However, in general, it is difficult to develop a systematic method to find a candidate CQLLF for the present case. This is an open question for future work.

## 4. Appendix

**Proof of Proposition 3.1:**

By Lemma 3.1, it is required that $A + C \leq 0$. However, we can show that $A + C \neq 0$. If $A + C = 0$, then $clo(\Omega_1 \cup \Omega_2) = \Re^2$ if and only if (again by Lemma 3.1) there exists some positive number $p > 0$ such that

$$d = p((1+k)c_1 + da) \quad (4.1)$$
$$-k = p(a + b + bk + (c_1 + c_2)d) \quad (4.2)$$
$$-d = p(c_2 + db). \quad (4.3)$$

Since in the present case $b > 0, a < 0, c_1 + c_2 > 0 (\Rightarrow d \neq 0)$, we know from (4.1) and (4.3), respectively, that

$$d = -\frac{pc_2}{pb + 1} \quad (4.4)$$
$$1 + k = \frac{1 - pa}{pc_1}d = -\frac{c_2(1 - pa)}{c_1(pb + 1)}. \quad (4.5)$$

Substituting (4.4) and (4.5) into (4.2), we have

$$(1+k)(1+pb) - (1+pb) + p(a+b) + pd(c_1 + c_2) = 0.$$

Equivalently, we obtain

$$(pa - 1)(pb + 1) - p^2c_1c_2 = 0,$$

which is a contradiction to (3.4). Therefore, the following condition is necessary

$$A + C < 0. \quad (4.6)$$

Suppose there exists $(k, d)$ such that $clo(\Omega_1 \cup \Omega_2) = \Re^2$. Under condition (3.4), fixing $d$ and increasing $k$ continuously, we know (by analyzing the region $\{(d,k) : 1 + k - d^2 > 0, A + C < 0\}$ under the following three cases $c_1 > 0$, $c_1 = 0$ and $c_1 < 0$, respectively) that there must exist some $k$ such that $clo(\Omega_1 \cup \Omega_2) = \Re^2$ and that two corresponding regions $\Omega_1$ and $\Omega_2$ share a common boundary, which is equivalent to the fact that there exists a real number $s \in [-\infty, +\infty]$ such that

$$-d + ks + ds^2 = 0, \quad A + Bs + Cs^2 = 0. \quad (4.7)$$

Since in the present case, $d \neq 0$ (otherwise, by (3.5), we know that the $x_1$-axis and the $x_2$-axis must be strictly inside $\Omega_2$, which reduces to $c_1 < 0$ and $c_2 < 0$, a contradiction to the fact that $c_1 + c_2 > 0$) and $C \neq 0$ (otherwise, $\Omega_2$ is a half plane; by Lemma 3.1, we know that it is impossible that $clo(\Omega_1 \cup \Omega_2) = \Re^2$), condition (4.7) is equivalent to the following equation (we can solve $s$ from (4.7) then substitute it into one equation of (4.7))

$$((A+C)^2 - B^2)d^2 + (C - A)Bdk + ACk^2 = 0.$$

Noting that $((A+C)^2 - B^2)d^2 + (C-A)Bdk + ACk^2 = (1 + k - d^2)(((a+b)^2 - (c_1+c_2)^2)d^2 + (a+b)(c_1 - c_2)dk - c_1c_2k^2)$, we have equivalently that

$$((a+b)^2 - (c_1 + c_2)^2)d^2 + (a+b)(c_1 - c_2)dk - c_1c_2k^2 = 0. \quad (4.8)$$

Suppose (4.6) and (4.8) are satisfied. Since $\Omega_1$ and $\Omega_2$ share a common boundary, if we can show that one vector does not belong to $\Omega_1$ but belongs to $\Omega_2$, then $clo(\Omega_1 \cup \Omega_2) = \Re^2$. Notice that when $k > 0$, then the line $\{x : x_2 = x_1\}$ $(x_1 \neq 0)$ is strictly outside $\Omega_1$, which means that it must lie inside $clo(\Omega_2)$, i.e.,

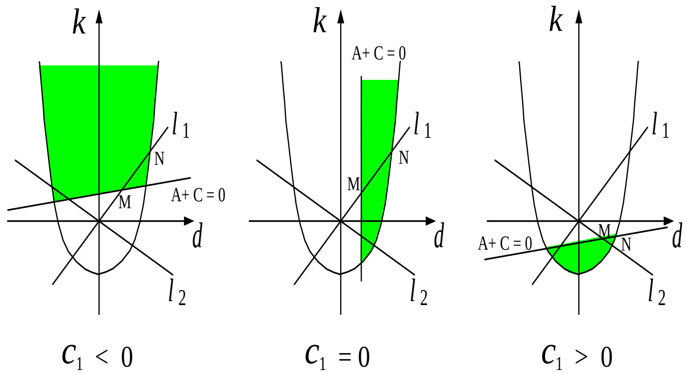


Figure 1: Line segment $\overline{MN}$ in the proof of Proposition 3.1 (shadowed area is the region $\{(d,k) : 1+k-d^2 > 0, A+C \le 0\}$)

$A+B+C \le 0$; when $k<0$, then the line $\{x : x_2 = -x_1\}$ ($x_1 \ne 0$) is strictly outside $\Omega_1$, which means that it must lie inside $clo(\Omega_2)$, i.e., $A-B+C \le 0$. We further show that $k=0$ is impossible. If $k=0$ ($\Rightarrow |d|<1$), then $\Omega_1$ is bounded by the two lines $\{x : x_2 = x_1\}$ and $\{x : x_2 = -x_1\}$. These must be inside $clo(\Omega_2)$, i.e., $(c_1+c_2+a+b)(1+d) \le 0$ and $(c_1+c_2-a-b)(1-d) \le 0$, which implies that $c_1+c_2 \le 0$, which is a contradiction to the fact that $c_1+c_2>0$. Therefore, we must also have

$$A+B+C \le 0\,(k>0) \quad \text{or} \quad A-B+C \le 0\,(k<0). \quad (4.9)$$

We thus have the following result.

**Proposition 4.1.** The existence of a CQLLF under condition (B2) is equivalent to the existence of $(d,k)$ such that (3.4), (4.6), (4.8) and (4.9) hold. □

We can further prove that Proposition 4.1 is equivalent to Proposition 3.1, i.e., condition (4.9) can be replaced by

$$A+B+C = 0\,(k>0) \quad \text{or} \quad A-B+C = 0\,(k<0). \quad (4.10)$$

First, by Proposition 4.1 we know that there exists $(k,d)$ such that (3.4), (4.6), (4.8) and (4.9) are satisfied. Let $\mathcal{S} = \{(d,k) : 1+k-d^2 > 0\}$ and let $l_1, l_2$ denote the two lines represented by equation (4.8). If line $A+C=0$ on the $(d,k)$-plane does not intersect $l_1, l_2$ within $\mathcal{S}$, noting that $(0,0)$ is not in $A+C=0$, we conclude that there exists no $(k,d)$ satisfying the conditions in Proposition 4.1 (refer, e.g., to Example 3.1), which is a contradiction. Therefore, $l_1, l_2$ must intersect $A+C=0$ inside $\mathcal{S}$.

There are at most two line segments (see, e.g., in Fig. 1, $\overline{MN}$) for each of the cases $c_1>0$, $c_1=0$ and $c_1<0$, such that all points on the segments satisfy (4.8) and $A+C \ge 0$. Let us consider all these segments. For line segment $\overline{MN}$ in Fig. 1, if it does not intersect line $A+B+C=0$ or line $A-B+C=0$, then either (i) $\overline{MN}$ lies inside the region $A+B+C<0\,(k>0)$ or $A-B+C<0\,(k<0)$, or (ii) $\overline{MN}$ lies inside the region $A+B+C>0\,(k>0)$ or $A-B+C>0\,(k<0)$. If (ii) is true, then all the points on $\overline{MN}$ do not satisfy the conditions in Proposition 4.1. If (i) is true, then we notice that point $M$ (at which line $A+C=0$ and either $l_1$ or $l_2$ intersect, with corresponding coordinate say, $(k_M, d_M)$, satisfies that $clo(\Omega_1 \cup \Omega_2) = \Re^2$ (check directly from (3.5) and (3.6)). However, this contradicts the claim that $A+C \ne 0$ in the proof of Proposition 4.1. Therefore, we know under the condition of Proposition 3.1 that there exists at least one segment that intersects the line $A+B+C=0\,(k>0)$ or the line $A-B+C=0\,(k<0)$, which concludes the proof. □

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