Using Lyapunov Matrices for Sliding Mode Design

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Abstract

In this manuscript, a property of Lyapunov matrices is proposed and its application to sliding mode design is addressed. It will be shown that the sliding modes which guarantee the desired sliding behavior can be obtained by manipulating Lyapunov matrices associated with the full order systems. The proposed approach enables us to adopt a variety of Lyapunov- (or Riccati-) based approaches for the sliding mode design. Applications to uncertain systems, systems with uncertain state delay, pole-clustering problems, multi-objective approach and *etc.* are discussed.

1 Introduction

Designing sliding modes that guarantee the desired performances has been one of the major issues in control theory. The typical approach to sliding mode design is to handle the reduced order system through the nonsingular transformation to the regular form (e.g., see [1, 2]). Based on the reduced order system obtained, it has been shown in [1] that the controllability of the nominal system guarantees the existence of sliding modes and the detectability of a certain reduced order system, which depends on the weighting matrix defining the quadratic performance index, is required for the so-called x-optimal design (which utilizes the linear quadratic regulator (LQR) method). Also, handling the reduced order system, many of the standard approaches to sliding mode control have been proposed (e.g., see [3, 4] or, [2] and the references therein).

Recently, attention has been paid to the application of the Riccati approach to the sliding mode design since the original effort was made in [1] by introducing the x-optimal design. In [5], it has been shown that a Lyapunov equation (or the algebraic Riccati equation in the LQR approach) can be adopted to yield appropriate sliding modes for a class of uncertain systems with matched uncertainties. Also, in [6], the Lyapunov approaches have been taken into account in order to cope with the mismatched uncertainties and assign the eigenvalues of the sliding modes in the prescribed region. More recently, it has been shown in [7, 8] that the Lyapunov matrix associated with the quadratic stability can be used for parameterizing the sliding mode in the presence of real parametric uncertainties. It is noted that the Riccati (or Lyapunov) approaches are very simple since the full order systems are handled instead of the reduced ones.

Motivated by the results in [8], the paper is devoted to propose a systematic design procedure based on the multiplier theories (e.g., see [9]), which have been extensively investigated for a variety of issues using the Lyapunov (or Riccati) inequalities. For example, uncertain delayed systems (e.g., [10, 11] and references therein), parametric uncertain systems (e.g., [12, 13] among many), pole-clustering problems ([14]) and *etc.* have been effectively dealt with using the multiplier approaches. The common interest of the multiplier approaches is to find the Lyapunov matrix satisfying matrix inequalities that constrain design objectives. It is well known that such multiplier approaches can be rewritten by the linear matrix inequalities (LMIs) using the Shur-complement and the change of variables (see [9]). Once the constraints are stated in LMIs, the design problem can be easily solved thanks to the convexity. Furthermore, the constraints of LMIs can be easily combined for the multi-objective approach in which the Lyapunov matrix is assumed to be common for all the desired objectives (e.g., see [15]). The multi-objective approach has been known to be a very useful concept for the controller design nevertheless the possible conservatism. More recently, an effort has been made in [16] to reduce the conservatism using the scalar scale that enters in the form of LMIs.

In the paper, we propose the methods to design sliding modes based on the multiplier approaches. First, it will be shown that the linear sliding mode can be parameterized by partitioning and augmenting the Lyapunov matrix of the well-known Lyapunov inequality in Section 1. Then, using the parameterization, the x-optimal approach is re-visited for illustrating the usefulness of the proposed approach. In Section 2, several issues that can be solved by the multiplier approaches will be taken into account. Also, we will address how to design the sliding mode based on the multi-objective approach. Finally, the conclusion follows in Section 3.

The notations used in the paper are fairly standard. Among them, $\| \bullet \|$ and $\lambda(\bullet)$ represents the Euclidean norm and the set of eigenvalues of the argument matrix, respectively. The inequality signs for matrices denote the sign-definiteness for the real symmetric matrices.

2 Main results

2.1 All Stabilizing Sliding Surfaces

Consider the system

$$\dot{x} = Ax + B(u + w) \tag{1}$$

where $x \in \Re^n$ and $u \in \Re^m$ are the state and the control input, respectively, and $w \in \Re^l$ is the disturbance of which each element is bounded as $|w_j(t)| \leq \overline{w}_j$, $\forall j \in [0, l]$, for the known $\overline{w}_j$. The stabilizability of the pair

(A, B) is assumed. And, for the simplicity of description, suppose the system is in the regular form [1, 2]

$$\begin{cases} \dot{x}_1 = A_{11}x_1 + A_{12}x_2 \\ \dot{x}_2 = A_{21}x_1 + A_{22}x_2 + B_2(u+w) \end{cases} \tag{2}$$

where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \begin{bmatrix} \Re^{n-m} \\ \Re^m \end{bmatrix}$ and B_2 is nonsingular. Without loss of generality, consider the sliding function

$$s(t) = Sx_1 + x_2 \tag{3}$$

for some $S \in \Re^{(n-m)\times m}$. It turns out, in [1], that the existence of sliding modes is guaranteed by the controllability of the pair (A, B) under which there exist some matrices S that make the matrix $A_{11} - A_{12}S$ stable. Hence, it is usual in sliding mode control that the reduced order system has been handled to obtain the stabilizing sliding function coefficient S. Now, let us present one of the main results in the following.

Theorem 1 *There exist some sliding modes if and only if there exist some $P > 0$ and $K \in \Re^{m\times n}$ satisfying*

$$(A - BK)^T P + P(A - BK) + Q < 0, \tag{4}$$

for a $Q \geq 0$. Moreover, for the feasible P, the sliding mode is given by

$$S = P_{22}^{-1} P_{12}^T \tag{5}$$

where P_{ij}'s are defined as

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} \in \begin{bmatrix} \Re^{(n-m)\times(n-m)} & \Re^{(n-m)\times m} \\ \Re^{m\times(n-m)} & \Re^{m\times m} \end{bmatrix}.$$

Proof. (Necessity) Let S denote the sliding function coefficient, which guarantees the stability of the matrix $A_{11} - A_{12}S$. Then, there should exist some $P_r > 0$ satisfying, for any $Q_r \geq 0$,

$$(A_{11} - A_{12}S)^T P_r + P_r (A_{11} - A_{12}S) + Q_r < 0. \tag{6}$$

Then, let the negative definite matrix $R < 0$ denote the left hand side of (6) for the future reference. Now, for an arbitrary $P_{22} > 0$, define the matrices

$$P_{12} = S^T P_{22}, \quad P_{11} = P_r + P_{12}^T P_{22}^{-1} P_{12}, \tag{7}$$

and

$$K = [\, K_1 \quad K_2 \,] \tag{8}$$

where, for an $\epsilon > 0$,

$$\begin{aligned} K_1 &= B_2^{-1} \left\{ A_{21} + P_{22}^{-1} P_{12}^T A_{11} + P_{22}^{-1} A_{12}^T P_r + \tfrac{\epsilon}{2} P_{22}^{-2} P_{12}^T \right\} \\ K_2 &= B_2^{-1} \left\{ A_{22} + P_{22}^{-1} P_{12}^T A_{12} + \tfrac{\epsilon}{2} P_{22}^{-1} \right\} \end{aligned}$$

Through some manipulations, it may be shown that

$$\begin{aligned} &T \left\{ (A - BK)^T P + P(A - BK) + Q \right\} T^T \\ &= \begin{bmatrix} R & 0 \\ 0 & -\epsilon I_m \end{bmatrix} < 0 \textit{ for } T = \begin{bmatrix} I_{n-m} & -P_{12}P_{22}^{-1} \\ 0 & I_m \end{bmatrix} \end{aligned} \tag{9}$$

which implies (4) due to the non-singularity of the transformation matrix.

(Sufficiency) Define $T_r := \left[I_{n-m}, \ -P_{12}P_{22}^{-1} \right]$. Then, pre- and post multiplying (4) by T_r and T_r^T, respectively, yields

$$\begin{aligned} \left(A_{11} - A_{12}P_{22}^{-1}P_{12}^T \right)^T P_r + P_r \left(A_{11} - A_{12}P_{22}^{-1}P_{12}^T \right) \\ + T_r Q T_r^T < 0 \end{aligned} \tag{10}$$

where $P_r = P_{11} - P_{12}^T P_{22}^{-1} P_{12}$, which is positive definite since $P > 0$. Hence, choosing $S = P_{22}^{-1} P_{12}^T$, the stability of the matrix $A_{11} - A_{12}S$ is shown. This completes the proof. (Q.E.D.)

Remark 1 *The feasibility of Lyapunov inequality (4) is the stabilizability of the pair (A, B). Hence, the stabilizability of the nominal system is the existence of sliding modes, which extends the controllability condition in [1].*

Theorem 1 shows that a certain property of the reduced order systems (*i.e.*, the stability issue in the above case) can be obtained by handling the full order systems not the reduced order systems. In order to provide more concrete example, we re-visit the x-optimal approach [1] in the following. Consider the performance index

$$J = \int_{t_s}^{\infty} x^T Q x \, dt \tag{11}$$

where t_s is the time when the sliding mode starts. With the cost function, we have the following result.

Theorem 2 *Suppose that the sliding function is given by $S = P_{22}^{-1} P_{12}^T$ for some P satisfying (4). Then the cost function (11) is bounded as*

$$J = \int_{t_s}^{\infty} x^T Q x \, dt < x_1(t_s)^T P_r x_1(t_s) \tag{12}$$

where $P_r = P_{11} - P_{12}P_{22}^{-1}P_{12}^T$.

Proof. Since the sliding function is given by $S = P_{22}^{-1} P_{12}^T$, it may be shown that

$$\begin{aligned} s(t) &= x_2 + Sx_1 = [\, P_{22}^{-1} P_{12}^T \quad I_m \,] x \\ &= P_{22}^{-1} [\, P_{12}^T \quad P_{22} \,] x = \left(P_{22}^{-1} B_2^{-T} \right) B^T P x, \end{aligned} \tag{13}$$

which implies $B^T P x = 0$ on $s(t) = 0$. Now, let us consider the derivative of a quadratic function $V = x^T P x$ for $t \geq t_s$ as

$$\begin{aligned} \dot{V} &= x^T \left\{ (A - BK)^T P + P(A - BK) \right\} x \\ &\quad + 2x^T PB \left(u + Fw + Kx \right) < -x^T Q x \end{aligned} \tag{14}$$

Integrating both sides in (14) *w.r.t.* time, we have

$$\begin{aligned} &\int_{t_s}^{\infty} x^T Q x \, dt < x(t_s)^T P x(t_s) \\ &= x_1(t_s)^T \left(P_{11} - P_{12}P_{22}^{-1}P_{12}^T \right) x_1(t_s) \end{aligned} \tag{15}$$

since $x_2(t_s) = -P_{22}^{-1}P_{12}^T x_1(t_s)$. This completes the proof. (Q.E.D.)

Remark 2 *It should be pointed out that the inequality (12) does not imply the conservatism. To see this, observe that the relationship of inequality has resulted from the application of the Lyapunov inequality (4) in (14). Also, Theorem 1 states the necessary and sufficient condition for the existence condition of sliding modes. Thus, it may be shown that the quadratic term $x_1(t_s)^T P_r x_1(t_s)$ is the least upper bound.*

Using the result of Theorem 2, the x-optimal design can be redefined based on the LMIs method [9] that utilizes the change of variables such as $Y := P^{-1}$ and $L := KP^{-1}$. Especially, note the matrix inversion property

$$Y = P^{-1} = \begin{bmatrix} (P_{11} - P_{12}P_{22}^{-1}P_{12}^T)^{-1} & \star \\ \star & \star \end{bmatrix} \tag{16}$$

to deal with the performance bound (12), where $\star$ positions are of no concern. Thus the x-optimal design is stated as follows: *Given some $Q \geq 0$, minimize γ w.r.t. $Y > 0$ and L satisfying*

$$\begin{bmatrix} AY + YA^T - BL - L^T B^T & YC_q \\ C_q^T Y & -I \end{bmatrix} < 0 \tag{17}$$

$$\begin{bmatrix} \gamma I & x_1(t_s) \\ x_1(t_s)^T & U_1 Y U_1^T \end{bmatrix} > 0 \tag{18}$$

where $Q = C_q C_q^T$ and $U_1 = [I_{n-m},\ 0_{(n-m)\times m}]$.

Remark 3 *The above LMIs problem is different from the x-optimal approach [1] in a few aspects. First, the weighting matrix Q is not necessarily invertible. Secondly, as to the solvability issue, the stabilizability of the nominal system is only needed (i.e., the detectability of a certain reduced order system depending on the weighting matrix Q is not required).*

Through the discussion above, it turns out that the property of the reduced order systems can be defined by simply handling the full order systems. It allows us to directly utilize a variety of Lyapunov (or Riccati) approaches in order to obtain linear sliding modes. Note that Lyapunov (or Riccati) approaches have been widely developed for dealing with uncertain delayed systems, parametric uncertain systems, pole-clustering problems, multi-objective approaches, and *etc.*

2.2 Application to Uncertain Delayed Systems

The state delayed systems has been of concern in some literature (e.g., see [2] and the references therein). While the complete solution remains unsolved yet, we here investigate the issue again to introduce the Lyapunov matrix-based approach.

Consider the system

$$\dot{x}(t) = Ax(t) + A_d x(t-\tau) + B(u+w) \tag{19}$$

where $0 \leq \tau \leq \tau_{max}$ for the known τ_{max}. Also, without loss of generality, we assume the system is in the regular form similar to (2).

We start with the following control law that renders the reachability condition:

$$u = \begin{cases} 0, & (\|s(t)\| = 0) \\ -B_2^{-1}\{GAx + \beta s + Z(t)sign(s)\}, & (\|s(t)\| > 0) \end{cases} \tag{20}$$

where $sign(s) = [sign(s_1), \cdots, sign(s_m)]^T$, $\beta > 0$, $G = [S,\ I_m]$ and $Z(t) = diag[z_1, \cdots, z_m]$ for z_i defined as

$$z_i = \sum_{j=1}^{n} |(GA_d)_{ij}|\overline{x}_j + \sum_{k=1}^{l} |(GB)_{ik}|\overline{w}_k \tag{21}$$

where $\overline{x}_j(t) \triangleq \sup_{\xi\in[t-\tau_{max},t]} |x_j(\xi)|$. To show the reachability condition, rewrite the sliding function behavior as

$$\begin{aligned} \dot{s} &= G\dot{x} \\ &= GAx + GA_d x(t-\tau) + B_2 u + GBw. \end{aligned} \tag{22}$$

Then, for the Lyapunov functional candidate $V_s = \frac{1}{2}s^T s$, it can be shown that $\dot{V}_s < -\beta\|s\|^2$ using the fact

$$\begin{aligned} s^T GA_d x(t-\tau) &= \sum_{i=1}^{m} s_i \sum_{j=1}^{n} (GA_d)_{ij} x_j(t-\tau) \\ &\leq \sum_{i=1}^{m} |s_i| \sum_{j=1}^{n} |(GA_d)_{ij}|\overline{x}_j(t). \end{aligned} \tag{23}$$

For addressing the existence of sliding modes, we rely upon the following lemma.

Lemma 1 *The system*

$$\dot{x} = A_c x + A_d x(t-\tau) \tag{24}$$

is quadratically stable if there exist some $P > 0$ and $H > 0$ satisfying

$$\begin{bmatrix} A_c^T P + PA_c + H & PA_d \\ A_d^T P & -H \end{bmatrix} < 0 \tag{25}$$

The above has been one of the standard approaches to handle the systems with the uncertain state delay while there may exist some conservatism. Note that the maximum amount of delay is not reflected. The quadratic stability can be shown by using the Lyapunov-Krasovskii functional (see [9] and the original reference therein):

$$V = x^T Px + \int_{t-\tau}^{t} x(\theta)^T Hx(\theta)d\theta \tag{26}$$

for some $P > 0$ and $H > 0$.

We can now present the following result.

Theorem 3 *There exist some sliding modes if there exist some $P > 0,\ H > 0$ and $K \in \Re^{m\times n}$ satisfying*

$$\begin{bmatrix} (A-BK)^TP + P(A-BK) + H + Q & PA_d \\ A_d^TP & -H \end{bmatrix} < 0 \quad (27)$$

for a $Q \geq 0$. Moreover, the sliding function coefficient is given by $S = P_{22}^{-1}P_{12}^T$ for the feasible P in (27) and, with this, it holds that

$$J = \int_{t_s}^{\infty} x^TQx\ dt < x_1(t_s)^TP_rx_1(t_s) \quad (28)$$

where $P_r = P_{11} - P_{12}P_{22}^{-1}P_{12}^T$.

Proof. For T_r (as defined in (10)), define the augmented matrix

$$T = \begin{bmatrix} T_r & 0 \\ 0 & T_r \end{bmatrix}. \quad (29)$$

Pre- and post multiplying (27) by T and T^T, respectively, it follows that

$$\begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^T & \Phi_{22} \end{bmatrix} < 0 \quad (30)$$

where $\Phi_{11} = (A_{11}-A_{12}S)^TP_r+P_r(A_{11}-A_{12}S)+T_r(Q+H)T_r^T$, $\Phi_{12} = P_r(A_{d,11} - A_{d,12}S)$ and $\Phi_{22} = -T_rHT_r^T$ for $S = P_{22}^{-1}P_{12}^T$. This implies, based on Lemma 1, the stability of the system

$$\dot{\xi} = (A_{11} - A_{12}S)\xi + (A_{d,11} - A_{d,12}S)\xi(t-\tau). \quad (31)$$

Note that (31) is the system behavior on the sliding mode, *i.e.,* $x_2 = -Sx_1$.

Now, the performance issue can be made using the similar argument through (13)-(14). That is, using the fact that $B^TPx = 0$ on the sliding mode, one may show

$$\begin{aligned} \dot{V} &= \eta^T \begin{bmatrix} (A-BK)^TP + P(A-BK) + H & PA_d \\ A_d^TP & -H \end{bmatrix} \eta \\ &< -x^TQx, \quad \text{where } \eta = \begin{pmatrix} x(t) \\ x(t-\tau) \end{pmatrix}. \end{aligned} \quad (32)$$

Then, integrating both sides with respect to the time results in (28). This completes the proof. (Q.E.D.)

It is noted that the inequality (27) can be rewritten by an LMI using the change of variables $Y := P^{-1},\ L := KP^{-1}$ and $\widehat{H} := PHP$ as follows:

$$\begin{bmatrix} AY + YA^T - BL - L^TB^T + \widehat{H} & YC_q & A_dY \\ C_q^TY & -I & 0 \\ YA_d^T & 0 & -\widehat{H} \end{bmatrix} < 0 \quad (33)$$

Hence, combining with (18) in order to bound the quadratic performance, the sliding mode can be obtained based on the guaranteed cost control through the convex search.

Using the Lyapunov approach, the similar result to Theorem 3 has been proposed in [5]. It has been shown, in [5], that the Lyapunov matrix that meets a certain matrix inequality associated with the full order system can be used for determining the sliding function coefficient without handling the reduced order system. Note, however, the performance issue has not been addressed in [5], and the obtained matrix inequality constraint is different from that of Theorem 3.

2.3 Application to Parametric Uncertain Systems

Sliding mode design for parametric uncertain systems has been addressed in [8] based on the Riccati approach. However, we re-state the issue here for the completeness of the paper and the introduction of the recently developed quadratic stability condition which uses the symmetric- and the skew symmetric scales.

Consider the uncertain system

$$\dot{x} = (A + \Delta A)\,x + B(u + w) \quad (34)$$

where ΔA represents the real parametric uncertainties of the form

$$\Delta A = MF(t)N \quad (35)$$

where $M,\ N^T \in \Re^{n\times h}$ and $F(t) = diag\,[\delta_1(t), \cdots, \delta_p(t)]$ for the Lebesgue measurable functions δ_i such that $|\delta_i(t)| \leq 1,\ \forall\ t \geq 0$. As to the reachability issue, it has been shown in [8] that the reachability condition is met by the control

$$u = \begin{cases} 0, & (\|s(t)\| = 0) \\ -B_2^{-1}\{GAx + \beta s + Z(t)sign(s)\}, & (\|s(t)\| > 0) \end{cases} \quad (36)$$

where $\beta > 0$ and $Z(t) = diag\,[z_1, \cdots, z_m]$ for z_i defined as

$$z_i = \sum_{j=1}^{h} |(GM)_{ij}(Nx)_j| + \sum_{k=1}^{l} |(GB)_{ik}|\overline{w}_k \quad (37)$$

This can be proven by showing that the derivative of the quadratic function $V = \frac{1}{2}s^Ts$ is made to be negative.

Now, for addressing the existence of sliding modes, the quadratic stability should be considered *a priori.*

Lemma 2 *The system*

$$\dot{x} = (A_c + \Delta A)\,x \quad (38)$$

is quadratically stable if there exist some $P > 0$, $X \in S_{sym}$ and $U \in S_{skew}$ satisfying,

$$\begin{aligned} &A_c^TP + PA_c + PMXM^TP \\ &+(N + PMU^T)X^{-1}(N + PMU^T)^T < 0 \end{aligned} \quad (39)$$

where $S_{sym} := \{X \mid XF = FX,\ X > 0\}$ and $S_{skew} := \{U \mid UF = FU,\ U = -U^T\}$.

The condition has been derived in [17] based on the $\mathcal{S}$-procedure and the realness of uncertainties, and applied to the L_2 disturbance attenuation problem [18]. See Appendix for a simpler proof using the quadratic bounding technique than in the references. It is noted that the usage of the skew symmetric scales (as well as the symmetric scales) effectively reduces the design conservatism in the presence of the multi-rank uncertain parameters (*i.e.,* the repeated uncertain parameters in $F(\bullet)$).

Theorem 4 *There exist sliding modes if there exist some $P > 0$, $K \in \Re^{m \times n}$, $X \in S_{sym}$ and $U \in S_{skew}$ satisfying, given a $Q \geq 0$,*

$$(A-BK)^T P + P(A-BK) + Q + PMXM^T P + (N + PMU^T)X^{-1}(N+PMU^T)^T < 0. \quad (40)$$

Moreover, using the sliding function coefficient $S = P_{22}^{-1} P_{12}^T$ for the feasible parameter, the performance index is bounded as

$$J = \int_{t_s}^{\infty} x^T Q x \; dt < x_1(t_s)^T P_r x_1(t_s) \quad (41)$$

where $P_r = P_{11} - P_{12} P_{22}^{-1} P_{12}^T$.

Proof. In order to save the space, we refer to [8] for the detailed procedures. The rough sketch of the proof is as follows. First, pre- and post multiply T_r and T_r^T (defined in (10)) by (40), respectively. Then, through some manipulations, one may show the quadratic stability of the reduced order uncertain system by choosing $S = P_{22}^{-1} P_{12}^T$. Also, the relation (41) can be shown by following the similar steps done in the proof of Theorem 2. This completes the proof. (Q.E.D.)

It is noted that that Theorem 3 is an extension of Theorems 1 and 2 to uncertain systems. It can be observed that Theorem 3 would be equivalent to the results of Theorems 1 and 2 in the absence of uncertainties, *i.e.*, $M = N = 0$. Also, in practice, the inequality (40) can be rewritten by an LMI using the change of variables and the Shur complement [9] as follows:

$$\begin{bmatrix} YA^T + AY - BL - L^T B^T + MXM^T & * & * \\ C_q^T Y & -I & * \\ N^T Y + UM^T & 0 & -X \end{bmatrix} < 0 \quad (42)$$

where $Y = P^{-1}$, $L = KP^{-1}$ and $Q = C_q C_q^T$. Hence, the x-optimal approach for the parametric uncertain systems can be easily obtained by replacing (17) with (42).

2.4 Multi-objective approach

Recently, much attention has been paid to design the controller that satisfies several performance criteria such as the H_∞ disturbance attenuation, the H_2 performance, the pole-clustering in the specified region and *etc*. Especially, it turns out that the multi-objective controllers can be designed with relative ease by searching the common Lyapunov matrix. The purpose of the section is to show the standard multi objective approach can be effectively adopted for designing the sliding modes.

Consider, for example, the x-optimal design with the eigenvalues of the sliding modes in the prescribed region. Note here that the design problem has two objectives, i.e., optimizing the quadratic performance and the pole placement in the specified region. To deal with the issue, we introduce the following result.

Theorem 5 *There exist some S so that the set of the eigenvalues of the matrix $A_{11} - A_{12}S$ belong to the set $\mathcal{Z}(c, \rho)$ for real scalars c and $\rho > 0$, that is,*

$$\lambda(A_{11} - A_{12}S) \subset \mathcal{Z}(c,\rho) := \{z \in \mathcal{C} \mid |z + c| < \rho\} \quad (43)$$

if and only if there exist some $K \in \Re^{m \times n}$ and $P > 0$ satisfying

$$\begin{bmatrix} \rho P & (A-BK)^T P + cP \\ P(A-BK) + cP & \rho P \end{bmatrix} > 0 \quad (44)$$

Also, setting as $S = P_{22}^{-1} P_{12}^T$ for the feasible P in the above, the pole-clustering property (43) holds.

Proof. See Appendix for the details. Instead, the rough sketch of the proof is given. Using the results in [14, 15], it is clear that (43) holds if and only if there exist some $P_r > 0$ and S satisfying

$$\begin{bmatrix} \rho P_r & * \\ P_r(A_{11} - A_{12}S) + cP_r & \rho P_r \end{bmatrix} > 0. \quad (45)$$

Hence, we must show the equivalence between the feasibility of the inequalities (44) and (45). First, the sufficiency can be easily shown using the down-sizing transformation matrix T defined in (29) with the same manner done in the proof of Theorem 4. Proving the necessity requires more elaborate manipulations for expanding the inequality (45) with the reduced oreder size to one with the full order, which can be accomplished by utilizing the full degree of freedom of K. This completes the proof. (Q.E.D.)

Remark 4 *Note that the inequality (44) is the necessary and sufficient condition for the existence of the full state feedback which places the closed loop poles in the specified region such that $\lambda(A - BK) \subset \mathcal{Z}(c, \rho)$.*

Then, combining the results in Theorem 2 and Theorem 5 with the assumption that the Lyapunov matrices (P's of (4) and (44)) are common, it is easy to have the following result.

Corollary 1 *Given some c, $\rho > 0$ and $Q \geq 0$, there exist some sliding modes S such that (i) $\lambda(A_{11} - A_{12}S) \subset \mathcal{Z}(c,\rho)$, and (ii) $\int_{t_s}^{\infty} x^T Q x dt < \gamma \|x_1(t_s)\|^2$ if there exist some $Y > 0$ and L satisfying inequalities*

$$\begin{bmatrix} \rho Y & YA^T - L^T B^T + cY \\ AY - BL + cY & \rho Y \end{bmatrix} > 0 \quad (46)$$

$$\begin{bmatrix} AY + YA^T - BL - L^T B^T & YC_q \\ C_q^T Y & -I \end{bmatrix} < 0 \quad (47)$$

$$\begin{bmatrix} \gamma I & I \\ I & U_1 Y U_1^T \end{bmatrix} > 0 \quad (48)$$

Then, for the feasible Y $(:= P^{-1})$, the sliding function is given by $S = P_{22}^{-1} P_{12}^T$.

The above shows how the various objectives can be effectively combined in a design problem. First, note that the formulation has the convexity for the design parameters, which can be solved using the LMIs technique. Also, the proposed approach does not require handling the reduced order system that has made the design complicated in some cases. As a result, a variety of results in the multiplier theories (using the Lyapunov matrices) can be adopted for the sliding mode design in the framework of multi-objective approach. Further applications remain as an active area of research.

3 Concluding Remarks

In this manuscript, the methods for sliding mode design have been newly proposed based on the multiplier approaches that use the Lyapunov matrices for the design constraints. It has been shown that the sliding mode can be designed by combining the partitions of the Lyapunov matrix that constrains the desired objective. Taking advantage of the construction technique, many of results that have been developed for the full state feedback synthesis in the area of multiplier theories are shown to be applicable to sliding mode design. The issues on the so-called x-optimal design, the state delayed systems, parametric uncertain systems and multi-objective approach were discussed.

A Proof of Lemma 2

For the quadratic function $V = x^T P x$, where $P > 0$, the quadratic stability can be proven by showing $\dot{V} < 0$. To do this, the following bounding technique is crucial:

$$PMFN + N^T F^T M^T P = PMF(N + UM^T P) + (N + UM^T P)^T F M^T P \quad (49)$$
$$\leq PMXM^T P + (N + UM^T P)^T X^{-1} (N + UM^T P) \quad (50)$$

Note that (49) is established thanks to the commuting property and the skew symmetricity, i.e., $FU = UF = -U^T F$. Also, (50) is the standard bounding technique for the block-diagonal uncertainties (e.g. see [8]). This completes the proof. (Q.E.D.)

B Proof of Theorem 6

(Sufficiency) Define as $T := \left[\begin{array}{c|c} T_r & 0 \\ \hline 0 & T_r \end{array}\right]$, where $T_r = \left[I_{n-m},\ -P_{12}P_{22}^{-1}\right]$. Then, pre- and post multiplying (44) by T and T^T, respectively, leads to (45) for the choice of $S = P_{22}^{-1}P_{12}^T$. (Necessity) First, consider the matrix $H := \left[\begin{array}{cc|cc} 0 & I_m & 0 & 0 \\ \hline 0 & 0 & 0 & I_m \end{array}\right]$, which makes the matrix $\begin{bmatrix} H \\ T \end{bmatrix}$ nonsingular. For brevity of the notation, let L_f and L_r denote the left hand sides of (44) and (45), respectively. Through some manipulations, it can be shown that

$$\begin{bmatrix} H \\ T \end{bmatrix} L_f \begin{bmatrix} H \\ T \end{bmatrix}^T = \left[\begin{array}{c|c} HL_fH^T & HL_fT^T \\ \hline TL_fH^T & TL_fT \end{array}\right] = \left[\begin{array}{cc|cc} \rho P_{22} & N & 0 & A_{12}^T P_r \\ N^T & \rho P_{22} & M & 0 \\ \hline 0 & M^T & \multicolumn{2}{c}{\multirow{2}{*}{L_r}} \\ P_r A_{12} & 0 & & \end{array}\right] \quad (51)$$

where $N = A_{12}^T P_{12} + (A_{22} - B_2K_2)^T P_{22} + cP_{22}$ and $M = P_{12}^T(A_{11} - A_{12}S) + P_{22}(A_{21} - B_2K_1) - P_{22}(A_{22} - B_2K_2)$. Note that N and M can be freely chosen thanks to K_1 and K_2. Since $L_r > 0$, the positive definiteness of the above transformed quantity is equivalent to that of the matrix

$$\begin{bmatrix} \rho P_{22} & N \\ N^T & \rho P_{22} \end{bmatrix} - \begin{bmatrix} 0 & A_{12}^T P_r \\ M & 0 \end{bmatrix} L_r^{-1} \begin{bmatrix} 0 & A_{12}^T P_r \\ M & 0 \end{bmatrix}^T \quad (52)$$

which can be always positive definite by selecting proper P_{22} and N. For example, choose $M = N = 0$ and a sufficiently large $P_{22} > 0$. This completes the proof. (Q.E.D.)

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