Observability of Perspective Dynamical Systems*

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Abstract. Perspective dynamical systems arise in machine vision, and the essential problem in such a system is how to determine any unknown states and /or any unknown parameters from its perspective observation. Considering simple perspective dynamical systems, we study the state observability and the parameter identifiability of such systems, using the differential geometric method, that is, the observability rank condition, which has been developed for general nonlinear systems, and present various necessary and/or sufficient conditions for observability and/or identifiability.

I. INTRODUCTION.

The essential problem in machine vision is how to determine the position of a moving rigid body and/or any unknown parameters characterizing the motion and shape of the body from knowledge of the associated optical flow [1]. Perspective dynamical systems arise from mathematically describing such machine vision problems, and this essential problem can be described in system theory terminology as the problem of determining any unknown state and /or of identifying any unknown parameters of such a system based on its perspective observations [2]-[6].

For perspective linear systems, the state observability problem has already been studied in [5], [6] wherein a new generalization of the Popov, Belevitch and Hautus (PBH) condition has been described. While the PBH rank test is an algebraic geometric technique and relies on the knowledge of the associated optical flow, in this paper, we study the local observability of perspective dynamical systems using the differential geometric method [7]-[10], that is, via computing the observability rank condition.

Since for any perspective dynamical system its complete state at each instant of time is not available due to the perspective observation, it is clear that any unknown parameters of such a system must be identified along with the state estimation. Therefore the parameter identifiability problem should be studied simultaneously with the state observability problem, and hence even for a simple perspective linear system the observability problem becomes quite complicated. So this paper considers only two very simple perspective dynamical systems, i.e., a perspective linear system [2], [3] and a perspective Riccati system [4], and presents various necessary and/or sufficient conditions for observability and identifiability for such systems.

II. PRELIMINARIES.

First, some notations are introduced, which are used throughout this paper. For a scalar-valued function $\lambda : \boldsymbol{R}^n \to \boldsymbol{R}$ and a column vector-valued function $f : \boldsymbol{R}^n \to \boldsymbol{R}^n$, define the following notations:

(2.1)
$$\begin{cases} d\lambda(x) := \dfrac{\partial \lambda}{\partial x}(x) = \begin{bmatrix} \dfrac{\partial \lambda}{\partial x_1}(x) & \dfrac{\partial \lambda}{\partial x_2}(x) & \cdots & \dfrac{\partial \lambda}{\partial x_n}(x) \end{bmatrix} \\ L_f \lambda(x) := \dfrac{\partial \lambda}{\partial x}(x) f(x) = \sum_{k=1}^{n} \dfrac{\partial \lambda}{\partial x_k}(x) f_k(x) \end{cases}$$

where $x = [x_1 \quad x_2 \quad \cdots \quad x_n]^T \in \boldsymbol{R}^n$ and $f = [f_1 \; f_2 \cdots f_n]^T$. $L_f \lambda(x)$ is called the **Lie derivative** of λ along f, and its higher derivatives $L_f^k \lambda(x)$ $(k = 0,1,2,\cdots)$ are recursively defined as

$$(2.2) \qquad \begin{cases} L_f^0 \lambda(x) := \lambda(x), \\ L_f^k \lambda(x) := \left(\dfrac{\partial}{\partial x} L_f^{k-1} \lambda(x) \right) f(x), \quad k = 1,2,\cdots. \end{cases}$$

Further, we use the following notation:

(2.3)
$$L_f d\lambda(x) := dL_f \lambda(x) = f^T(t) \left[\frac{\partial (d\lambda)}{\partial x}(x) \right]^T + d\lambda(x) \frac{\partial f}{\partial x}(x).$$

Now, we consider a nonlinear system, having no input, of the form

$$(2.4) \qquad \mathrm{S_N} : \begin{cases} \dot{x}(t) = f(x(t)), \quad x(0) = x_0 \in \boldsymbol{R}^n \\ y(t) = h(x(t)) \end{cases}$$

where $x(t) \in \boldsymbol{R}^n$ and $y(t) \in \boldsymbol{R}^m$ with $m < n$ are the state and the observation vectors, respectively, and $f : \boldsymbol{R}^n \to \boldsymbol{R}^n$, $h : \boldsymbol{R}^n \to \boldsymbol{R}^m$ $(h = [h_1 \quad h_2 \quad \cdots \quad h_m]^T)$ are column vector-valued rational functions. (More generally, these functions can be extended to meromorphic functions; that is,

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those functions represented as ratios of analytic functions having only singular points of poles. However, for our purpose rational functions are sufficient.)

First, the following definition is given.

(2.5) Definition.

(i) System (2.4) is said to be **observable** at a point $\hat{x}\in \boldsymbol{R}^n$, if there exist an open set $U\subset \boldsymbol{R}^n$ of $\hat{x}$ and positive integers $k_1,k_2,\cdots,k_m$ satisfying $k_1+k_2+\cdots+k_m=n$ such that for arbitrary $x\in U$ the set of row vectors defined by

$$\{L_f^{k-1}dh_i(x)\,|\,i=1,\cdots,m;k=1,\cdots,k_m\} \qquad (1)$$

is linearly independent.

(ii) System (2.4) is said to be **generically observable** if it is observable at all the points $\hat{x}\in \boldsymbol{R}^n$ except the common zeros of a finite number of some nonzero scalar-valued rational functions of x. □

The definition of observability above is due to Krener and Respondek [8]. It is well known that for general nonlinear systems global or complete observability cannot usually be expected, and therefore local or generical observability would be suitable notions. In fact, the observability in Definition (2.5) implies the local weak observability [7], [10], which intuitively means that any sufficiently close states can be instantaneously distinguished. Further, the generical observability means that the system is observable at almost every point in $\boldsymbol{R}^n$, i.e., at every point belonging to some open dense subset of $\boldsymbol{R}^n$. Although there have appeared alternative definitions in the literature [7]-[9], our definition given above is a most suitable one to our purpose.

III. Simple Perspective Systems and the Problem Statement

First, we introduce the notion of perspective systems as follows.

(3.1) Definition. System (2.4) is called **perspective** if the observation vector y is given as a rational function in x of the form

$$y=h(x)=\begin{bmatrix}\dfrac{C_1x}{C_{m+1}x}\\ \vdots \\ \dfrac{C_mx}{C_{m+1}x}\end{bmatrix},\quad C=\begin{bmatrix}C_1\\ \vdots\\ C_m\\ C_{m+1}\end{bmatrix}\in \boldsymbol{R}^{(m+1)\times n} \qquad (1)$$

where $1\le m<n$ and $C_k\in \boldsymbol{R}^{1\times n}$ $(k=1,\cdots,m+1)$. □

It is well known in machine vision that when a feature point $x(t)=[x_1(t)\quad x_2(t)\quad x_3(t)]^T$ of a rigid body moving in the 3-dimensional space $\boldsymbol{R}^3$ is observed with a CCD camera, the observation obtained on the image plane is essentially represented as the form

(3.2) $\qquad y(t)=h(x(t))=[x_1(t)/x_3(t)\quad x_2(t)/x_3(t)]^T$,

which is a special form of Definition (3.1), and called the **optical flow**. See, e.g., [1]-[4]. An essential problem in machine vision is to estimate its motion and to locate the surface from the observed information of several feature points selected on the surface of the body.

The following are simple examples of perspective systems, which have been studied in machine vision [1]-[4]. The first example is given by

(3.3)

$$\mathrm{S_L}:\begin{cases}\dfrac{d}{dt}\begin{bmatrix}x_1(t)\\x_2(t)\\x_3(t)\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}+\begin{bmatrix}0&-\omega_3&\omega_2\\ \omega_3&0&-\omega_1\\ -\omega_2&\omega_1&0\end{bmatrix}\begin{bmatrix}x_1(t)\\x_2(t)\\x_3(t)\end{bmatrix},\\[2ex] \begin{bmatrix}y_1(t)\\y_2(t)\end{bmatrix}=\begin{bmatrix}x_1(t)/x_3(t)\\x_2(t)/x_3(t)\end{bmatrix},\quad x_3(t)\ne 0\end{cases}$$

and will be called a **perspective linear system** in this investigation, where $\omega=[\omega_1\quad \omega_2\quad \omega_3]^T$ represents the angular velocity vector and $b=[b_1\quad b_2\quad b_3]^T$ the translation velocity vector. The second one is a little bit more general than the first one, and is given by

$$\text{(3.4)}\quad \mathrm{S_R}:\begin{cases}\dfrac{d}{dt}\begin{bmatrix}x_1(t)\\x_2(t)\\x_3(t)\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}+\begin{bmatrix}a_{11}&a_{12}&a_{13}\\ a_{21}&a_{22}&a_{23}\\ a_{31}&a_{32}&a_{33}\end{bmatrix}\begin{bmatrix}x_1(t)\\x_2(t)\\x_3(t)\end{bmatrix}\\[2ex] \qquad +\begin{bmatrix}c_1&c_2&c_3&0&0&0\\ 0&c_1&0&c_2&c_3&0\\ 0&0&c_1&0&c_2&c_3\end{bmatrix}\begin{bmatrix}x_1^2(t)\\x_1(t)x_2(t)\\x_3(t)x_1(t)\\x_2^2(t)\\x_2(t)x_3(t)\\x_3^2(t)\end{bmatrix},\\[2ex] \begin{bmatrix}y_1(t)\\y_2(t)\end{bmatrix}=\begin{bmatrix}x_1(t)/x_3(t)\\x_2(t)/x_3(t)\end{bmatrix},\quad x_3(t)\ne 0\end{cases}$$

and will be called a **perspective Riccati system** [4].

In what follows, we consider the essential problem of machine vision for these two perspective systems $\mathrm{S_L},\mathrm{S_R}$, but restricting ourselves to the case where only a single feature point is observed and no surface location is concerned. So we focus our attention to the state estimation and/or parameter identification problems based on all the available information. More precisely, the problem we are concerned with can be stated as follows: Based on the continuously observed perspective observation

(3.5) $\qquad Y(t):=\{y(\tau)=[y_1(\tau)\quad y_2(\tau)]^T\,|\,0\le\tau\le t\}$

or a part of it, under what conditions can the state $x(t)=[x_1(t)\quad x_2(t)\quad x_3(t)]^T$ and/or the parameters $\omega=[\omega_1\quad\omega_2\quad\omega_3]^T$, $b=[b_1\quad b_2\quad b_3]^T$ or a part of the parameters be uniquely determined ? Theoretical point of view, this problem can be answered by investigating the observability and/or identifiability of nonlinear systems $\mathrm{S_L},\mathrm{S_R}$.

IV. OBSERVABILITY OF PERSPECTIVE LINEAR SYSTEMS

First, we consider the observability for perspective linear system $\mathrm{S_L}$ given by (3.3). For notational convenience, let

$$\begin{cases} f(x) := \Omega x + b, \quad \Omega := \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}, \quad b := \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \\ h(x) := \begin{bmatrix} h_1(x) \\ h_2(x) \end{bmatrix} = \begin{bmatrix} x_1/x_3 \\ x_2/x_3 \end{bmatrix}, \quad x_3 \neq 0. \end{cases}$$

Then, from Definition (2.5)(i), in order to check observability for system $\mathrm{S_L}$ it suffices to investigate the linear independence of the set of row vectors given by

(4.2)
$$\{dh_1(x), dh_2(x), L_f dh_1(x), L_f dh_2(x), L_f^2 dh_1(x), L_f^2 dh_2(x)\}.$$

It is easily seen that

(4.3) $$dh_1(x) = \begin{bmatrix} \frac{1}{x_3} & 0 & -\frac{x_1}{x_3^2} \end{bmatrix}, \quad dh_2(x) = \begin{bmatrix} 0 & \frac{1}{x_3} & -\frac{x_2}{x_3^2} \end{bmatrix}$$

and hence $dh_1(x)$ and $dh_2(x)$ are linearly independent at every $x \in R^n$ with $x_3 \neq 0$. It is not difficult to see that the other vectors in (4.2) can be represented in the form

(4.4) $$L_f^j dh_i(x) = [\alpha_{ij}(x) \quad \beta_{ij}(x) \quad \gamma_{ij}(x)], \quad i, j = 1,2,$$

where $\alpha_{ij}(x), \beta_{ij}(x), \gamma_{ij}(x)$ are some rational functions of $x = [x_1 \quad x_2 \quad x_3]^T$. Now, define the matrices

(4.5) $$M_{L(i,j)}(x) := \begin{bmatrix} dh_1(x) \\ dh_2(x) \\ L_f^j dh_i(x) \end{bmatrix} = \begin{bmatrix} 1/x_3 & 0 & -x_1/x_3^2 \\ 0 & 1/x_3 & -x_2/x_3^2 \\ \alpha_{ij}(x) & \beta_{ij}(x) & \gamma_{ij}(x) \end{bmatrix}, \quad i, j = 1,2.$$

Then, their determinants can be easily computed as

(4.6) $$\det M_{L(i,j)}(x) = \frac{1}{x_3^3}\{\alpha_{ij}(x)x_1 + \beta_{ij}(x)x_2 + \gamma_{ij}(x)x_3\}, \quad i, j = 1,2$$

and using the MAPLE one can easily obtain

(4.7) $$\begin{cases} \det M_{L(1,1)}(x) = (b_3x_1 - b_1x_3)/x_3^5 \\ \det M_{L(1,2)}(x) = (b_3\omega_2x_3^2 + 4b_1b_3x_3 + b_2\omega_3x_3^2 \\ \quad + 4b_3\omega_2x_1^2 - 4b_3^2x_1 - 3b_1\omega_2x_1x_3 \\ \quad + 2b_1\omega_1x_2x_3 - 2b_3\omega_3x_2x_3 \\ \quad + b_2\omega_1x_1x_3 - 4b_3\omega_1x_1x_2)/x_3^6 \\ \det M_{L(2,1)}(x) = (b_3x_2 - b_2x_3)/x_3^5 \\ \det M_{L(2,2)}(x) = (b_1\omega_3x_3^2 + b_3\omega_1x_3^2 - 4b_2b_3x_3 \\ \quad + 4b_3\omega_1x_2^2 + 4b_3^2x_2 + b_1\omega_2x_2x_3 \\ \quad + 2b_2\omega_2x_1x_3 - 3b_2\omega_1x_2x_3 \\ \quad - 2b_3\omega_3x_1x_3 - 4b_3\omega_2x_1x_2)/x_3^6 \end{cases}$$

First, the following lemma can be proved.

(4.8) Lemma.

(i) Arbitrary three vectors in (4.2) are linearly dependent for every $x \in R^3$ with $x_3 \neq 0$ if and only if

$$\det M_{L(i,j)}(x) = 0, \quad \forall x \in R^3 \text{ with } x_3 \neq 0, \quad i, j = 1,2. \tag{1}$$

(ii) Eq. (1) is satisfied if and only if $b = 0$.

Proof. For the statement (i), the necessity is obvious, and hence only the sufficiency is proved. Assume Eq. (1) holds. Since the vectors $dh_1(x), dh_2(x)$ in Eq. (4.3) are linearly independent for every $x \in R^3$ with $x_3 \neq 0$ and are rational functions, it follows from Eq. (1) that every vector $L_f^j dh_i(x)$ $(i, j = 1,2)$ can be represented as a linear combination of $dh_1(x), dh_2(x)$ in the form

$$L_f^j dh_i(x) = a_{ij}(x)dh_1(x) + b_{ij}(x)dh_2(x), \quad i, j = 1,2$$

where $a_{ij}(x), b_{ij}(x)$ $(i, j = 1,2)$ are some rational functions. Therefore, the set of vectors given in Eq. (4.2) has no more than two linearly independent vectors, and hence the sufficiency is proved.

The statement (ii) is easily verified from Eq. (4.7). In fact, since every term in $\det M_{L(i,j)}(x)$ contains one of b_1, b_2, b_3, $b = 0$ implies Eq. (1). Conversely, Eq. (1), in particular, $\det M_{L(1,1)}(x) \equiv 0$ and $\det M_{L(2,1)}(x) \equiv 0$ imply $b_1 = b_2 = b_3 = 0$. □

Now, using Lemma (4.8), the following theorem can be proved.

(4.9) Theorem. Perspective linear system $\mathrm{S_L}$ given by Eq. (3.3) is generically observable if and only if $b \neq 0$.

Proof. It follows from Lemma (4.8) that $\det M_{L(i,j)}(x)$ is not identically zero for some i, j if and only if $b \neq 0$. Therefore, by Definition (2.5) the condition $b \neq 0$ is equivalent to System $\mathrm{S_L}$ being observable at every point $x \in R^3$ except the zeros of this $\det M_{L(i,j)}(x)$. Finally, since $\det M_{L(i,j)}(x)$ is a rational function, it follows that the condition $b \neq 0$ is equivalent to System $\mathrm{S_L}$ being generically observable. □

If $b \neq 0$ and

(4.10)
$$\mathcal{N}_L := \{x \in R^3 \mid \det M_{L(i,j)}(x) = 0 \text{ and } x_3 \neq 0, \ i, j = 1,2\},$$

then $\mathcal{N}_L \neq \phi$ and is a strictly proper and very small subset of R^3 (in fact, the complement $\mathcal{N}_L^c$ is an open and dense subset of R^3), and system $\mathrm{S_L}$ is locally observable at every point $x \in \mathcal{N}_L^c$.

Next, we consider a single output case in System $\mathrm{S_L}$, that is, the case where only one of the components $y_1(t), y_2(t)$ is available, so that for $i = 1,2$ $\mathrm{S_L}$ is reduced to the following systems:

(4.11)

$$\mathrm{S}_{\mathrm{L}}^{(i)}:\begin{cases}\dfrac{d}{dt}\begin{bmatrix}x_1(t)\\x_2(t)\\x_3(t)\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}+\begin{bmatrix}0&-\omega_3&\omega_2\\\omega_3&0&-\omega_1\\-\omega_2&\omega_1&0\end{bmatrix}\begin{bmatrix}x_1(t)\\x_2(t)\\x_3(t)\end{bmatrix},\\ y_i(t)=\dfrac{x_i(t)}{x_3(t)},\quad x_3(t)\neq 0.\end{cases}$$

In order to investigate the observability of $\mathrm{S}_{\mathrm{L}}^{(i)}$, it suffices to check the rank of the matrices

$$(4.12)\qquad M_L^{(i)}(x):=\begin{bmatrix}dh_i(x)\\L_f dh_i(x)\\L_f^2 dh_i(x)\end{bmatrix},\quad i,j=1,2.$$

After a patient computation, we get

(4.13)

$$\begin{cases}\det M_L^{(1)}(x)=\dfrac{1}{x_3^6}\{-(\omega_2\omega_3 b_3+\omega_1^2 b_2+\omega_1\omega_2 b_1)x_1^2\\ \quad +2(\omega_1\omega_3 b_3+\omega_1^2 b_1)x_1x_2-2\omega_1\omega_3 b_2 x_1 x_3+2(\omega_1\omega_3 b_1\\ \quad +\omega_3^2 b_3)x_2x_3-(\omega_2\omega_3 b_3+\omega_1\omega_2 b_1+\omega_3^2 b_2)x_2^2\\ \quad +2(\omega_3 b_3^2+\omega_1 b_1 b_3)x_1-2(\omega_3 b_1 b_3+\omega_1 b_1^2)x_3\}\\ \det M_L^{(2)}(x)=\dfrac{1}{x_3^6}\{-2(\omega_2\omega_3 b_3+\omega_2 b_2)x_1x_2\\ \quad -2(\omega_2\omega_3 b_2+\omega_3^2 b_3)x_1x_3+\omega_1\omega_2 b_2+(\omega_1\omega_3 b_3+\omega_2^2 b_1\\ \quad +\omega_1\omega_2 b_2)x_2^2+2\omega_2\omega_3 b_1 x_2 x_3+(\omega_1\omega_3 b_3+\omega_3^2 b_1)x_3^2\\ \quad +2(\omega_2 b_2 b_3+\omega_3 b_3^2)x_2-2(\omega_3 b_2 b_3+\omega_2 b_2^2)x_3\}.\end{cases}$$

Clearly, when $\det M_L^{(i)}(x)$ is not identically zero, defining

$$\mathcal{N}_L^{(i)}:=\{x\in \boldsymbol{R}^3 \mid \det M_L^{(i)}(x)=0 \text{ and } x_3\neq 0\},\ i=1,2,$$

we have

$$\det M_L^{(i)}(x)=3,\quad \forall x\notin \mathcal{N}_L^{(i)} \text{ with } x_3\neq 0,\quad i=1,2,$$

and hence $\mathrm{S}_{\mathrm{L}}^{(i)}$ is generically observable. Further, it is clear that $\mathrm{S}_{\mathrm{L}}^{(i)}$ is not generically observable if and only if $\det M_L^{(i)}(x)$ is identically zero, i.e., $\det M_L^{(i)}(x)\equiv 0$. A simple but patient computation gives that $\det M_L^{(1)}(x)\equiv 0$ if and only if any one of the following conditions (i)-(v) is satisfied:

$$(4.14)\qquad\begin{cases}\text{(i)}\ \ \omega_1=0 \text{ and } \omega_2=0\\ \text{(ii)}\ \ \omega_1=0, b_2=0 \text{ and } b_3=0\\ \text{(iii)}\ \ \omega_3=0, b_1=0 \text{ and } b_2=0\\ \text{(iv)}\ \ b_1=0, b_2=0 \text{ and } b_3=0\\ \text{(v)}\ \ b_2=0 \text{ and } \omega_1 b_1+\omega_3 b_3=0\end{cases}$$

and similarly $\det M_L^{(2)}(x)\equiv 0$ if and only if any one of the following conditions (i)-(v) is satisfied:

$$(4.15)\qquad\begin{cases}\text{(i)}\ \ \omega_2=0 \text{ and } \omega_3=0\\ \text{(ii)}\ \ \omega_2=0, b_1=0 \text{ and } b_3=0\\ \text{(iii)}\ \ \omega_3=0, b_1=0 \text{ and } b_2=0\\ \text{(iv)}\ \ b_1=0, b_2=0 \text{ and } b_3=0\\ \text{(v)}\ \ b_1=0 \text{ and } \omega_2 b_2+\omega_3 b_3=0.\end{cases}$$

Therefore, the following theorem has been shown.

(4.16) Theorem. Concerning the perspective linear system $\mathrm{S}_{\mathrm{L}}^{(i)}$ with a single output given by Eq. (4.11), the following statements hold.

(i) $\mathrm{S}_{\mathrm{L}}^{(1)}$ is generically observable if and only if none of the conditions (i)-(v) given in Eq. (4.14) is satisfied.

(ii) $\mathrm{S}_{\mathrm{L}}^{(2)}$ is generically observable if and only if none of the conditions (i)-(v) given in Eq. (4.15) is satisfied. □

Now, we consider the observability of perspective Riccati system S_{R} given by Eq. (3.4) in the same manner as for the perspective linear system S_{L}. First, defining

$$(4.17)\qquad\begin{cases}f(x):=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}+\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\\ \qquad\quad +\begin{bmatrix}c_1&c_2&c_3&0&0&0\\0&c_1&0&c_2&c_3&0\\0&0&c_1&0&c_2&c_3\end{bmatrix}\begin{bmatrix}x_1^2\\x_1x_2\\x_3x_1\\x_2^2\\x_2x_3\\x_3^2\end{bmatrix},\\ h(x):=\begin{bmatrix}h_1(x)\\h_2(x)\end{bmatrix}=\begin{bmatrix}x_1/x_3\\x_2/x_3\end{bmatrix},\quad x_3\neq 0\end{cases}$$

we can show for this Riccati system that the matrices $M_{R(i,j)}(x)$ corresponding to Eq. (4.5) has the same form, and their determinants are computed as follows:

(4.18)

$$\begin{cases}\det M_{R(1,1)}(x)=(b_3x_1-b_1x_3)/x_3^5\\ \det M_{R(1,2)}(x)=-\{-4b_1b_3x_3+(a_{11}b_1-a_{13}b_3-2a_{33}b_1\\ \quad +a_{12}b_2)x_3^2+4a_{31}b_3x_1^2+4b_3^2x_1-2a_{11}b_3x_1x_3\\ \quad -2a_{12}b_3x_2x_3-3a_{31}b_1x_1x_3-2a_{32}b_1x_2x_3\\ \quad -a_{32}b_2x_1x_3+4a_{32}b_3x_1x_2+3a_{33}b_3x_1x_3\}/x_3^6\\ \det M_{R(2,1)}(x)=(b_3x_2-b_2x_3)/x_3^5\\ \det M_{R(2,2)}(x)=-\{(a_{21}b_1+a_{22}b_2-a_{23}b_3-2a_{33}b_2)x_3^2\\ \quad -4b_2b_3x_3+4a_{32}b_3x_2^2+4b_3^2x_2-a_{31}b_1x_2x_3\\ \quad -2a_{22}b_3\omega_3x_2x_3-2a_{21}b_3x_1x_3-2a_{31}b_2x_1x_3\\ \quad -3a_{32}b_2x_2x_3+4a_{31}b_3x_1x_2+3a_{33}b_3x_2x_3\}/x_3^6\end{cases}$$

As before, it is not difficult to see that $\det M_{R(i,j)}(x) \equiv 0\ (\forall i, j = 1,2)$ if and only if $b = 0$, and therefore, the following theorem is readily proved.

(4.19) Theorem. Perspective Riccati system S_R given by Eq. (3.4) is generically observable if and only if $b \neq 0$. □

It is very interesting to note that comparing the above with Theorem (4.9) both S_L and S_R have the same necessary and sufficient conditions for generical observability.

Now, we can construct single output perspective Riccati systems $S_R^{(i)}\ (i = 1,2)$, corresponding to those $S_L^{(i)}\ (i = 1,2)$, and investigate their observability. Although the argument becomes much more complicated than before, we can get necessary and sufficient conditions for $S_R^{(i)}\ (i = 1,2)$ to be generically observable. In fact, the final result is stated without details in the following theorem.

(4.20) Theorem.

(i) $S_R^{(1)}$ is generically observable if and only if the parameters a_{ij}, b_i, c_i satisfy none of the following conditions (a)-(g):

$$
\left\{
\begin{array}{ll}
\text{(a)} & b_3^2 c_2 - a_{12}a_{32}b_3 + a_{31}a_{32}b_1 - a_{32}^2 b_2 + a_{22}a_{32}b_3 = 0 \\
\text{(b)} & a_{12}a_{32}b_3 - a_{32}^2 b_1 = 0 \\
\text{(c)} & b_1 b_3 c_2 - a_{11}a_{32}b_1 + a_{32}a_{33}b_1 + 2a_{12}a_{32}b_2 \\
& \quad + a_{11}a_{12}b_3 - a_{12}a_{33}b_3 - a_{12}a_{22}b_3 - a_{22}a_{32}b_1 = 0 \\
\text{(d)} & a_{12}^2 b_3 - a_{12}a_{32}b_3 = 0 \\
\text{(e)} & -b_1^2 c_2 - a_{12}^2 b_2 + a_{12}a_{13}b_3 - a_{13}a_{32}b_1 + a_{12}a_{22}b_1 = 0 \\
\text{(f)} & a_{32}b_1 b_3 - a_{12}b_3^2 = 0 \\
\text{(g)} & a_{12}b_1 b_3 - a_{32}b_1^2 = 0.
\end{array}
\right.
$$

(ii) $S_R^{(2)}$ is generically observable if and only if the parameters a_{ij}, b_i, c_i satisfy none of the following conditions (a)-(g):

$$
\left\{
\begin{array}{ll}
\text{(a)} & a_{21}a_{31}b_3 - a_{31}^2 b_2 = 0 \\
\text{(b)} & a_{21}^2 b_3 - a_{21}a_{31}b_2 = 0 \\
\text{(c)} & a_{21}a_{32}b_3 + a_{31}^2 b_1 - a_{31}a_{32}b_2 - a_{11}a_{31}b_3 + b_3^2 c_1 = 0 \\
\text{(d)} & -2a_{21}a_{31}b_1 - a_{21}a_{22}b_3 + a_{21}a_{33}b_3 + a_{22}a_{31}b_2 \\
& \quad - a_{31}a_{33}b_2 + a_{11}a_{21}b_3 + a_{11}a_{31}b_2 - 2b_2 b_3 c_1 = 0 \\
\text{(e)} & b_2^2 c_1 + a_{21}^2 b_1 - a_{21}a_{23}b_3 - a_{11}a_{21}b_2 + a_{23}a_{31}b_2 = 0 \\
\text{(f)} & a_{21}b_3^2 - a_{31}b_2 b_3 = 0 \\
\text{(g)} & a_{31}b_2^2 - a_{21}b_2 b_3 = 0.\ \square
\end{array}
\right.
$$

It is not very difficult to show that if $c_k = 0\ (k = 1,2,3)$ and $a_{ii} = 0\ (i = 1,2,3)$ with $a_{ij} = -a_{ji}\ (i \neq j)$ then Theorem (4.20) reduces to Theorem (4.16).

V. Identifiability of Perspective Linear Systems

In this section, we consider the parameter identifiability problem for the perspective linear system S_L. By this identifiability problem, we mean the problem of whether or not some unknown parameters of the system can be determined uniquely from the perspective observation

$$(5.1)\qquad Y(t) := \{ y(\tau) = [y_1(\tau) \quad y_2(\tau)]^T \mid 0 \le \tau \le t \}.$$

Of course, in this problem, since the complete state $x(t)$ is not available, the parameter identifiability problem must be considered along with the state observability problem.

To avoid too cumbersome argument, we consider only the following two simplified identifiability problems.

(5.2) Problem 1. It is assumed that the angular velocity vector $\omega = [\omega_1 \quad \omega_2 \quad \omega_3]^T$ is known, and that when expressing the translation velocity vector as $b = \| b \| [\hat{b}_1 \quad \hat{b}_2 \quad \hat{b}_3]^T$ the unit vector $\hat{b} := [\hat{b}_1 \quad \hat{b}_2 \quad \hat{b}_3]^T$ is known, but only the magnitude $\| b \|$ is unknown. □

Letting $x_4 := \| b \|$ and setting $dx_4/dt = 0$ to represent that $\| b \|$ is a constant, Problem 1 is to check the state observability of the following perspective linear system:

$$
(5.3)\quad S_{Lb}: \left\{
\begin{array}{l}
\dfrac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 & \hat{b}_1 \\ \omega_3 & 0 & -\omega_1 & \hat{b}_2 \\ -\omega_2 & \omega_1 & 0 & \hat{b}_3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}, \\
\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} x_1(t)/x_3(t) \\ x_2(t)/x_3(t) \end{bmatrix}, \quad x_3(t) \neq 0 .
\end{array}
\right.
$$

(5.4) Problem 2. It is assumed that the translation velocity vector $b = [b_1 \quad b_2 \quad b_3]^T$ is known, and that when expressing the angular velocity vector as $\omega = \| \omega \| [\hat{\omega}_1 \quad \hat{\omega}_2 \quad \hat{\omega}_3]^T$ only the magnitude $\| \omega \|$ is unknown. □

In this time, letting $x_4 := \| \omega \|$ and setting $dx_4/dt = 0$, Problem 2 is equivalent to checking the state observability of the following perspective nonlinear system:

$$
(5.5)\quad S_{L\omega}: \left\{
\begin{array}{l}
\dfrac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & -\hat{\omega}_3 & \hat{\omega}_2 & 0 \\ \hat{\omega}_3 & 0 & -\hat{\omega}_1 & 0 \\ -\hat{\omega}_2 & \hat{\omega}_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} x_1(t)x_4(t) \\ x_2(t)x_4(t) \\ x_3(t)x_4(t) \\ x_4(t) \end{bmatrix}, \\
\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} x_1(t)/x_3(t) \\ x_2(t)/x_3(t) \end{bmatrix}, \quad x_3(t) \neq 0 .
\end{array}
\right.
$$

It should be noticed that S_{Lb} is a perspective linear system, but $S_{L\omega}$ is a perspective nonlinear system.

As before, to check the observability of $S_{Lb}, S_{L\omega}$, it suffice to check the linear independence of

$$\{dh_1, dh_2, L_f dh_1, L_f dh_2, L_f^2 dh_1, L_f^2 dh_2, L_f^3 dh_1, L_f^3 dh_2\}.$$

Again, letting

$$(5.6)\quad \begin{cases} L_f^j dh_i(x) = [\alpha_{ij}(x) \quad \beta_{ij}(x) \quad \gamma_{ij}(x) \quad \delta_{ij}(x)], \\ \qquad\qquad i = 1,2;\ j = 1,2,3 \\ M_{Lp(i,j)}^{(k,l)}(x) := \begin{bmatrix} \frac{1}{x_3} & 0 & -\frac{x_1}{x_3^2} & 0 \\ 0 & \frac{1}{x_3} & -\frac{x_2}{x_3^2} & 0 \\ \alpha_{ij}(x) & \beta_{ij}(x) & \gamma_{ij}(x) & \delta_{ij}(x) \\ \alpha_{kl}(x) & \beta_{kl}(x) & \gamma_{kl}(x) & \delta_{kl}(x) \end{bmatrix}, \\ \qquad\qquad (i,j) \neq (k,l), \quad p = b \text{ or } \omega \end{cases}$$

one can obtain

$$(5.7)\quad \det M_{Lp(i,j)}^{(k,l)}(x)$$
$$= \delta_{kl}(x) \det M_{L(i,j)}(x) - \delta_{ij}(x) \det M_{L(k,l)}(x)$$
$$= \frac{1}{x_3^3}\left\{ x_1 \det \begin{bmatrix} \alpha_{ij}(x) & \alpha_{kl}(x) \\ \delta_{ij}(x) & \delta_{klj}(x) \end{bmatrix} \right.$$
$$\left. + x_2 \det \begin{bmatrix} \beta_{ij}(x) & \beta_{klj}(x) \\ \delta_{ij}(x) & \gamma_{kl}(x) \end{bmatrix} + x_3 \det \begin{bmatrix} \gamma_{ij}(x) & \gamma_{lj}(x) \\ \delta_{ij}(x) & \delta_{kl}(x) \end{bmatrix} \right\},$$
$$(i,j) \neq (k,l), \quad p = b \text{ or } \omega$$

where $M_{L(i,j)}(x)$ is the matrix developed in Eq. (4.5).

For System S_{Lb}, further calculation shows that

$$(5.8)\qquad \det M_{Lb(i,j)}^{(k,l)}(x) \equiv 0, \quad \forall (i,j) \neq (k,l).$$

Thus, System S_{Lb} is never identifiable (observable), and hence the unknown parameter $\| b \|$ is never identifiable from the perspective observation given by (5.1). Therefore, to make $\| b \|$ identifiable, it is necessary to have more information, for instance, by observing more than one feature point or using more than a single CCD camera. This problem will be investigated elsewhere in the future.

On the other hand, for System $S_{L\omega}$, it can be shown that

$$(5.9)\qquad \det M_{Lb(i,j)}^{(k,l)}(x) \neq 0, \quad \forall (i,j) \neq (k,l),$$

that is, the determinant $\det M_{Lb(i,j)}^{(k,l)}(x)$ cannot be identically zero, and hence $\| \omega \|$ is generically identifiable (observable).

Summarizing the results obtained in this section, we have the following theorem.

(5.10) Theorem. The unknown parameter $\| b \|$ in Problem 1 is never identifiable from the perspective observation given by (5.1), but the unknown parameter $\| \omega \|$ in Problem 2 is generically identifiable. □

VI. Concluding Remarks

Considering very simple perspective dynamical systems, the generical observability of such systems was investigated, using the observability rank condition developed for general nonlinear systems, and various necessary and/or sufficient conditions for observability were obtained. In particular, it was first proved that a necessary and sufficient condition for the perspective linear system S_L to be generically observable is simply $b \neq 0$, and surprisingly that this result is also true for the perspective Riccati system S_R. Further it was shown that, even for the single output perspective systems $S_L^{(i)}$ and $S_R^{(i)}$ $(i = 1,2)$, the generical state observability is obtained for almost all parameters characterizing the systems. Finally it was shown that the unknown parameter $\| b \|$ of the translation vector b is never identifiable, but the unknown parameter $\| \omega \|$ of the angular velocity vector is generically identifiable.

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