Robust Kalman Filter Design

Xing Zhu, Yeng Chai Soh and Lihua Xie

School of Electrical & Electronic Engineering
Nanyang Technological University
Nanyang Avenue, Singapore 639798
E-mail:eycsoh@ntu.edu.sg

Abstract

In this paper, the problem of finite and infinite horizon robust Kalman filtering for uncertain discrete-time systems is studied. The system under consideration is subject to time-varying norm-bounded parameter uncertainty in both the state and output matrices. The problem addressed is the design of linear filters having an error variance with a guaranteed upper bound for any allowed uncertainty. A novel technique is developed for robust filter design. This technique gives necessary and sufficient conditions to the design of robust filters over finite and infinite horizon.

1 Introduction

One of the fundamental problems in control systems and signal processing is the estimation of the state variables of a dynamic system through available noisy measurements. In the past three decades, this problem has attracted the interests of many researchers and one of the popular methods is based on the minimization of the variance of the estimation error, i.e. the celebrated Kalman filtering approach (see, e.g. [1]). The filtering algorithm requires the knowledge of a perfect dynamic model for the signal generating system, and that the noise sources are white processes with known statistics. Thus, the standard Kalman filter may not be robust against modeling uncertainty and disturbances. This has motivated many studies on robust Kalman filter design, which may probably yield a suboptimal solution with respect to the nominal system, but it will guarantee an upper bound to the filtering error covariance in spite of large parameter uncertainties.

In recent years, several results have been derived on the design of such robust estimators that give an upper bound to the error variance for any allowed modeling uncertainty (see, e.g., [2] – [3]). The continuous-time case has been discussed in [4] and references therein. In [4], both the finite and infinite-horizon filtering problems were addressed and necessary and sufficient conditions for the existence of robust filters with an optimalized upper bound for the error variance were given. As for the discrete-time case, [2], [5] and [6] only focus on the design of the stationary robust filters, while in [3], only the finite-horizon case was discussed by using semidefinite programming method. Although in [7], both the finite and infinite-horizon robust filters were discussed, conditions for the design of such filters are only sufficient.

In this paper, we consider the problem of robust *a prior* Kalman filtering for discrete-time systems with norm-bounded parameter uncertainty in both the state and output matrices. The process and measurement noises are assumed to be stationary white signals with known statistics. The problem addressed is the design of linear filters that yield an estimation error variance with a guaranteed upper bound for all admissible uncertainties. Both the finite-horizon and infinite-horizon cases are investigated, and necessary and sufficient conditions for the existence of such filters are given.

The remainder of this paper is organized as follows: In Section 2, we give the problem formulation. In Section 3, the problem of robust Kalman filtering for discrete-time systems with parameter uncertainty over finite-horizon is studied. The infinite-horizon case is discussed in Section 4. A numerical example is illustrated in Section 5. Finally, conclusions are drawn in Section 6.

Notations: Most of the notations used in this paper are fairly standard. The superscript 'T' denotes matrix transposition. $\Re^n$ denotes the n-dimensional Euclidean space and $||\cdot||$ refers to Euclidean vector norm. $l_2[0,N]$ stands for the space of square summable vector sequence over $[0,N]$, and $||\cdot||_2^2$ is the $l_2[0,N]$ norm defined by $||\cdot||_2^2 := \sum_0^N ||\cdot||^2$.

2 Problem Formulation

Consider the following class of uncertain discrete-time system

$$x_{k+1} = (A+\Delta A_k)x_k + B\omega_k \quad (2.1)$$
$$y_k = (C+\Delta C_k)x_k + v_k \quad (2.2)$$

where $x_k \in \Re^n$ is the system state, $\omega_k \in \Re^q$ is the process noise, $y_k \in \Re^m$ is the measurement, $v_k \in \Re^m$ is the measurement noise, and A, B and C are known real matrices with appropriate dimensions.

In addition, ΔA_k and ΔC_k are unknown matrices which represent time-varying parameter uncertainties. These uncertainties are assumed to be of the following structure

$$\begin{bmatrix} \Delta A_k \\ \Delta C_k \end{bmatrix} = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} F_k E \quad (2.3)$$

where $F_k \in \Re^{i\times j}$ is an unknown real time-varying matrix satisfying

$$F_k^T F_k \le I, \quad k \ge 0 \quad (2.4)$$

and H_1, H_2 and E are known real constant matrices of appropriate dimensions that specify how the elements of the nominal matrices A and C are affected by uncertainty in F_k.

Next we shall adopt the following assumptions for the process and measurement noises:

Assumption 2.1. *For all integers k and $l \ge 0$,*

(i) $E(\omega_k) = 0,\ E(\omega_k\omega_l^T) = W\delta(k-l),\ W \ge 0;$

(ii) $E(v_k) = 0,\ E(v_k v_l^T) = V\delta(k-l),\ V > 0;$

(iii) $E(\omega_k v_l^T) = 0.$ □

In the above, $E(\cdot)$ denotes the expectation and $\delta(k)$ is the Kronecker Delta.

In this paper, we are concerned with the design of robust Kalman filters in both the finite horizon and infinite horizon cases for the uncertain system (2.1)-(2.2).

For the finite horizon case, the system matrices in (2.1)-(2.2) can be allowed to be time-varying, i.e., A, B and C are replaced by A_k, B_k and C_k, respectively. However, for simplicity, we shall only derive the results for constant system matrices. Our objective is to design a robust filter of the form

$$\hat{x}_{k+1} = A_{fk}\hat{x}_k + K_{fk}y_k, \quad \hat{x}_0 = 0 \quad (2.5)$$

with the estimation error defined by

$$e_k = x_k - \hat{x}_k$$

where A_{fk} and K_{fk} are time-varying matrices to be determined in order that the variance of the estimation error $e_k = x_k - \hat{x}_k$ is guaranteed to be smaller than a certain bound for all uncertainty matrices F_k satisfying (2.4), i.e., the estimation error dynamics satisfies

$$E\big[(x_k-\hat{x}_k)(x_k-\hat{x}_k)^T\big] \le S_k$$

and S_k is minimized in some sense.

For the infinite horizon case, the system matrices A, B, C, H_1, H_2 and E are confined to be constant matrices and our objective is to design a time-invariant robust filter (2.5) where $A_{fk} \equiv A_f$ and $K_{fk} \equiv K_f$ $(k \ge 0)$ are constant matrices to be determined in order that the estimation error dynamics are asymptotically stable and there exists a symmetric nonnegative definite matrix S such that in the steady state,

$$E\big[(x_k-\hat{x}_k)(x_k-\hat{x}_k)^T\big] \le S$$

for all uncertainty matrices F_k satisfying (2.4).

3 Finite-Horizon Robust Filter Design

In this section, a solution to the guaranteed cost robust filtering problem over finite-horizon $[0,N]$ will be given using a Riccati equation approach.

3.1 Preliminaries

First, we assume that the initial condition $x(0)$ is a zero mean Gaussian random variable x_0 independent of the noise ω_k and v_k, and with an unknown covariance matrix that satisfies the following assumption:

Assumption 3.1.

(i) $E[x_0x_0^T] \le \bar{S}_0$, where $\bar{S}_0 = \bar{S}_0^T > 0$ is a known matrix;

(ii) $rank[A \quad H_1 \quad BW^{\frac{1}{2}}] = n.$ □

Next, in terms of system (2.1)-(2.2) and filter (2.5), the state-space equations for the estimation error e_k are as follows [i]:

$$\xi_{k+1} = (A_{c1} + H_{c1}F_kE_{c1})\xi_k + G\eta_k \quad (3.1)$$
$$e_k = L\xi_k \quad (3.2)$$

where

$$\xi_k = \begin{bmatrix} e_k \\ \hat{x}_k \end{bmatrix},\ \xi_0 = \begin{bmatrix} x_0 \\ 0 \end{bmatrix},\ \eta_k = \begin{bmatrix} \omega_k \\ v_k \end{bmatrix},$$

[i] In order to simplify expression, we drop the subscript k in A_f and K_f.

$$A_{c1} = \begin{bmatrix} A - K_f C & A - A_f - K_f C \\ K_f C & A_f + K_f C \end{bmatrix},$$

$$H_{c1} = \begin{bmatrix} H_1 - K_f H_2 \\ K_f H_2 \end{bmatrix},\ E_{c1} = \begin{bmatrix} E & E \end{bmatrix},$$

$$G = \begin{bmatrix} B & -K_f \\ 0 & K_f \end{bmatrix},\ L = \begin{bmatrix} I & 0 \end{bmatrix}. \qquad (3.3)$$

Definition 3.1. *The filter (2.5) is said to be a quadratic filter associated with a guaranteed cost matrix* $\Sigma_k = \Sigma_k^T \geq 0$ *if for some* $\epsilon_k > 0$, Σ_k *satisfies the following difference Riccati equation (DRE):*

$$\begin{aligned} \Sigma_{k+1} &= A_{c1}\Sigma_k A_{c1}^T + A_{c1}\Sigma_k E_{c1}^T(\epsilon_k^{-1}I - E_{c1}\Sigma_k E_{c1}^T)^{-1} \\ &\quad \cdot E_{c1}\Sigma_k A_{c1}^T + \epsilon_k^{-1}H_{c1}H_{c1}^T + G\bar{W}G^T \qquad (3.4) \end{aligned}$$

and such that $I - \epsilon_k E_{c1}\Sigma_k E_{c1}^T > 0$, *where* $\Sigma_0 = diag\{\bar{S}_0, 0\}$ *and* $\bar{W} = diag\{W, V\}$.

Note that since $\Sigma_0 = \Sigma_0^T \geq 0$, it follows that if for some $\epsilon_k > 0$ a solution to (3.4) exists over $[0, N]$, then it is symmetric positive semidefinite.

Next we recall a linear matrix inequality result which will be needed in the proof of our main result.

Lemma 3.1. **([8], [9])** *Let* $A \in \Re^{n\times n}$, $H \in \Re^{n\times i}$, $E \in \Re^{j\times n}$ *and* $Q = Q^T \in \Re^{n\times n}$ *be given matrices.*

(1) Then there exists a real matrix $\Sigma = \Sigma^T > 0$ *such that*

$$(A + HF_kE)\Sigma(A + HF_kE)^T + Q < 0$$

for all F_k *satisfying* $F_k^T F_k \leq I$, *if and only if there exists a scalar* $\epsilon > 0$ *such that* $\frac{I}{\epsilon} - E\Sigma E^T > 0$ *and*

$$\begin{aligned} A\Sigma A^T + A\Sigma E^T(\frac{1}{\epsilon}I - E\Sigma E^T)^{-1}E\Sigma A^T + \frac{1}{\epsilon}HH^T & \\ +Q &< 0 \end{aligned}$$

(2) Then there exists a real matrix $\Sigma = \Sigma^T \geq 0$ *such that*

$$(A + HF_kE)\Sigma(A + HF_kE)^T + Q \leq 0$$

for all F_k *satisfying* $F_k^T F_k \leq I$, *if there exists a scalar* $\epsilon > 0$ *such that* $\frac{I}{\epsilon} - E\Sigma E^T > 0$ *and*

$$\begin{aligned} A\Sigma A^T + A\Sigma E^T(\frac{1}{\epsilon}I - E\Sigma E^T)^{-1}E\Sigma A^T + \frac{1}{\epsilon}HH^T & \\ +Q &= 0 \end{aligned}$$

Then we give the following result which shows that a quadratic estimator will provide a known guaranteed cost.

Lemma 3.2. *Consider the uncertain system (2.1)-(2.2) satisfying Assumption 2.1 and let (2.5) be a given quadratic filter associated with a guaranteed cost matrix* $\Sigma_k = \Sigma_k^T \geq 0$. *Then the covariance matrix of* ξ_k *of the error system (3.1)-(3.2) satisfies the bound*

$$E[\xi_k\xi_k^T] \leq \Sigma_k\ , \quad \forall k \in [0, N]$$

for all admissible uncertainties. Furthermore,

$$E[e_k e_k^T] \leq L\Sigma_k L^T = \Sigma_{11,k}\ , \quad \forall k \in [0, N] \qquad (3.5)$$

where $\Sigma_{11,k} \in R^{n\times n}$ *is the (1,1) block of the matrix* Σ_k *and* e_k *is the estimation error as defined in (2.6).*

Next, we introduce the following two DREs which will be related to our main results.

$$\begin{aligned} P_{k+1} &= AP_kA^T + AP_kE^T\Big(\frac{I}{\epsilon_k} - EP_kE^T\Big)^{-1}EP_kA^T \\ &\quad + \frac{1}{\epsilon_k}H_1H_1^T + BWB^T,\ P_0 = \bar{S}_0 \qquad (3.6) \\ S_{k+1} &= AQ_kA^T - \Big(AQ_kC^T + \frac{1}{\epsilon_k}H_1H_2^T\Big) \\ &\quad \cdot\big(R_{\epsilon_k} + CQ_kC^T\big)^{-1}\Big(AQ_kC^T + \frac{1}{\epsilon_k}H_1H_2^T\Big)^T \\ &\quad + \frac{1}{\epsilon_k}H_1H_1^T + BWB^T\ ,\quad S_0 = \bar{S}_0 \qquad (3.7) \end{aligned}$$

where $\epsilon_k > 0$ is a parameter to be chosen and

$$\begin{aligned} Q_k^{-1} &= S_k^{-1} - \epsilon_k E^T E \\ R_{\epsilon_k} &= V + \epsilon_k^{-1}H_2H_2^T \qquad (3.8) \end{aligned}$$

Proposition 3.1. *The DRE (3.7) can be re-expressed as follows:*

$$\begin{aligned} S_{k+1} &= AS_kA^T - \begin{bmatrix} AS_kE^T & AS_kC^T + \frac{1}{\epsilon_k}H_1H_2^T \end{bmatrix} \\ &\quad \cdot \begin{bmatrix} -\frac{1}{\epsilon_k}I + ES_kE^T & ES_kC^T \\ CS_kE^T & R_{\epsilon_k} + CS_kC^T \end{bmatrix}^{-1} \\ &\quad \cdot \begin{bmatrix} AS_kE^T & AS_kC^T + \frac{1}{\epsilon_k}H_1H_2^T \end{bmatrix}^T \\ &\quad + \frac{1}{\epsilon_k}H_1H_1^T + BWB^T \end{aligned}$$

which can be equivalently rewritten in a clearer form:

$$\begin{aligned} S_{k+1} &= AS_kA^T - (AS_k\hat{C}_k^T + \hat{B}_k\hat{D}_k^T) \\ &\quad \cdot(\hat{C}_kS_k\hat{C}_k^T + \hat{R}_k)^{-1}(AS_k\hat{C}_k^T + \hat{B}_k\hat{D}_k^T)^T \\ &\quad + \hat{B}_k\hat{B}_k^T \qquad (3.9) \end{aligned}$$

where $\hat{C}_k = \begin{bmatrix} \sqrt{\epsilon_k}C \\ \sqrt{\epsilon_k}E \end{bmatrix}$, $\hat{B}_k = \begin{bmatrix} BW^{\frac{1}{2}} & \frac{1}{\sqrt{\epsilon_k}}H_1 & 0 \end{bmatrix}$, $\hat{D}_k = \begin{bmatrix} \tilde{D}_k \\ 0 \end{bmatrix}$, $\hat{R}_k = \begin{bmatrix} \tilde{D}_k\tilde{D}_k^T & 0 \\ 0 & -I \end{bmatrix}$, $\tilde{D}_k = \begin{bmatrix} 0 & H_2 & \sqrt{\epsilon_k}V^{\frac{1}{2}} \end{bmatrix}$. □

In the following lemma, we shall show that the existence of P_k and S_k is guaranteed by the existence of Σ_k to (3.4).

Lemma 3.3.

(i) *Under Assumption 3.1, if for a given filter of (2.5) and for some scalar $\epsilon_k > 0$, the DRE (3.4) has a bounded solution Σ_k over $[0, N]$ and such that $I - \epsilon_k E_{c1} \Sigma_k E_{c1}^T > 0$, then there exists a bounded solution $P_k = P_k^T > 0$ to the DRE (3.6) over $[0, N]$ for the same $\epsilon_k > 0$, and such that $P_k^{-1} - \epsilon_k E^T E > 0$.*

(ii) *Under Assumption 3.1, if for some scalar $\epsilon_k > 0$, the DRE (3.6) has a bounded solution P_k over $[0, N]$ and such that $P_k^{-1} - \epsilon_k E^T E > 0$, then there exists a bounded solution $S_k = S_k^T > 0$ to the DRE (3.7) over $[0, N]$ for the same $\epsilon_k > 0$ and such that $S_k^{-1} - \epsilon_k E^T E > 0$. Furthermore, $P_k \geq S_k > 0$ over $[0, N]$.*

Remark 3.1. From the proof of Lemma 3.3, we have that if for some $\epsilon_k > 0$ there exists a bounded solution Σ_k to the DRE (3.4) over $[0, N]$, then the solution P_k to the DRE (3.6) over $[0, N]$ exists and is equivalent to $\hat{\Sigma}_k = \Sigma_{11,k} + \Sigma_{12,k} + \Sigma_{21,k} + \Sigma_{22,k}$. □

3.2 Finite-horizon Robust Filters

Our next theorem presents a necessary and sufficient condition for the existence of a quadratic filter.

Theorem 3.1. *Consider that the uncertain system (2.1)-(2.2) satisfies Assumptions 2.1 and 3.1. Then there exists a robust quadratic filter for the system that minimizes the bound on the error variance in (3.5) if and only if for some $\epsilon_k > 0$, there exists a solution $P_k = P_k^T > 0$ over [0,N] to the DRE (3.6) with $P_0 = \bar{S}_0$, and such that $P_k^{-1} - \epsilon_k E^T E > 0$.*

Under this condition, an optimal quadratic guaranteed cost a priori filter is given by

$$\hat{x}_{k+1} = (A + \Delta A_{ek})\hat{x}_k + K_f[y_k - (C + \Delta C_{ek})\hat{x}_k], \quad \hat{x}_0 = 0 \tag{3.10}$$

where

$$\Delta A_{ek} = \epsilon_k A S_k E^T (I - \epsilon_k E S_k E^T)^{-1} E \tag{3.11}$$

$$\Delta C_{ek} = \epsilon_k C S_k E^T (I - \epsilon_k E S_k E^T)^{-1} E \tag{3.12}$$

$$K_f = (A Q_k C^T + \frac{1}{\epsilon_k} H_1 H_2^T)(R_{\epsilon_k} + C Q_k C^T)^{-1} \tag{3.13}$$

and $S_k = S_k^T > 0$ is a solution of DRE (3.7) over [0,N] with $S_0 = \bar{S}_0$ and satisfies $Q_k^{-1} = S_k^{-1} - \epsilon_k E^T E > 0$.

Moreover, the optimal guaranteed cost is

$$E[(x_{k+1} - \hat{x}_{k+1})(x_{k+1} - \hat{x}_{k+1})^T] \leq S_{k+1} \tag{3.14}$$

Remark 3.2. It should be commented that although the filter parameters of (3.10) does not depend on the solution of the DRE (3.6), in order for this filter to provide a bound on the error variance, it is not enough to just find a solution to the DRE (3.7). □

Remark 3.3. It is worth noting that if there is no parameter uncertainty in system (2.1)–(2.2), that is $H_1 = 0$, $H_2 = 0$, $E = 0$, then the robust filter of Theorem 3.1 reduces to the standard finite-horizon Kalman filter for the nominal system of (2.1)–(2.2). Since in this case with the stability of A, a bounded solution to DRE (3.6) always exists and DRE (3.7) recovers the Riccati difference equation in the standard Kalman filtering. □

Proposition 3.2. *For the difference Riccati equation (3.6), if $\bar{\epsilon}_k$ is the supremum of ϵ_k such that $\bar{\epsilon}_k E P_k E^T < I$, then under Assumption 3.1, the difference Riccati equation (3.6) exists a positive definite solution for any $\epsilon_k \in (0, \bar{\epsilon}_k)$.* □

Proposition 3.3. *In case of robust filter over finite-horizon, ϵ_k is a scaling parameter. As a matter of fact, such a parameter could be exploited to improve the filter performance. In addition, if ϵ_k is such that difference Riccati equation (3.6) admits a positive definite solution, then ϵ_k also gives rise to a positive definite solution for difference Riccati equation (3.7).* □

Proposition 3.4. *For $\epsilon_k \in (0, \bar{\epsilon}_k)$, $tr(P_k)$ and $tr(S_k)$ are both convex functions of ϵ_k.* □

4 Infinite-Horizon Robust Filter Design

In this section, we discuss the design of stationary guaranteed cost robust filters over infinite-horizon. Attention is focused on the asymptotic variance of the estimation error. Note that now the filter is also required to be asymptotically stable.

We assume that the system (2.1) is quadratically stable. The definition is as follows,

Definition 4.1. **([2])** *The system (2.1) is said to be quadratically stable if there exists a symmetric positive definite matrix P such that*

$$(A + \Delta A_k)^T P (A + \Delta A_k) - P < 0$$

for all admissible uncertainties ΔA_k.

and we also adopt the following assumption throughout this section

Assumption 4.1. $rank[A \;\; H_1 \;\; BW^{\frac{1}{2}}] = n.$ □

Definition 4.2. *The filter (2.5) is said to be a quadratic stable filter associated with a guaranteed cost matrix*

$\Sigma_k = \Sigma_k^T \geq 0$ *if for some* $\epsilon > 0$, Σ *satisfies the following algebraic Riccati equation (ARE):*

$$\begin{aligned}\Sigma &= A_{c1}\Sigma A_{c1}^T + A_{c1}\Sigma E_{c1}^T(\epsilon_k^{-1}I - E_{c1}\Sigma E_{c1}^T)^{-1}E_{c1}\Sigma A_{c1}^T \\ &\quad +\epsilon^{-1}H_{c1}H_{c1}^T + G\bar{W}G^T \qquad (4.1)\end{aligned}$$

and such that $\hat{A} = A_{c1} + A_{c1}\Sigma E_{c1}^T(\epsilon^{-1}I - E_{c1}\Sigma E_{c1}^T)^{-1}E_{c1}$ *is asymptotically stable.*

Next, we give the following result which is the stationary counterpart of Lemma 3.2.

Lemma 4.1. *Suppose the uncertain system (2.1)-(2.2) is quadratically stable and satisfies Assumption 2.1, and let (2.5) be a given quadratic stable filter. Then, the covariance matrix of* ξ_k *of the error system (3.1)-(3.2) satisfies the following bound in steady state, i.e. as* $k \to \infty$

$$E[\xi_k \xi_k^T] \leq \Sigma \ , \quad \forall k \in [0, N]$$

Furthermore,

$$E[e_k e_k^T] \leq L\Sigma L^T = \Sigma_{11} \ , \quad \forall k \in [0, N] \qquad (4.2)$$

where $\Sigma_{11} \in R^{n\times n}$ *is the (1,1) block of the matrix* Σ.

We introduce the following two algebraic Riccati equations (ARE) which will be related to our main results.

$$\begin{aligned}P &= APA^T + APE^T\big(\frac{1}{\epsilon}I - EPE^T\big)^{-1}EPA^T \\ &\quad +\frac{1}{\epsilon}H_1H_1^T + BWB^T \qquad (4.3)\end{aligned}$$

$$\begin{aligned}S &= AQA^T - \big(AQC^T + \frac{1}{\epsilon}H_1H_2^T\big)\big(R_\epsilon + CQC^T\big)^{-1} \\ &\quad \cdot\big(AQC^T + \frac{1}{\epsilon}H_1H_2^T\big)^T \\ &\quad +\frac{1}{\epsilon}H_1H_1^T + BWB^T \qquad (4.4)\end{aligned}$$

where $Q^{-1} = S^{-1} - \epsilon E^T E$ and $R_\epsilon = V + \frac{1}{\epsilon}H_2H_2^T$.

Remark 4.1. Similar to the finite-horizon case, the algebraic Riccati equation (4.4) can be re-expressed as follows:

$$\begin{aligned}S &= ASA^T - (AS\hat{C}^T + \hat{B}\hat{D}^T)(\hat{C}S\hat{C}^T + \hat{R})^{-1} \\ &\quad (AS\hat{C}^T + \hat{B}\hat{D}^T)^T + \hat{B}\hat{B}^T \qquad (4.5)\end{aligned}$$

where $\hat{C} = \begin{bmatrix} \sqrt{\epsilon}C \\ \sqrt{\epsilon}E \end{bmatrix}$, $\hat{B} = \begin{bmatrix} BW^{\frac{1}{2}} & \frac{1}{\sqrt{\epsilon}}H_1 & 0 \end{bmatrix}$, $\hat{D} = \begin{bmatrix} \tilde{D} \\ 0 \end{bmatrix}$, $\hat{R} = \begin{bmatrix} \tilde{D}\tilde{D}^T & 0 \\ 0 & -I \end{bmatrix}$, $\tilde{D} = \begin{bmatrix} 0 & H_2 & \sqrt{\epsilon}V^{\frac{1}{2}} \end{bmatrix}$. □

Definition 4.3. (Stabilizing Solution)([10]) *A real positive symmetric matrix* P *(or* S*) is said to be a stabilizing solution to* ARE (4.3) *(or* ARE (4.4)*) if* P *(or* S*) satisfies* ARE (4.3) *(or* ARE (4.4)*) and the matrix* $\hat{A}_1 = A - APE^T(EPE^T - \frac{I}{\epsilon})^{-1}E$ *(or* $\hat{A}_2 = A - (AS\hat{C}^T + \hat{B}\hat{D}^T)(\hat{C}S\hat{C}^T + \hat{R})^{-1}\hat{C}$*) is Schur stable, i.e. all its eigenvalues lie in the open unit circle.* □

The following result is a stationary counterpart of Lemma 3.3.

Lemma 4.2.

(i) *Under Assumption 4.1, if for a given filter of (2.5) and for some scalar* $\epsilon > 0$, *the ARE (4.1) has a stabilizing solution* $\Sigma = \Sigma^T \geq 0$ *and such that* $I - \epsilon E_{c1}\Sigma E_{c1}^T > 0$, *then there exists a stabilizing solution* $P = P^T \geq 0$ *to the ARE (4.3) for the same* ϵ, *and such that* $P^{-1} - \epsilon E^T E > 0$.

(ii) *Under Assumption 4.1, if for some scalar* $\epsilon > 0$, *the ARE (4.3) has a stabilizing solution* $P = P^T \geq 0$ *and such that* $P^{-1} - \epsilon E^T E > 0$, *then there exists a stabilizing solution* $S = S^T \geq 0$ *to the ARE (4.4) for the same* ϵ, *and such that* $S^{-1} - \epsilon_k E^T E > 0$. *Furthermore,* $P \geq S$.

The next theorem provides a solution to the robust filtering problem for the uncertain system (2.1)-(2.2) in the steady state case.

Theorem 4.1. *Consider the uncertain system (2.1)-(2.2) satisfying Assumptions 2.1 and 4.1. Then there exists a stable quadratic robust filter if and only if for some* $\epsilon > 0$, *there exists a stabilizing solution* $P = P^T \geq 0$ *over* $[0, \infty]$ *to the ARE (4.3) and such that* $P^{-1} - \epsilon E^T E > 0$.

Under this condition, an optimal quadratic stable filter is given by

$$\hat{x}_{k+1} = (A + \Delta A_e)\hat{x}_k + K_f\big[y_k - (C + \Delta C_e)\hat{x}_k\big] \qquad (4.6)$$

where

$$\begin{aligned}\Delta A_e &= \epsilon ASE^T\big(I - \epsilon ESE^T\big)^{-1}E \\ \Delta C_e &= \epsilon CSE^T\big(I - \epsilon ESE^T\big)^{-1}E \\ K_f &= \big(AQC^T + \frac{1}{\epsilon}H_1H_2^T\big)\big(R_\epsilon + CQC^T\big)^{-1}\end{aligned}$$

and $S = S^T \geq 0$ *is a stabilizing solution of ARE (4.4) satisfying* $Q^{-1} = S^{-1} - \epsilon E^T E > 0$.

Moreover, the optimal guaranteed cost is

$$E\big[(x_{k+1} - \hat{x}_{k+1})(x_{k+1} - \hat{x}_{k+1})^T\big] \leq S$$

Remark 4.2. It can be shown using monotonicity results on the algebraic Riccati equation that if system (2.1)-(2.2) is quadratically stable, then there exists an $\bar{\epsilon} > 0$ such that for any $\epsilon \in (0, \bar{\epsilon}]$, there exists a stabilizing solution to ARE (4.3) ([8]). And $\min_\epsilon trace(S)$ is a convex optimization problem. □

5 Numerical Example

We consider the following uncertain discrete-time system ([2]),

$$x_{k+1} = \begin{bmatrix} 0 & -0.5 \\ 1 & 1+\delta \end{bmatrix} x_k + \begin{bmatrix} -6 \\ 1 \end{bmatrix} \omega_k \quad (5.1)$$

$$y_k = \begin{bmatrix} -100 & 10 \end{bmatrix} x_k + v_k \quad (5.2)$$

where δ is an uncertain parameter satisfying $|\delta| \leq 0.3$. Note that the above system is of the form of system (2.1)-(2.2) with $H_1 = \begin{bmatrix} 0 \\ 10 \end{bmatrix}, H_2 = 0, E = \begin{bmatrix} 0 & 0.03 \end{bmatrix}$, $W = 1$ and $V = 1$.

We now want to estimate the signal $z_k = Lx_k$ with $L = [1 \;\; 0]$, i.e., only partial state needs to be estimated. In such case, the guaranteed cost is given by $tr(LSL^T)$.

We apply the stationary filter design procedure of Section 4 to the system (5.1)-(5.2). By considering Remark 4.2, we obtain $\bar{\epsilon} = 1.38$. The optimal parameter also equals to $\bar{\epsilon}$, i.e. $\epsilon_{opt} = 1.38$ with the corresponding optimal guaranteed cost $tr(LS_{\epsilon_{opt}}L^T) = 69.2$. And the actual cost with $\delta = 0$, $\delta = 0.3$ and $\delta = -0.3$ are 51.06, 54.44 and 52.80, respectively. All these results are better than those in [2].

The corresponding optimal robust filter is given by

$$\hat{x}_{k+1} = \bar{A}\hat{x}_k + K_f[y_k - \bar{C}\hat{x}_k]$$

where $\bar{A} = \begin{bmatrix} 0 & -0.71 \\ 1.00 & 1.27 \end{bmatrix}$, $\bar{C} = \begin{bmatrix} -100 & 27.94 \end{bmatrix}$ and $K_f = \begin{bmatrix} -0.007 \\ 0.005 \end{bmatrix}$. We also have

$$P_s(\epsilon_{opt}) = \begin{bmatrix} 225.5 & -195.3 \\ -195.3 & 390.4 \end{bmatrix}$$

and

$$S_s(\epsilon_{opt}) = \begin{bmatrix} 69.2 & -79.0 \\ -79.0 & 234.1 \end{bmatrix}.$$

6 Conclusion

In this paper, we give a solution to the problem of finite and infinite horizon robust Kalman filtering for uncertain discrete-time systems. Necessary and sufficient conditions to the design of robust filters are obtained. Results of this paper are demonstrated by a numerical example. It shows that our result is better than that of [2].

References

[1] B. Anderson and J. Moore, *Optimal filtering*. Englewood Cliffis, N.J.: Prentice-Hall, 1979.

[2] L. Xie, Y. C. Soh, and C. E. de Souza, "Robust Kalman filtering for uncertain discrete-time systems," *IEEE Trans. on Automatic Control*, vol. 39, pp. 1310–1314, 1994.

[3] M. Fu, C. E. de Souza, and Z. Luo, "Finite horizon robust Kalman filter design," in *Proc. 38th IEEE Conf. on Decision and Control*, (Phoenix, Arizona), pp. 4555–4560, 1999.

[4] U. Shaked and C. E. de Souza, "Robust minimum variance filtering," *IEEE Trans. on Signal Processing*, vol. 43, pp. 2474–2483, 1995.

[5] I. Petersen and D. McFarlane, "Optiaml guaranteed cost filtering for uncertain discrete-time linear systems," *Int. J. Robust & Nolinear Control*, vol. 6, pp. 267–280, 1996.

[6] Z. Wang, J. Zhu, and H. Unbehauen, "Robust filter design with time-varying parameter uncertainty and error variance constranits," *Int. J. Control*, vol. 72, no. 1, pp. 30–38, 1999.

[7] Y. Theodor and U. Shaked, "Robust discrete-time minimum-variance filtering," *IEEE Trans. on Signal Processing*, vol. 44, pp. 181–189, 1996.

[8] L. Xie and Y. C. Soh, "Guaranteed cost control of uncertain discrete-time systems," *Control Theory and Advanced Technology*, vol. 10, no. 4, pp. 1235–1251, 1995.

[9] C. E. de Souza, M. Fu, and L. Xie, "H_∞ analysis and synthesis of discrete-time systems with time-varying uncertainty," *IEEE Trans. on Automatic Control*, vol. 38, pp. 459–462, 1993.

[10] C. E. de Souza and L. Xie, "Robust H_∞ filtering," *Control and Dynamic System*, vol. 65, pp. 323–377, 1994.