Optimal Control Strategy for High Jump based on Complementarity Modeling

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Abstract

This paper proposes an optimal control strategy for a jumping robot system based on the complementarity modeling. We consider a variable constraint jumping system consisting of a robot part and an environment(catapult) part. Such a system can be efficiently modeled as a complementarity system, in the sense that the discontinuous phenomena such as collision or separation are handled in a unified framework.

First of all, we give a simple criterion to judge the contact/taking-off condition based on the complementarity modeling. Secondly, we formulate an optimal control problem to maximize the peak height in a jump and give a numerical solution; due to the pre-specified input limitation, the resulting control is of bang-bang type. Finally the optimal controller is analytically reconsidered and is implemented as a switching state feedback law.

1 Introduction

On various occasion of robotic application, we confront the interaction of robots with their environments such as walls, grounds or other robots. One of the most significant instances is a *jumping robot,* which is designed to perform taking-off from and/or landing at the ground. Control problems of such a system is challenging indeed, especially from the viewpoint of control engineering, since the variation of contact conditions play an essential role in taking-off or landing motions. Many works have been done from various viewpoints, e.g., mechanism design[8], taking-off control[7] and orientation control in flight phase[5]. Jumping problem in general is supposed to consist of a jumping robot part and an environment part, where they may contact with or separated from each other.

This sort of jumping system can be categorized into the variable constraint mechanical systems, in that the mechanical constraint of the system varies as the contact condition between the robot and the environment changes. In most of existing attempts for such systems, each contact condition (say a constraint *mode*) has been dealt with distinctly: e.g., Berkemeier and Fearling[7] considered a logic diagram to determine the constraint forces at each instant, in addition to the continuous equation of motion. As against them, this study adopts a unified approach to handle these variable constraint mechanical systems (Brogliato et al.[3] also have done some pioneer works on impact mechanics and juggling robots). The variation of constraint is also regarded as a hybrid nature of the system, in the sense that the dynamics consists of a mixture of a continuous part(the evolution of continuous equation of motion) and a discrete part(the logical switching of the constraints).

As a modeling framework for a class of hybrid systems, the *complementarity approach* is proposed by van der Schaft and Schumacher[11]. This approach models the discrete part of the system dynamics as bilinear inequality constraints, called *complementarity conditions.* In the case of our jumping system, we may set the complementarity condition to indicate that either the force acting on the robot from the environment or the distance between the robot and the environment should be kept zero. One of the prominent features of this approach is that the *mode selection problem* by solving which the mode switching condition and the well-posedness of the system is indicated, is formulated as an Linear Complementarity Problem(LCP) which is well investigated in mathematical programming.

Based on this approach, this paper provides twofold contributions. At first, we prepare a complementarity modeling of the jumping system in Section 2 and clarify the criterion to judge the contact/taking-off condition in Section 3 by solving the corresponding mode selection problem. The second contribution is to derive an optimal control strategy for high jump. Section 4 formulates an optimal control problem which aims at maximizing the peak-height in a single jump, subject to a limitation of input magnitude. Under a simplification of the problem which reduces the environment to the rigid ground, the optimal bang-bang type controller is analytically derived and is implemented as a switching state feedback controller in Section 5. Finally, we provide concluding remarks in Section 6.

2 Complementarity Modeling of Contact/Taking-off Systems

Let us consider a class of contact/taking-off mechanical systems consisting of two parts, namely a robot and an environment, in a vertical plane as depicted in Fig. 1.

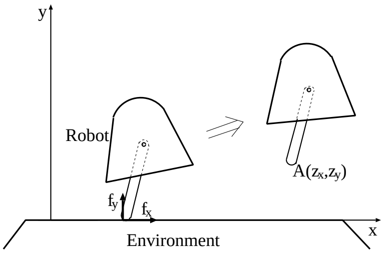

Fig.1 A class of contact/taking-off mechanical systems

This system can be regarded as a variable constraint control system in the sense that the mechanical constraints varies as the contact condition changes. Dynamics of this sort of systems are usually described as two distinct equations; e.g., one for the separate condition and the other for the contact condition. In contrast to the conventional approach, we adopt the complementarity modeling[6][11] to unify the distinct equations.

Let $\xi \in R^q$ be the generalized coordinate vector of the whole system, including the position and joint variables of both the robot and the environment. The position of A, or the lower end of the robot, is written as a pair of function $(z_x(\xi), z_y(\xi))$, and obviously z_y remains nonnegative. Suppose that A is the only point of contact with the environment. Thus the vertical constraint force $f_y \geq 0$ acts on the point A from the environment only when the $z_y = 0$, due to this mechanical contact constraint. The horizontal Coulomb friction force f_x also acts on A from the environment only when $z_y = 0$.

Instead of considering the slide of the point A on the surface, we impose here an limitation on the magnitude of horizontal friction force so as *not to cause the horizontal slide:*

$$|f_x(t)| \le \min\{\gamma f_y(t), \delta\} \tag{1}$$

where γ is the inclination of the friction cone and $\delta > 0$ is a sufficiently small constant. From now on, we assume that $z_x(t) = 0, \dot{z}_x(t) = 0$ while $z_y(t) = 0$. In other words, we will only consider the motion without horizontal slide.

Using the auxiliary variables (f_x, f_y) and (z_x, z_y), the equation of motion can be represented by

$$M(\xi)\ddot{\xi} + C(\xi, \dot{\xi})\dot{\xi} + G(\xi) = \Phi_\xi^T(\xi) f(t) + Fu(t) \tag{2}$$

$$f(t) := [\ f_x,\ f_y\ ]^T, \quad \Phi_\xi(\xi) := \begin{bmatrix} \frac{\partial z_x}{\partial \xi} \\ \frac{\partial z_y}{\partial \xi} \end{bmatrix}$$

and $M(\xi)$, $C(\xi, \dot{\xi})$ and $G(\xi)$ respectively denote the inertial matrix, the viscosity and Colioris coefficient and the potential force.

Taking $x(t) = [\xi^T(t), \dot{\xi}^T(t)]^T$ as the state vector, we have the following complementarity system expression:

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & I \\ 0 & -M^{-1}(\xi)C(\xi,\dot{\xi}) \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ -M^{-1}(\xi)G(\xi) \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 \\ M^{-1}(\xi)\Phi_\xi^T(\xi) \end{bmatrix} f(t) + \begin{bmatrix} 0 \\ M^{-1}(\xi)F \end{bmatrix} u(t) & (3) \\ z(t) &\ge 0, \quad w(t) \ge 0, \quad z(t)w(t) = 0, & (4) \end{aligned}$$

where the complementarity input and output are defined by $w(t) := f_y,\ z(t) := z_y$ respectively.

Remark: Note that this expression slightly differs from standard complementarity systems [11] in that $w(t)$ does not include f_x. Indeed, f_x is determined by the following rule: it is the solution of non-slide constraint $z_x(t) \equiv 0$, i.e., $z_x(t) = 0$ and $\dot{z}_x(t) = 0$ if $z_y = 0$, and it remains zero while $z_y > 0$. Therefore we may concentrate on the problem of computing z_y and f_y.

[Example] Now let us consider a jumping system as depicted in Fig. 2. The robot part is made of two masses, a spring, a damper and an actuator, while the environment part is made of a mass, a spring and a damper connected to the ground. This system performs only linear motion in the vertical direction. Let us set the generalized coordinate of the system as $\xi(t) = [x_1 \ \ x_2 \ \ x_e]^T$ and set the state $x(t) = [\xi^T(t), \dot{\xi}^T(t)]^T$.

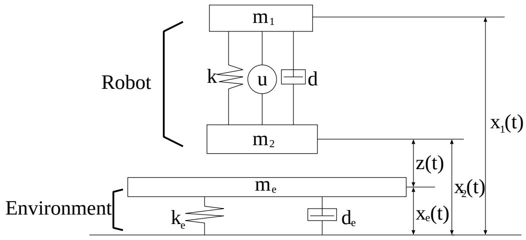

Fig.2 Linear jumping system

Let $z(t)$ and $w(t)$ be the distance between the robot and the environment and the force which acts on the robot from the environment, respectively. Then the complementarity modeling leads us to the following linear complementarity system[6]:

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & I \\ A_{21} & A_{22} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ b_1 \end{bmatrix} w(t) + \begin{bmatrix} 0 \\ b_2 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ b_3 \end{bmatrix} & (5) \\ z(t) &= [c_1 \ \ 0]x(t) & (6) \\ z(t) &\ge 0, \quad w(t) \ge 0, \quad z(t)w(t) = 0 & (7) \end{aligned}$$

Note that we have nothing to do with the horizontal friction in this model, by virtue of the assumption (1). Eq. (5) involves the bias term consists of the gravity term and the natural length of spring. In the description above, Eq. (7) allows the following physical interpretation. When the robot is separated from the environment ($z(t) > 0$), the robot is free from constraint force from the environment ($w(t) = 0$). When the robot is in touch with the environment ($z(t) = 0$), the robot is affected by constraint force from the environment ($w(t) \ge 0$).

3 The Contact/Taking-off Condition

In view of the complementarity modeling, we can see that the jumping system has two modes; one is the *contact mode* in which the distance between the robot keeps contact with the environment ($z = 0$) and the constraint forces act on them ($w \ge 0$), and the other is the *non-contact mode* in which the distance is strictly positive ($z > 0$) and there arise no constraint forces ($w = 0$).

The above nature leads us to the following questions: (1)What kind of events cause the change of mode, or what is the condition for mode switching? (2)Under what condition the motion is determined uniquely? They are stated as *mode selection problem* in [11] and proven to be reduced to a series of Linear Complementarity Problems(LCP's) which are well developed in the field of mathematical programming.

As we are concerned in the taking-off motion of the robot from the environment, what should be investigated is the mode switching from the contact mode to the non-contact mode. In this section, a simple criterion is derived via the corresponding mode selection problem, which enables us to determine whether the robot in the contact mode keeps the mode or takes off from the environment.

When the robot is in touch with the environment, the following two conditions are satisfied:

$$\begin{aligned} [\, z_x, z_y \,]^T &= 0 & (8) \\ \frac{d}{dt}[\, z_x, z_y \,]^T &= \Phi_\xi \dot{\xi} = 0 & (9) \end{aligned}$$

Then, we should solve the mode selection problem using the following second order differential equation:

$$\begin{aligned} \frac{d^2}{dt^2}\left[\ z_x \quad z_y\ \right]^T &= \frac{d\Phi_\xi}{dt}\dot{\xi} + \Phi_\xi \ddot{\xi} \\ &= \frac{d\Phi_\xi}{dt}\dot{\xi} + \Phi_\xi M^{-1}(-G - C + u) + \Phi_\xi M^{-1} \Phi_\xi^T f & (10) \end{aligned}$$

As assumed above, $\ddot{z}_x(t) = 0$ is satisfied when $z_y = 0$. Hence, Set $\ddot{z}_x(t) = 0$, then $\ddot{z}_y$ can be expressed as an affine function of f_y, i.e.,

$$\ddot{z}_y = \beta(x, u) + \alpha(x) f_y \tag{11}$$

by eliminating f_x in Eq. (10).

For a given time τ_0, at which the system is in the contact mode, i.e., $z(\tau_0) = \dot{z}(\tau_0) = 0$, the mode selection problem is reduced to the following LCP.

[Scalar LCP]:

$$\begin{aligned} &given \ : \ \alpha(x(\tau_0)), \beta(x(\tau_0), u(\tau_0)) \\ &find \ : \ \ddot{z}(\tau_0), w(\tau_0) \\ &\quad s.t. \ \ \ddot{z}(\tau_0) = \beta(x(\tau_0), u(\tau_0)) + \alpha(x(\tau_0))w(\tau_0) \\ &\qquad \ddot{z}(\tau_0) \geq 0, \quad w(\tau_0) \geq 0, \quad \ddot{z}(\tau_0)w(\tau_0) = 0 \end{aligned}$$

Once we have the solutions $\ddot{z}(\tau_0)$ and $w(\tau_0)$, they would be the answer to the mode selection problem in the sense that they determine whether the contact mode is being kept ($\ddot{z} = 0$) or is being switched ($w = 0$). According to the criterion for the uniqueness of the LCP's solution [4], we can conclude that the mode selection problem has a unique solution if $\alpha > 0$. It is easily seen that this LCP always has a unique solution for the system (3), since

$$\Phi_\xi M^{-1} \Phi_\xi^T > 0$$

which implies $\alpha > 0$ holds.

Therefore, the solution of the mode selection problem is summarized as follows:

$$\begin{array}{llll} if & \beta(\tau_0) < 0 & then & \ddot{z}(\tau_0) = 0, \ w(\tau_0) = -\beta/\alpha \\ if & \beta(\tau_0) = 0 & then & \ddot{z}(\tau_0) = w(\tau_0) = 0 \\ if & \beta(\tau_0) > 0 & then & \ddot{z}(\tau_0) = \beta, \ w(\tau_0) = 0. \end{array}$$

Note that f_x is computed as

$$f_x = \begin{cases} \bar{f}_x & \text{i}f \ z_y = 0 \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

using z_y and f_y determined through this argument, where $\bar{f}_x$ is the solution of $\ddot{z}_x = 0$.

In order to investigate the taking-off motion rigorously, let t_0 and t_a be respectively the initial time and the taking-off time. Therefore the following conditions are imposed:

1. β should be kept nonpositive for $[t_0, t_a)$ to keep the contact with the environment, i.e.,
$$\beta(x(t), u(t)) \leq 0 \ ; \quad t \in [t_0, t_a) \tag{13}$$
2. β should be strictly positive at $t = t_a$ to take off from the environment, i.e.,
$$\beta(x(t_a), u(t_a)) > 0 \tag{14}$$

[Example] We consider the vertical linear jumping system (5)-(7). When the robot is in touch with the environment, the following two conditions are satisfied:

$$z(t) = [c_1 \ \ 0]x(t) = x_2(t) - x_e(t) = 0 \tag{15}$$

$$\dot{z}(t) = [c_1 \ \ 0]\dot{x}(t) = \dot{x}_2(t) - \dot{x}_e(t) = 0. \tag{16}$$

Hence, $\ddot{z}(t)$ can be represented by

$$\begin{aligned} \ddot{z}(t) &= [c_1 \ \ 0]\ddot{x}(t) \\ &= \{[c_1 A_{21} \ \ c_1 A_{22}]x(t) + c_1 b_2 u(t) + c_1 b_3\} + \{c_1 b_1\} w(t) \\ &= \beta + \alpha w(t) \qquad (17) \\ \alpha &:= c_1 b_1 \qquad (18) \\ \beta &:= [c_1 A_{21} \ \ c_1 A_{22}]x(t) + c_1 b_2 u(t) + c_1 b_3 \qquad (19) \end{aligned}$$

The mode selection problem is solved by applying the contact/taking-off condition given as Eq. (13) and Eq. (14).

4 Optimal Jumping Control

In the rest of the paper, we concentrate on a linear jumping system as shown in Fig.2. There are a lot of types of interesting control problems for the robot to jump. We are going to formulate an optimal jumping control problem to maximize the highest reachable point of the center of gravity of the robot under the control effort constraint $u_{min} \leq u(t) \leq u_{max}$ as one of them. The contact/taking-off condition obtained in the previous section will be taken into account in the formulation, and the optimality condition can be derived based on the maximum principle.

4.1 Problem Formulation

The height of the center of gravity of the robot is

$$x_g(t) = \frac{m_1 x_1(t) + m_2 x_2(t)}{m_1 + m_2}. \tag{20}$$

Let $x_g(t_f)$ be the highest reachable point of the center of gravity. Note that it only depends on the states at the moment of taking-off ($x_g(t_a)$ and $\dot{x}_g(t_a)$), since no external force affects the robot after that moment. Then, we can readily see that $x_g(t_f)$ is expressed as

$$x_g(t_f) = x_g(t_a) + \frac{\dot{x}_g^2(t_a)}{2g}, \tag{21}$$

if $\dot{x}_g(t_a) > 0$ holds. Hence, the highest jumping control can be formulated as an optimal control problem on the time interval $t_0 \leq t \leq t_a$ instead of the whole time interval $t_0 \leq t \leq t_f$.

When the robot is in touch with the environment, i.e., $x_e = x_2$ and $\dot{x}_e = \dot{x}_2$ hold, Fig.2 can be reduced to Fig.3.

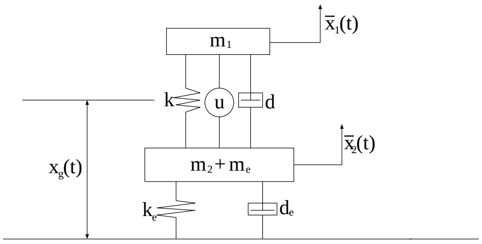

Fig.3 The jumping system when the robot is in touch with the environment

Let $\bar{x}_1(t)$ and $\bar{x}_2(t)$ be the displacements from the equilibrium points of k and k_e respectively and introduce a new state vector $x_c(t) = [\bar{x}_1 \ \ \bar{x}_2 \ \ \dot{\bar{x}}_1 \ \ \dot{\bar{x}}_2]^T$. Then, the state equation can be represented by

$$\dot{x}_c(t) = A_c x_c(t) + B_c u(t), \qquad x_c(t_0) = x_{c0} \tag{22}$$

where

$$\begin{aligned} A_c &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k}{m_1} & \frac{k}{m_1} & -\frac{d}{m_1} & \frac{d}{m_1} \\ \frac{k}{m_2+m_e} & -\frac{k+k_e}{m_2+me} & \frac{d}{m_2+me} & -\frac{d+d_e}{m_2+me} \end{bmatrix}, \\ B_c &= [0 \ \ 0 \ \ \tfrac{1}{m_1} \ \ -\tfrac{1}{m_2+m_e}]^T. \end{aligned}$$

Taking the conditions (13) and (14) into account, the input and the state are required to satisfy the following two constraints:

$$\beta(x_c(t), u(t)) \leq 0, \quad u_{min} \leq u(t) \leq u_{max}, \quad t \in [t_0, t_a). \tag{23}$$

The following constraints must be also satisfied at the time $t = t_a$:

$$\beta(x_c(t_a), u(t_a)) > 0, \quad u_{min} \leq u(t_a) \leq u_{max}. \tag{24}$$

Here, the reachable point of the center of gravity is expressed as

$$\begin{aligned} x_g(t_f) &= \frac{m_1 \bar{x}_1(t_a) + m_2 \bar{x}_2(t_a)}{m_1 + m_2} \\ &+ \frac{1}{2g}\left(\frac{m_1 \dot{\bar{x}}_1(t_a) + m_2 \dot{\bar{x}}_2(t_a)}{m_1 + m_2}\right)^2 + \mathrm{C}, \end{aligned} \tag{25}$$

where C is a constant determined from the physical parameters. As a summary, the problem of optimal high jump is formulated as follows:

[Optimal high jump control problem]:
Given a robot system described by (22). Find the optimal control input $\bar{u}(t)$ that maximizes the cost functional

$$J = \phi(x_c(t_a)) = p^T x_c(t_a) + x_c^T(t_a) P x_c(t_a), \tag{26}$$

under the constraints of the state and the control input (23) and (24), where

$$\begin{aligned} p^T &:= \left[\, \tfrac{m_1}{m_1+m_2} \quad \tfrac{m_2}{m_1+m_2} \quad 0 \quad 0 \,\right], \\ P &:= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{m_1^2}{2g(m_1+m_2)^2} & \frac{m_1 m_2}{2g(m_1+m_2)^2} \\ 0 & 0 & \frac{m_1 m_2}{2g(m_1+m_2)^2} & \frac{m_2^2}{2g(m_1+m_2)^2} \end{bmatrix}. \end{aligned}$$

4.2 Optimality Condition

The Hamiltonian H corresponding to the optimal high jump control problem is described as

$$H = \lambda^T (A_c x_c + B_c u), \tag{27}$$

where the vector λ associates with $x_c(t)$. According to the maximum principle [9], the optimality condition is given as follows:

$$\begin{aligned} \dot{\lambda} &= -H_{x_c}^T - \mu \beta_{x_c}^T && (28) \\ \mu &\geq 0, \quad \mu\beta = 0 \\ \dot{x}_c &= A_c x_c + B_c u && (29) \\ x_c(t_0) &= x_{c0} && (30) \\ \lambda^T(t_a) &= -[\phi_{x_c}]_{t=t_a}, && (31) \end{aligned}$$

where the suffix x_c denotes the partial differentiation by x_c. If $\beta = 0$, μ must satisfy

$$H_u + \mu \beta_u = 0, \tag{32}$$

where the suffix u denotes the partial differentiation by u. The function $H(x_c(t), u(t), \lambda(t))$ of the variable u attains its minimum at the point $u = \bar{u}(t)$ for all $t \in [t_0, t_a)$, namely,

$$H(x_c(t), \bar{u}(t), \lambda(t)) = \min_{u \,\in\, U} H(x_c(t), u, \lambda(t)), \tag{33}$$

where

$$U := \{u \in \mathbf{R} \mid \beta(x_c, u) \leq 0, \quad u_{min} \leq u \leq u_{max}\}. \tag{34}$$

Solving the differential equations (28), (29) and (32) under the boundary conditions (30) and (31) lead to the optimal solutions.

4.3 Numerical Solution

Now let us solve the optimal high-jump control problem numerically by applying the Sakawa-Shindo's algorithm [10].

Mechanical constants are: $m_1 = 4.0[kg]$, $m_2 = 1.0[kg]$, $m_e = 1.0[kg]$, $k = 20[N/m]$, $k_e = 40[N/m]$, $d = 1.0[Ns/m]$, $d_e = 1.0[Ns/m]$, $l_0 = 4.0[m]$ and $l_e = 5.0[m]$, where l_0 and l_e denote the natural lengths of k and k_e respectively. We set the taking-off time as $t_a = 4.0[s]$, the initial condition as $x_{c0} = 0$, and the input limitation as $u_{max} = 30[N]$ and $u_{min} = -30[N]$. The optimal control input of bang-bang type is obtained as shown in Fig.4,

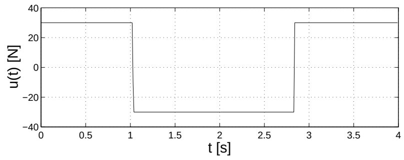

Fig.4 Optimal control input

Fig.5 illustrates the time response of β when the optimal control input is applied to the system (22). We can see from Fig.5 that $\beta(t)$ remains negative on the time interval $t_0 \leq t < t_a$, which implies that the robot is kept in touch with the environment on the interval as required. An input satisfying $\beta > 0$ at the time $t = t_a$ makes the robot separate from the environment. In this example, if we choose the input as $u(t_a) = -30[N]$, the system satisfies $\beta = 23.4 > 0$.

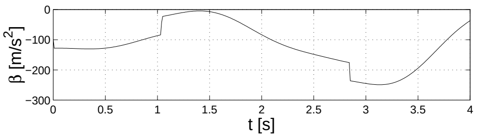

Fig.5 Response of β

Fig.6 shows the simulation results based on the above investigations. We chose the input as $u(t) = -30[N]$ after the time t_a. The highest reachable point of the center of gravity was $x_g(t_f) = 11.0[m]$ at the time $t_f = 4.83[s]$ as shown in the simulation result Fig.6.

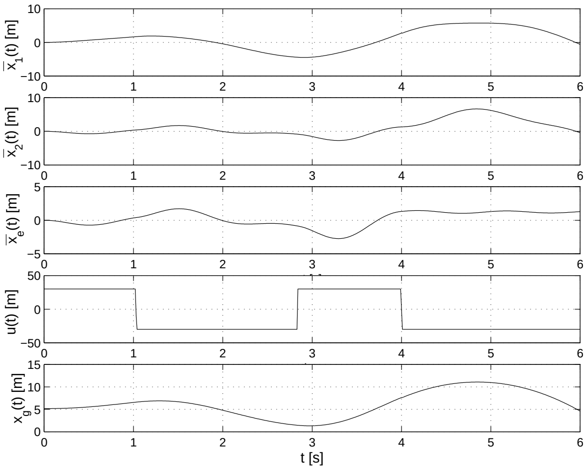

Fig.6 Simulation results

5 Analytic Solution

The numerical solution in the previous section is of open-form, and of course, depends on initial states. It is desirable to obtain a closed form analytic solution or an optimal switching rule.

From now on, we suppose that the environment is fixed as depicted in Fig. 7, so that the order of the system considered would be reduced to two for simplicity.

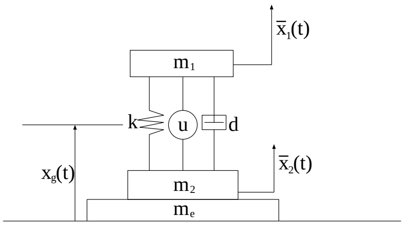

Fig.7 The jumping system with fixed environment

5.1 Optimal Input

Let us start with deriving the analytic condition for the co-state vector $\lambda(t)$. Choose a new state vector as $x_c(t) = [\bar{x}_1(t) \;\; \dot{\bar{x}}_1(t)]^T$ ($\bar{x}_2(t) = \dot{\bar{x}}_2(t) = 0$). The system state equation (22) is rewritten as

$$\begin{aligned} \dot{x}_c(t) &= A_c x_c(t) + B_c u(t) \\ &= \begin{bmatrix} 0 & 1 \\ -\frac{k}{m_1} & -\frac{d}{m_1} \end{bmatrix} x_c(t) + \begin{bmatrix} 0 \\ \frac{1}{m_1} \end{bmatrix} u(t). \end{aligned} \qquad (35)$$

Let $\lambda(t) := [\lambda_1(t) \;\; \lambda_2(t)]^T$, $\lambda_0 := [\lambda_{10} \;\; \lambda_{20}]^T := \lambda(t_0)$ and $\lambda_a := [\lambda_{1a} \;\; \lambda_{2a}]^T := \lambda(t_a)$. Suppose $t_0 = 0$ without loss of generality and the magnitude of the input is restricted as $|u(t)| \leq u_m$ where u_m is a positive constant.

Note that, in the definition of the Hamiltonian (27), the coefficient of the input u is $\lambda^T B_c$. Hence, according to Eq. (33), the optimal control input is determined by the sign of $\lambda^T B_c$. For example, the optimal control input for the system (35) is determined by the sign of λ_2 since $\lambda^T B_c = \frac{1}{m_1}\lambda_2$ where $m_1 > 0$. Assume that β is strictly negative for the sake of simplicity, the optimal control input $\bar{u}(t)$ can be obtained as of bang-bang type

$$\bar{u}(t) = -\mathrm{sgn}(\lambda_2(t))u_m. \qquad (36)$$

Since $\lambda(t)$ obeys the linear differential equation

$$\dot{\lambda}(t) = -A_c^T \lambda(t), \qquad (37)$$

so its solution $\lambda(t)$ is explicitly given by

$$\lambda(t) = e^{-A_c^T t}\lambda_0. \qquad (38)$$

The second element $\lambda_2(t)$ is derived from the following differential equation:

$$\ddot{\lambda}_2 - 2\zeta\omega\dot{\lambda}_2 + \omega^2\lambda_2 = 0 \qquad (39)$$
$$\left(2\zeta\omega = \frac{d}{m_1}, \quad \omega^2 = \frac{k}{m_1}\right)$$

Here, the nature of the solution of λ_2 depends on the sign of $\zeta^2 - 1$. Hence, we will separately investigate the two cases of $0 \leq \zeta \leq 1$ (under damping) and $\zeta > 1$ (over damping) for the derivation of input switching rule. We only give the result of under damping case for the shortage of paper, however, the other can can be dealt with in the same manner.

Let the real part of the solution of the characteristic equation for Eq. (39) be η_1 and the imaginary part be η_2, i.e.,

$$\eta_1 = \zeta\omega, \quad \eta_2 = \sqrt{1-\zeta^2}\omega. \qquad (40)$$

Then the solution of Eq. (39) is written as

$$\begin{aligned} \lambda_2(t) &= \sqrt{(\frac{\eta_1\lambda_{20} - \lambda_{10}}{\eta_2})^2 + \lambda_{20}^2} e^{\eta_1 t}\sin(\eta_2 t - \psi) \qquad (41) \\ \psi &= \sin^{-1}\{\frac{-\lambda_{20}}{\sqrt{(\{\eta_1\lambda_{20} - \lambda_{10})/\eta_2\}^2 + \lambda_{20}^2}}\} \\ & \qquad (0 \leq \psi < 2\pi). \end{aligned}$$

Remenber that the number of input switchings, say l, is determined by the $\mathrm{sgn}(\lambda_2)$ and let the input-switching times be $t_1, \ldots, t_l$. What can be seen is that the interval of respective input-switching times from t_1 to t_l is π/η_2. Hence, the *first* switching time t_1 can be expressed as

$$t_1 = \begin{cases} \psi/\eta_2 & (0 \leq \psi < \pi) \\ (\psi - \pi)/\eta_2 & (\pi \leq \psi < 2\pi) \end{cases} \qquad (42)$$

Note also that we should switch the control input every π/η_2 after the time t_1.

When the control input given by Eq. (36) is applied to the robot, analytical optimal trajectory is illustrated Fig.8.

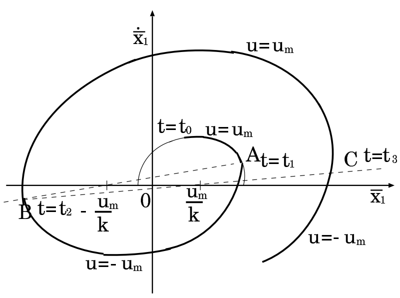

Fig.8 Analytical optimal trajectory

With given physical parameters and initial conditions, we must derive the initial input and the initial switching condition. As for λ_0 and λ_a, we have

$$\lambda_0 = e^{A_c^T t_a}\lambda_a \qquad (43)$$

where Eq. (31) gives us the following expression on λ_a:

$$\lambda_a^T = \begin{bmatrix} -\frac{m_1}{m_1+m_2} & -\frac{m_1^2}{g(m_1+m_2)^2}\dot{\bar{x}}_1(t_a) \end{bmatrix}. \qquad (44)$$

Let us proceed to eliminating $\dot{\bar{x}}_1(t_a)$ from (44). Note that $\dot{\bar{x}}_1(t_a)$ depends on the choice of the initial input $u(0)$ and ψ.

Since the interval of respective input-switching times from t_1 to t_l is π/η_2, the number of input switchings is restricted to two when physical parameters and the time t_a are given. Suppose the time t_a is in the interval

$$s\frac{\pi}{\eta_2} < t_a \leq (s+1)\frac{\pi}{\eta_2} \qquad (45)$$

where s is a non-negative integer. Then the number of input switchings l should be s or $s+1$, one of which is even and the other is odd. If the initial input $u(0)$ is positive, i.e., $u(0) = u_m$ and ψ is in $[0, \pi)$, the number of input switchings denoted by l_{even} is even. Then, assuming $d = 0$ for the sake of simplicity, $\dot{\bar{x}}_1(t_a)$ is given by

$$\begin{aligned} \dot{\bar{x}}_1(t_a) &= -2\frac{u_m}{k}l_{even}\omega\frac{\lambda_{2a}}{\sqrt{(\lambda_{1a}/\omega)^2 + \lambda_{2a}^2}} \\ & \quad -(\bar{x}_1(0) - \frac{u_m}{k})\omega\sin\omega t_a + \dot{\bar{x}}_1(0)\cos\omega t_a. \end{aligned} \qquad (46)$$

(we omit its derivation for lack of space). On the other hand, if $u(0) = -u_m$ and $\psi \in [\pi, 2\pi)$, then the number of input switchings denoted by l_{odd} is odd and $\dot{\bar{x}}_1(t_a)$ is given by

$$\begin{aligned}\dot{\bar{x}}_1(t_a) &= -2\frac{u_m}{k}l_{odd}\omega\frac{\lambda_{2a}}{\sqrt{(\lambda_{1a}/\omega)^2+\lambda_{2a}^2}} \\ &\quad -(\bar{x}_1(0)+\frac{u_m}{k})\omega\sin\omega t_a + \dot{\bar{x}}_1(0)\cos\omega t_a. \quad (47)\end{aligned}$$

In the rest cases, namely $u(0) = u_m$ and $\psi \in [\pi, 2\pi)$, or $u(0) = -u_m$ and $\psi \in [0, \pi)$, $\dot{x}_g(t_a)$ fails to be positive. These cases are therefore out of our consideration.

Substituting (46) or (47) into (44) yields a quartic equation of λ_{2a}. Solving the equation, we can calculate λ_{2a}. Substituting (41), (43) and (44) renders the optimal solution.

At the present stage, we have a bang-bang type optimal control input as a time series (36). Now let us turn to convert it to a switching rule dependent on the state variables in order to obtain a feedback type optimal switching rule. At the initial switching time t_1, $\bar{x}_1$ and $\dot{\bar{x}}_1$ satisfies the following condition:

$$\begin{aligned}S_1 &:= [\{\bar{x}_1(t_1)+\frac{u_m}{k}\}^2\omega + \frac{\dot{\bar{x}}_1^2(t_1)}{\omega}]^{-1} \\ &\quad [\dot{\bar{x}}_1(t_1)\{(\bar{x}_1(0)+\frac{u_m}{k})\cos\omega t_a + \frac{\dot{\bar{x}}_1(0)}{\omega}\sin\omega t_a\} \\ &\quad -\{\bar{x}_1(t_1)+\frac{u_m}{k}\}\{-(\bar{x}_1(0)+\frac{u_m}{k})\omega\sin\omega t_a \\ &\quad +\dot{\bar{x}}_1(0)\cos\omega t_a\}] - \sin\omega(t_a-t_1) = 0, \quad (48)\end{aligned}$$

if $\psi \in [0, \pi)$. (See the Appendix for the derivation). We omit the case $\psi \in [\pi, 2\pi)$ to avoid the redundancy. Since the initial switching time t_1 is expressed as Eq. (42), substituting (42) into (48) yields $S_1(\bar{x}_1, \dot{\bar{x}}_1) = 0$. What should be noted here is that the condition $S_1(\bar{x}_1, \dot{\bar{x}}_1) = 0$ depends only on the state variables $\bar{x}_1$ and $\dot{\bar{x}}_1$, it can be regarded as a *feedback switching rule.*

We can see that the next switching time t_2 is $t_1 + \frac{\pi}{\omega}$ from the periodic nature of $\lambda_2(t)$ in Eq. (41). Hence, the next switching rule $S_2(\bar{x}_1, \dot{\bar{x}}_1) = 0$ is obtained based on the similar consideration as above. The succeeding switching times $t_3, \cdots$ are also converted to state-based switching rules $S_i(\bar{x}_1, \dot{\bar{x}}_1) = 0$ by iterating this type of operation. As a consequence, the optimal input switching rule, for given taking-off time and input limitation, is derived as follows:

[Step 1] Solve the equation of λ_{2a} (Eq. (44) with Eq. (46) and Eq. (44) with Eq. (47)).

[Step 2] Select the initial control input u_m or $-u_m$ using the solution of Step 1.

[Step 3] Keep the initial control input until the state condition $S_1(\bar{x}_1, \dot{\bar{x}}_1) = 0$ is satisfied.

[Step 4] If the switching occurs, compute the next switching condition $S_2(\bar{x}_1, \dot{\bar{x}}_1) = 0$. Repeat this procedure until $t = t_a$.

[Step 5] At the time t_a, the robot jumps by switching the input to an input which holds $\beta > 0$.

Let us turn to see a simulation result. Physical consants are: m_1=3.0$[kg]$, $m_2 = 2.0[kg]$, $k = 16.0[N/m]$, d=0$[Ns/m]$, $x_{c0} = 0$, $t_a = 2.0[s]$, u_{max}= 40$[N]$, $u_{min} = -40[N]$. Fig.9 shows the optimal input switching condition $S_1(\bar{x}_1, \dot{\bar{x}}_1) = 0$ as an ellipse and the optimal trajectory of $\bar{x}_1(t), \dot{\bar{x}}_1(t)$.

The time response of β on the time interval $t_0 \leq t < t_a$ is shown in Fig.10. We can see from Fig.10 that β remains negative, which implies that the robot is kept in touch with the environment on the interval.

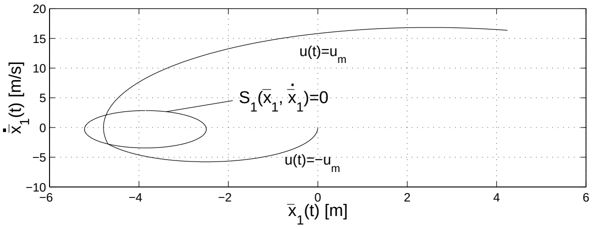

Fig.9 The input switching surface and the optimal trajectory

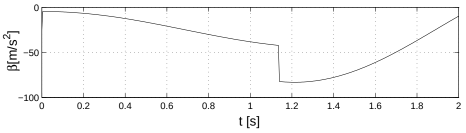

Fig.10 Response of β with the hard environment

6 Conclusion

In this paper, we proposed an optimal control strategy for a jumping system based on the complementarity modeling approach. We derived the criterion to judge the contact/taking-off condition at first, then formulated the optimal control problem for the jumping system to maximize the peak-height of a jump. Solving the problem numerically, we obtained the optimal bang-bang type controller under the limitation of input magnitude. Finally the optimal controller is analytically reconsidered and is implemented as a switching state feedback law.

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