A Computational Approach to Approximate Input/State Feedback Linearization

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Abstract

We propose a novel computational approach to the approximate input/state feedback linearization problem by interpolating a finite number of local linear coordinate transforms and static state feedback designs. For a class of single-input nonlinear systems, the approximate approach allows the main assumptions underlying exact input/state feedback linearization (involutivity, smoothness and controllability everywhere) to be relaxed. Moreover, the present approach relies only on simple numeric linear algebraic computations, in strong constrast to the exact input/state feedback linearization approach that relies on the solution of a partial differential equation and other symbolic compuations. In contrast to related approaches to approximation feedback linearization, the feedback design need not be restricted to a neighborhood of the equilibrium manifold. It is shown that the approximation error goes to zero uniformly as the resolution of the state space partitioning increases. Explicit expressions for the approximation error allows the accuracy and robustness of the design to be assessed.

1 Introduction

Here we consider the problem of non-linear controller design based on single-input state-space models of the form

$$\dot{x} = f(x) + g(x)u \tag{1}$$

where $x \in X \subset R^n$, $u \in R$, f and g are functions, and X is a compact set. In the standard approach to exact input/state feedback linearization [1, 2] one seeks a coordinate transform $\xi = T(x)$ and a static state feedback $u = \alpha(x, v)$, where $v \in R$ is an external input, such that the closed loop is rendered linear

$$\dot{\xi} = A\xi + Bv \tag{2}$$

and (A, B) is a given reachable system. In this work we will assume that (A, B) is given in the controller canonical form

$$A = \begin{pmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{3}$$

The system (1) is known to be exactly input/state feedback linearizable in a neighborhood of a state x_0 if and only if there exists an output function λ such that the system (1) with $y = \lambda(x)$ has exact relative degree n in the neighborhood of x_0, e.g. [1, 2]. In the exact input/state feedback linearization problem, it is required to determine such a λ in order to find the coordinate transform. However, finding λ is in general very difficult, since it corresponds to solving a set of partial differential equations. The purpose of the present work is to provide a much simpler computational approach that relies only on local linear approximation of the nonlinear function f and linear algebraic computations at the cost of an approximate solution. Various approaches to approximate feedback linearization are well known [3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. The present approach leads to a design based simpler computations and equal or less restrictive assumptions than the other approaches to exact or approximate feedback linearization known to the authors. In particular, like in exact feedback linearization but in constrast to all the above mentioned approaches to approximate feedback linearization, approximate linearization is achieved also at transient states far from the equilibrium manifold.

2 Feedback Design and Analysis

The basic idea is simple and can informally be stated as follows. At a finite number of fixed states we approximate the non-linear model (1) by local linear models, resulting in a number of linear local models. For each such local model we design a linear coordinate transform and a linear static feedback that transforms the local linear model into the desired linear system given by (A, B). Next, we define a semi-global feedback and coordinate transform by interpolating the local linear feedbacks and coordinate transforms.

2.1 Local Linear Models

The first step in the feedback design is to choose a set of points $D = \{x_1, x_2, ..., x_N\} \subset X \subset R^n$. Notice in particular that these points are not assumed to be equilibrium points for the system (1). We define

$$F_i = \frac{\partial f}{\partial x}(x_i), \qquad f_i = f(x_i) - F_i x_i, \quad G_i = g(x_i)$$

and assume that all the local linear systems (F_i, G_i) are reachable, for $i = 1, 2, ..., N$. This leads to the local linear model

$$\dot{x} = F_i x + G_i u + f_i \tag{4}$$

2.2 Local Feedback Design

Next, we transform the linear vector-field $F_i x + G_i u + f_i$ into the desired linear vector-field $A\xi + Bv$ using elementary linear algebraic methods.

We apply an linear and invertible local coordinate transform

$$\xi = T_i x + t_i \tag{5}$$

that transforms the local model (4) into the controller canonical form and cancels the constant term f_i as far as possible:

$$\begin{aligned} \dot{\xi} &= T_i F_i T_i^{-1}(\xi - t_i) + T_i G_i u + T_i f_i \\ &= T_i F_i T_i^{-1}\xi + T_i G_i u - T_i F_i T_i^{-1} t_i + T_i f_i \end{aligned}$$

In other words, we choose T_i such that $\tilde{F}_i = T_i F_i T_i^{-1}$, and $\tilde{G}_i = T_i G_i$ are in the canonical form

$$\tilde{F}_i = \begin{pmatrix} -\tilde{a}_{i,1} & -\tilde{a}_{i,2} & \cdots & -\tilde{a}_{i,n-1} & -\tilde{a}_{i,n} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

$$\tilde{G}_i = (1, 0, 0, ..., 0)^T$$

where the eigenvalues of F_i and $\tilde{F}_i$ must necessarily be the same. Defining the controllability matrices

$$\begin{aligned} W_{c,i} &= \left(G_i | F_i G_i | F_i^2 G_i | \cdots | F_i^{n-1} G_i\right) \\ \tilde{W}_{c,i} &= \left(\tilde{G}_i | \tilde{F}_i \tilde{G}_i | \tilde{F}_i^2 \tilde{G}_i | \cdots | \tilde{F}_i^{n-1} \tilde{G}_i\right) \end{aligned}$$

it is clear that $T_i = \tilde{W}_{c,i} W_{c,i}^{-1}$ is the desired matrix, see e.g. [2]. Moreover, we would like to choose t_i such that $\tilde{f}_i - \tilde{F}_i t_i$ is zero, where $\tilde{f}_i = T_i f_i$. However, it is immediately clear from the structure of $\tilde{F}_i$ that $\text{rank}(\tilde{F}_i) = n - 1$ when $\tilde{a}_{i,n} = 0$, while $\text{rank}(\tilde{F}_i) = n$ otherwise. When $\tilde{a}_{i,n} = 0$ we may choose

$$t_{i,1} = \tilde{f}_{i,2}, \quad ..., \quad t_{i,n-1} = \tilde{f}_{i,n}, \quad t_{i,n} = 0$$

which gives $\tilde{f}_i - \tilde{F}_i t_i = \tilde{\xi}_i$, where $\tilde{\xi}_i = (\tilde{\xi}_{i,1}, 0, 0, ..., 0)^T$ and $\tilde{\xi}_{i,1} = \tilde{f}_{i,1} + \tilde{f}_{i,2}\tilde{a}_{i,1} + ... + \tilde{f}_{i,n}\tilde{a}_{i,n-1}$. Otherwise, the modification

$$t_{i,n} = \left(\tilde{f}_{i,1} - \sum_{j=2}^{n} f_{i,j}\tilde{a}_{i,j-1}\right) / \tilde{a}_{i,n}$$

is feasible and leads to $\tilde{\xi}_i = 0$. Hence, the coordinate transform (5) will transform the local model (4) into

$$\dot{\xi} = \tilde{F}_i \xi + \tilde{G}_i u + \tilde{\xi}_i \tag{6}$$

Next, we define a local linear static state feedback

$$u = M_i \xi + l_i v + h_i \tag{7}$$

for the linear model (6), where l_i and h_i are scalars, and $M_i = (m_{i,1}, m_{i,2}, ..., m_{i,n})$. Substituting the feedback (7) into (6) gives the local closed loop dynamics

$$\dot{\xi} = \left(\tilde{F}_i + \tilde{G}_i M_i\right)\xi + \tilde{G}_i l_i v + \left(\tilde{G}_i h_i + \tilde{\xi}_i\right)$$

and from (3) we derive the design equations

$$\begin{aligned} h_i &= -\tilde{\xi}_{i,1} && (8) \\ l_i &= 1 && (9) \\ m_{i,j} &= \tilde{a}_{i,j} - a_j && (10) \end{aligned}$$

for $i = 1, 2, ..., N$ and $j = 1, 2, ...n$.

2.3 Interpolation functions

Having completed the local designs for all points in D, we proceed by designing interpolation functions for the set of local feedbacks and coordinate transforms. Clearly, each local design is valid in a neighborhood of a fixed state $x_i \in D$. To be able to guarantee that the interpolated local designs gives a close approximation to exact feedback linearization in X, one may assume that D covers X approximately uniformly, in the sense that for all $x \in X$ there is a $x_i \in D$ that is close in the Euclidean metric. The local feedbacks are given weight according to some interpolation-functions $w_1, ..., w_N : X \to R$. Assume the interpolation functions have compact support, and define

$$\delta = \max_{i \in \{1,2,...,N\}} \sup_{x \in \{x' | w_i(x') > 0\}} ||x - x_i||_2 \tag{11}$$

which can be interpreted as the granularity of the state space discretization. For any point $x \in X$, define the index set of neighbours in D as follows:

$$I(x) = \{i \in \{1, 2, ..., N\} \mid w_i(x) > 0\} \tag{12}$$

Assume the interpolation functions have the following properties for all $i = 1, 2, ..., N$:

$$\begin{aligned} w_i(x) &\geq 0, && \text{for all } x \in X && (13) \\ \sum_{j=1}^{N} w_j(x) &= 1, && \text{for all } x \in X && (14) \\ w_i(x) & && \text{is differentiable} && (15) \\ \delta &\to 0 && \text{as } N \to \infty && (16) \\ |I(x)| &\leq I_0, && \text{for all } N \geq 1 && (17) \\ \sup_{x \in X} \left|\left|\frac{dw_i(x)}{dx}\right|\right|_2 &\leq \frac{K_w}{\delta}, && \text{for all } N \geq 1 && (18) \end{aligned}$$

where K_w and I_0 are constants that are independent of N. Note that (16) and (17) are only possible if the interpolation functions have compact support. Hence, the semi-global approximately linearizing feedback and coordinate transform are $\alpha : X \times R \to R$

$$u = \alpha(x, v) = \sum_{i=1}^{N} \left(M_i T(x) + l_i v + h_i\right) w_i(x) \tag{19}$$

and $T : X \to T(X)$ defined by

$$\xi = T(x) = \sum_{i=1}^{N} (T_i x + t_i) w_i(x) \tag{20}$$

It is assumed that T is a bijective mapping. Notice that this fundamental assumption may restrict X to be a subset of the desired operating range of the system.

2.4 Approximation properties

The purpose of this section is to study the error of approximation, i.e. the deviation between the transformed system (1), (19) and (20) and the specified linear system (2). The desired dynamics are defined in transformed and original coordinates

$$\begin{aligned} \dot{\xi} &= \mathcal{D}_t(\xi, v) = A\xi + Bv \\ \dot{x} &= \mathcal{D}_o(x, v) = \frac{dT^{-1}}{d\xi}(T(x))\left(AT(x) + Bv\right) \end{aligned}$$

while the actual dynamics are in transformed and original coordinates

$$\begin{aligned}\dot{\xi} &= \mathcal{F}_t(\xi, v) = \frac{dT}{dx}(T^{-1}(\xi)) \\ &\quad \cdot \left(f(T^{-1}(\xi)) + g(x)\alpha(T^{-1}(\xi), v)\right) \\ \dot{x} &= \mathcal{F}_o(x, v) = f(x) + g(x)\alpha(x, v)\end{aligned}$$

When the approximately linearized global closed loop dynamics are written in transformed and original coordinates as

$$\dot{\xi} = \mathcal{F}_t(\xi, v) = \mathcal{D}_t(\xi, v) + \varepsilon(\xi, v) \quad (21)$$

$$\dot{x} = \mathcal{F}_o(x, v) = \mathcal{D}_o(x, v) + \epsilon(x, v) \quad (22)$$

it follows directly that the approximation error can be directly computed in transformed coordinates at all $\xi \in T(X)$ and $v \in R$

$$\varepsilon(\xi, v) = \mathcal{F}_t(\xi, v) - \mathcal{D}_t(\xi, v) \quad (23)$$

or equivalently in original coordinates for all $x \in X$ and $v \in R$

$$\epsilon(x, v) = \mathcal{F}_o(x, v) - \mathcal{D}_o(x, v) \quad (24)$$

Next, the main result in the paper shows that the approximation errors (23) and (24) goes to zero uniformly as $\delta \to 0$.

Theorem 1 *Suppose:*

1. *The functions f and g are Lipschitz on the compact set X and differentiable at all points in D.*
2. *The linear systems (A, B) and (F_i, G_i) are reachable for $i = 1, 2, ..., N$.*
3. *The parameters of (19) - (20) are selected according to the procedure in section 2.2.*
4. *The interpolation functions $w_1, ..., w_N$ satisfy the assumptions (13)-(18).*
5. *T is bijective.*

Then there exists positive constants c_0, c_1, c_2 and c_3 (independent of δ) such that

$$\sup_{\xi \in T(X), v \in R} ||\varepsilon(\xi, v)||_2 \leq c_0\delta + c_1\delta|v| \quad (25)$$

$$\sup_{x \in X, v \in R} ||\epsilon(x, v)||_2 \leq c_2\delta + c_3\delta|v| \quad (26)$$

The proof can be found in the appendix. Theorem 1 shows that approximate feedback linearization is achieved, and that the approximation error scales with the granularity of the state space discretization δ. The number of points in D should be sufficiently large and chosen such that the nonlinearities are captured in order to guarantee that ε and ϵ are small. Accurate transient response of the control system requires points x_i that are not equilibrium points to be included in D. As we have seen, this does not complicate the design of the local state feedbacks and transforms.

2.5 Outer control loop

Suppose the control objective is to make the scalar output $y = h(x)$ track a given reference $y^\star$ while rejecting noise and disturbances. The approximate feedback linearization technique described above transforms the system into an approximately linear system $\dot{\xi} = A\xi + Bv + \varepsilon(\xi, v)$ where ε is due to modelling error, approximation error, disturbances and noise. In order to meet this control objective, the output y must be related to the non-physical state-vector ξ, and an outer control loop must be designed for this system to guarantee low sensitivity with respect to ε. Since the uncertainty $\varepsilon(\xi, v)$ is in the form (26), standard stability theory for perturbed systems [13] can be applied to address the robust design of the outer feedback loop.

Tracking is achieved by a static feedforward from the reference. For a given $y^\star$ we define a corresponding equilibrium state $x^\star$ and control input $u^\star$ according to

$$0 = f(x^\star) + g(x^\star)u^\star, \qquad y^\star = h(x^\star)$$

We must assume $y^\star$ is feasible, i.e. this set of $n + 1$ equations in $n + 1$ variables has at least one solution $x^\star(y^\star)$ and $u^\star(y^\star)$. Now we get from (19) and (9) that a suitable static feedforward is

$$v^\star(y^\star) = u^\star(y^\star) - \sum_{i=1}^{N} (M_i T(x^\star(y^\star)) + h_i) w_i(x^\star(y^\star))$$

In addition, to meet the control objectives related to rejection of disturbances and noise, we may add an outer feedback loop, for example a static state feedback

$$v = v^\star(y^\star) + K(\xi - T(x^\star(y^\star))) \quad (27)$$

2.6 On-line Design

Above we have computed local linear approximations to the system and designed local linear state feedbacks and coordinate transform off-line at a finite number of states. An alternative approach is to do the design on-line at the current state for each sample instant. The same compuational procedures as described in section 2.2 are applicable. The main difference is that the current state $x(t)$ should be substituted for x_i in the equations. Note that there is no loss of acccuracy due to interpolation. However, as in exact feedback linearization it must be assumed that the system is controllable and f is differentiable at every state, such that local control design can be performed at any feasible state.

3 Examples

3.1 Example: Non-smooth system

Consider the non-smooth non-linear system

$$\dot{x}_1 = -\text{sat}(x_1) + x_2^2/2 + u \quad (28)$$

$$\dot{x}_2 = |x_1| - \exp(x_2/3) \quad (29)$$

where

$$\text{sat}(x_1) = \begin{cases} 1, & x_1 > 1 \\ x_1, & |x_1| \leq 1 \\ -1, & x_1 < -1 \end{cases} \quad (30)$$

Since it is non-smooth, it is not exactly feedback linearizable. We consider a design with a uniform grid of design points (272 points). Linearization and design at the "singular points" where the derivative does not exist is avoided.

Figure 1 illustrates the vector field for the approximately linearized dynamics in the original coordinates for $v = 0$, Parts 1 and 2 show the two components of the vector field $\mathcal{F}_o(\cdot, 0)$ in original coordinates. Parts 3 and 4 illustrates the two components of the approximation error vector field $\epsilon(\cdot, 0)$ in original coordinates. Visual comparison of the vector fields suggest that close approximation was achieved.

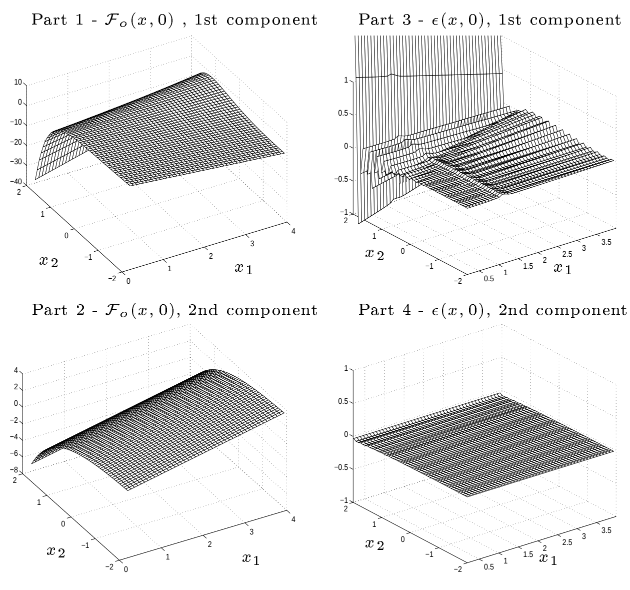

Figure 1: Vector fields of non-smooth system.

3.2 Example: Inverted pendulum on a cart

Consider the following model of a invereted pendulum on a cart ([2], pp. 85)

$$\dot{x} = \nu \tag{31}$$

$$\dot{\nu} = u \tag{32}$$

$$\dot{\phi} = \omega \tag{33}$$

$$\dot{\omega} = 9.8 \sin(\phi) - cos(\phi) u \tag{34}$$

where x is the cart position, ν is the cart velocity, ϕ is the angle of the pendulum and ω is the angular velocity of the pendulum. It is known that this system is not exactly feedback linearizable, [2]. The control problem is to control that state to equilibrium setpoints $(x^\star, 0, 0, 0)$. The approximate state-feedback linearization approach has been applied to this problem using on-line design as outlined in section 2.6. The desired linear dynamics are defined by

$$A = \begin{pmatrix} -11 & -44 & -76 & -48 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{35}$$

which has eigenvalues at $-2, -2, -3$ and -4. An outer feedback loop and feedforward is designed as described in section 2.5. In particular, we choose the poles of the closed loop at $-2, -3, -4$ and -5, which leads to $K = (-3, \ -27, \ -78, \ -72)$. Figure 2 illustrates tracking performance in the original state space with coordinates x. We observe that stable asymptotic tracking of the reference is achieved.

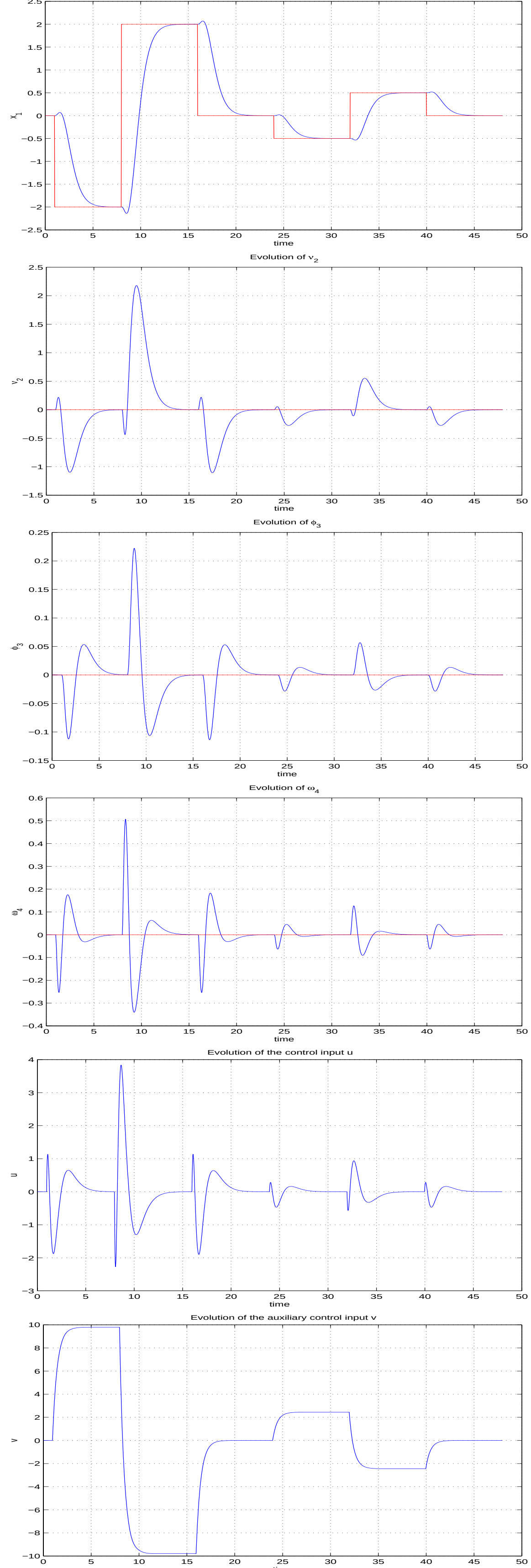

Figure 2: Inverted pendulum: The tracking of a reference trajectory is illustrated.

Appendix A

Definition. A function $\varphi(x)$ is said to be $\mathcal{O}_X(\delta^n)$ for some integer $n \geq 1$ if there exists a constant $K > 0$ such that

$$\sup_{x\in X} ||\varphi(x)/\delta^n||_2 \quad \leq \quad K \tag{36}$$

A function $\varphi(x,v)$ is said to be $\mathcal{O}_X(\delta^n, v)$ for some integer $n \geq 1$ if there exists constants $K, \kappa > 0$ such that

$$\sup_{x\in X} ||\varphi(x,v)/\delta^n||_2 \quad \leq \quad K\delta + \kappa\delta||v||_2 \tag{37}$$

□

Proof of Theorem 1: The time derivative of ξ is

$$\dot{\xi} \quad = \quad \frac{dT}{dx}(x)\dot{x} \; = \; \left(\sum_{l=1}^N T_l w_l(x)\right)\dot{x} + \varepsilon_T(x,v) \tag{38}$$

where

$$\varepsilon_T(x,v) \quad = \quad \left(\sum_{i=1}^N (T_i x + t_i)\frac{dw_i^T}{dx}(x)\right)(f(x) + g(x)\alpha(x,v))$$

Lemma 2 shows that ε_T scales proportionally to δ such that

$$\dot{\xi} \quad = \quad \left(\sum_{j=1}^N (F_j x + f_j + G_j u + \varepsilon_{f,j}(x,v))w_j(x)\right) \cdot \left(\sum_{l=1}^N T_l w_l(x)\right) + \mathcal{O}_X(\delta, v)$$

and

$$\varepsilon_{f,j}(x,v) \quad = \quad f(x) - F_j x - f_j + (g(x) - G_j)\alpha(x,v)$$

It follows from Lemmas 1, 3 and 6 that

$$\dot{\xi} \quad = \quad \sum_{l=1}^N\sum_{j=1}^N (T_l F_j x + T_l f_j + T_l G_j u)\, w_j(x) w_l(x) + \mathcal{O}_X(\delta, v)$$

Next,

$$\dot{\xi} \quad = \quad \sum_{l=1}^N\sum_{j=1}^N \left(T_l F_j \sum_{i=1}^N \left(T_i^{-1}(\xi - t_i)w_i(x) + \varepsilon_{T^{-1}}(x)\right) + T_l f_j + T_l G_j u\right) w_j(x) w_l(x) + \mathcal{O}_X(\delta, v)$$

and from Lemmas 3 and 4 it is clear that

$$\dot{\xi} \quad = \quad \sum_{l=1}^N\sum_{j=1}^N\sum_{i=1}^N \left(T_l F_j T_i^{-1}(\xi - t_i) + T_l f_j + T_l G_j u\right) \cdot w_j(x) w_l(x) w_i(x) + \mathcal{O}_X(\delta, v) \tag{39}$$

Lemma 1 can be used to establish that

$$\sum_{l=1}^N\sum_{i=1}^N\sum_{j=1}^N T_l F_j T_i^{-1} w_i(x) w_j(x) w_l(x) \quad = \quad \sum_{j=1}^N T_j F_j T_j^{-1} w_j(x) + \mathcal{O}_X(\delta, v)$$

and using similar arguments on the terms $T_l f_j$, $T_l G_j$, and $T_l F_j T_i^{-1} t_i$ we arrive at

$$\dot{\xi} \quad = \quad \sum_{j=1}^N \left(\tilde{F}_j(\xi - t_j) + \xi_j + \tilde{G}_j u\right) w_j(x) + \mathcal{O}_X(\delta, v)$$

Substituting the feedback (19) into the above equation gives

$$\dot{\xi} \quad = \quad \sum_{i=1}^N\sum_{j=1}^N \left(\left(\tilde{F}_j + \tilde{G}_j M_i\right)\xi + \left(\xi_j + \tilde{G}_j h_i\right) + \tilde{G}_j l_i v\right) \cdot w_i(x) w_j(x) + \mathcal{O}_X(\delta, v)$$

Using Lemma 1 and similar arguments as above,

$$\dot{\xi} \quad = \quad \sum_{j=1}^N \left(\left(\tilde{F}_j + \tilde{G}_j M_j\right)\xi + \left(\xi_j + \tilde{G}_j h_j\right) + \tilde{G}_j l_j v\right) w_j(x) + \mathcal{O}_X(\delta, v)$$

From the design equations (8)-(10), we get

$$\dot{\xi} \quad = \quad A\xi + Bv + \mathcal{O}_X(\delta, v) \tag{40}$$

Thus, $\varepsilon(\xi, v) = \mathcal{O}_{T(X)}(\delta, v)$ and from Lemma 6 we know that $T(X)$ is compact, and T is a bounded mapping. It follows that

$$\epsilon(x,v) \quad = \quad \varepsilon(T^{-1}(x), v) \; = \; \mathcal{O}_X(\delta, v) \tag{41}$$

which completes the proof.

Lemma 1 *For all* $i \in \{1,2,...,N\}$, $||F_i||, ||x_i||_2, ||T_i||, ||t_i||_2, ||T_i^{-1}||, ||f_i||_2, ||G_i||_2, ||M_i||_2, |h_i|$, *and* $||\xi_i||_2$ *are bounded. Moreover, for all* $i,j \in \{1,2,...,N\}$, *all the following functions are* $\mathcal{O}_X(\delta)$: $||F_i - F_j||w_i(x)w_j(x)$, $||x_i - x_j||_2 w_i(x)w_j(x)$, $||T_i - T_j||w_i(x)w_j(x)$, $||t_i - t_j||_2 w_i(x)w_j(x)$, $||f_i - f_j||_2 w_i(x)w_j(x)$, $||M_i - M_j||_2 w_i(x)w_j(x)$, $||\xi_i - \xi_j||_2 w_i(x)w_j(x)$, $||G_i - G_j||_2 w_i(x)w_j(x)$ *and* $|h_i - h_j|w_i(x)w_j(x)$.

Proof. The first statement follows from the compactness of X and since f is Lipschitz. Note that the existence of T_i^{-1} is implied by the reachability assumption. Next, we observe that $||x_i - x_j||_2 \leq 2\delta$ for all i,j such that $w_i(x)w_j(x) > 0$. This means that there exists constants K_1, K_2 and K_3 such that $||F_i - F_j||w_i(x)w_j(x) \leq K_1\delta$, $||G_i - G_j||_2 w_i(x)w_j(x) \leq K_2\delta$ and $||f_i - f_j||w_i(x)w_j(x) \leq K_3\delta$ since the functions f and g are Lipschitz. Note that the parameters of T_i, t_i, M_i and h_i depend continuously on the parameters of f_i, F_i and x_i. Hence, from (8)-(10) it follows that $||M_i - M_j||$ and $||h_i - h_j||_2$ satisfy the same type of bounds.

Lemma 2 $\varepsilon_T(x,v) = \mathcal{O}_X(\delta, v)$

Proof. Let $\tilde{t} : X \to R^n$ be a vector field that is differentiable twice and satisfies

$$\frac{d\tilde{t}}{dx}(x_i) = T_i, \quad \text{and} \quad \tilde{t}(x_i) = t_i + T_i x_i \tag{42}$$

for all $i = 1,2,...,N$. Such a function exists since it might be constructed using multivariate polynomials of sufficiently high order. Thus, by Taylor series expansion

$$\tilde{t}(x) \quad = \quad \tilde{t}(x_i) + \frac{d\tilde{t}}{dx}(x_i)(x - x_i) + \mathcal{O}_X(||x - x_i||_2^2) \tag{43}$$

$$\quad = \quad t_i + T_i x_i + T_i(x - x_i) + \mathcal{O}_X(||x - x_i||_2^2) \tag{44}$$

$$\quad = \quad T_i x + t_i + \mathcal{O}_X(||x - x_i||_2^2) \tag{45}$$

and

$$\varepsilon_T(x,v) = \sum_{i=1}^{N} \left(\tilde{t}(x) + \mathcal{O}_X(||x-x_i||_2^2)\right) \frac{dw_i^T}{dx}(x) \cdot (f(x) + g(x)\alpha(x,v))$$

Recall that (14) implies

$$\sum_{i=1}^{N} \frac{dw_i}{dx}(x) = 0, \qquad \text{for all } x \in X \tag{46}$$

such that

$$\varepsilon_T(x,v) = \sum_{i \in I(x)} \mathcal{O}_X(||x-x_i||_2^2) \frac{dw_i^T}{dx}(x) \cdot (f(x) + g(x)\alpha(x,v))$$

Consequently,

$$||\varepsilon_T(x,v)||_2 \leq \mathcal{O}_X(\delta^2) \cdot |I(x)| \cdot \max_{i=1,2,\dots N} \left|\left| \frac{dw_i}{dx}(x) \right|\right|_2 \cdot ||f(x) + g(x)\alpha(x,v)||_2$$

and the theorem follows from Lemma 5 and (13)-(18).

Lemma 3 *There exists constants K_0 and K_1 such that*

$$\sup_{x \in \{x' \ | \ w_i(x') > 0\}} ||\varepsilon_{f,j}(x,v)||_2 \leq K_0\delta + K_1\delta|v| \tag{47}$$

Proof. The result follows by application of Taylor's theorem at x_i (note that f and g are Lipschitz, and differentiable at x_i) and Lemma 5.

Lemma 4 *$x = \sum_{i=1}^{N} T_i^{-1}(\xi - t_i)w_i(x) + \varepsilon_{T^{-1}}(x)$, where $\varepsilon_{T^{-1}}(x) = \mathcal{O}(\delta)$.*

Proof. The result follows from Lemma 1, since

$$\begin{aligned} \varepsilon_{T^{-1}}(x) &= x - \sum_{i=1}^{N} T_i^{-1} \left(\sum_{k=1}^{N} (T_k x + t_k) w_k(x) - t_i \right) w_i(x) \\ &= \sum_{i=1}^{N} \sum_{k=1}^{N} \left(x - T_i^{-1} T_k x + t_k - t_i \right) w_i(x) w_k(x) \\ &= \sum_{i=1}^{N} \sum_{k=1}^{N} \left(T_i^{-1}(T_i - T_k)x + (t_k - t_i) \right) w_i(x) w_k(x) \end{aligned}$$

Lemma 5 *There exist constants K_5 and K_6 such that*

$$\sup_{x \in X} ||f(x) + g(x)\alpha(x,v)||_2 \leq K_5 + K_6|v| \tag{48}$$

Proof. Since f is Lipschitz (assume the Lipschitz constant is L), for arbitrary $x' \in X$

$$||f(x)||_2 \leq ||f(x')||_2 + L||x||_2 + L||x'||_2 \tag{49}$$

From the triangle inequality we get

$$||\alpha(x,v)||_2 \leq \sum_{i=1}^{N} (||M_i|| \cdot ||T(x)|| + ||l_i||_2|v| + ||h_i||_2)\, w_i(x)$$

Using Lemmas 1 and 6, the result follows.

Lemma 6 *The mappings T and T^{-1} are bounded, i.e. there exists constant K_7, K_8, K_9 and K_{10} such that*

$$\sup_{x \in X} ||T(x)|| \leq K_7 + K_8||x||_2 \tag{50}$$

$$\sup_{\xi \in T(X)} ||T^{-1}(\xi)|| \leq K_9 + K_{10}||\xi||_2 \tag{51}$$

Moreover, $T(X)$ is a compact set.

Proof. Boundedness of T follows from Lemma 1. Boundedness of T^{-1} then follows from Lemma 4. Compactness of $T(X)$ follows directly from the compactness of X and the boundedness of T.

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