Fast Algorithms for Exact and Approximate Feasibility of Robust LMIs[1]

GIUSEPPE CALAFIORE[2] AND BORIS POLYAK[3]

Abstract

In this paper, we discuss fast randomized algorithms for determining an admissible solution for robust linear matrix inequalities (LMIs) of the form $F(x,\Delta) \preceq 0$, where x is the optimization variable and Δ is the uncertainty, which belongs to a given set $\mathbf{\Delta}$. The proposed algorithm is based on uncertainty randomization: it finds a solution in a finite number of iterations with probability one, if a strong feasibility condition holds. Otherwise, it computes a candidate solution which minimizes the expected value of a suitably selected feasibility indicator function. The theory is illustrated by examples of application to uncertain linear inequalities and quadratic stability of interval matrices.

1 Introduction

In this paper, we discuss the problem of determining feasible or approximately feasible solutions to robust Linear Matrix Inequality (LMI) constraints, using fast iterative methods based on uncertainty randomization. The definitions of feasible and approximately feasible solutions will be detailed in the sequel.

Robust LMI constraints arise naturally from standard LMI problems, when uncertainty is present in the data matrices. More formally, let $\mathbb{S}$ be a prescribed subspace of $\mathbb{R}^{p,q}$, which accounts for structure in the uncertainty (e.g. block-diagonal), then we shall consider structured norm-bounded uncertainty

$$\mathbf{\Delta} \equiv \mathbf{\Delta}_\rho \doteq \{\Delta \in \mathbb{S} : \|\Delta\| \leq \rho\}, \tag{1}$$

where $\|\cdot\|$ denotes any matrix norm, and ρ denotes the uncertainty radius. Alternatively, the uncertainty may be described by a set of finite cardinality N_v, i.e.

$$\mathbf{\Delta} \equiv \mathbf{\Delta_f} = \{\Delta_1, ..., \Delta_{N_v}\}. \tag{2}$$

A robust LMI is defined as the matrix inequality

$$F(x,\Delta) \preceq 0; \;\; \forall \Delta \in \mathbf{\Delta}, \;\; x \in \mathbb{R}^m, \tag{3}$$

where

$$F(x,\Delta) = F_0(\Delta) + \sum_{i=1}^{m} x_i F_i(\Delta), \tag{4}$$

and where $F_i(\Delta) = F_i^T(\Delta) \in \mathbb{R}^{n,n}$ are functions of the uncertainty Δ. Let $\mathcal{X} \subseteq \mathbb{R}^m$ be a (non-empty) convex and closed set, then the robust feasibility problem is posed as

$$\text{Find } x \in \mathcal{X} \text{ such that } F(x,\Delta) \preceq 0, \;\; \forall \Delta \in \mathbf{\Delta}. \tag{5}$$

A vector $\tilde{x}$ that satisfies (3) is said a *robustly feasible* solution.

To motivate our developments, we notice that the constraint (3) is a convex constraint on x, but the number N_v of such constraints, in the case of discrete uncertainty (2), or the number of vertices in the case (1) with a polytopic matrix norm, can be extremely large and determining a robustly feasible solution is in general NP-hard, [9]. When the uncertainty Δ enters $F(x,\Delta)$ in a linear fractional form (LFT), the robust feasibility problem may be dealt with using the recently emerged techniques of robust semidefinite programming, [4, 9]. Even in the special (albeit quite general) case of LFT uncertainty, robust SDP techniques provide in general only sufficient conditions for robust feasibility. Also, these techniques transform the original uncertain problem in a convex problem of much larger size. When uncertainty is present, even problems that are originally of small size result in convex optimization problems that push to the limits the existing SDP solver codes. As an example, the problem of assessing quadratic stability for an interval matrix of dimension n, may be cast as a problem of finding a common solution to $N_v = 2^{n^2}$ Lyapunov inequalities for the vertex matrices, [6]. For $n = 4$, this results in an LMI problem involving a symmetric matrix of dimension $M \times M$, with $M = 4 \times 2^{16} = 262,144$.

Another problem related to (5), is that there may actually exist no robustly feasible solution. In this case, it may be of interest to determine a solution that satisfies the constraints in some averaged sense, to be more clearly defined later. A similar approach for approximate feasibility has recently been proposed by Barmish and Scherbakov in [3].

In this paper, we present simple and efficient algorithms for the computation of feasible or approximately feasible solutions for robust LMIs. The proposed algorithms

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[1]Part of this work was carried on while B. T. Polyak was visiting IRITI-CNR supported by a NATO-CNR Guest Fellowship.

[2]Dipartimento di Automatica e Informatica, Politecnico di Torino – Corso Duca degli Abruzzi, 24; 10129 Torino, Italy. e-mail: calafiore@polito.it.

[3]Institute for Control Science, Russian Academy of Sciences – Moscow 117806, Russia. e-mail: boris@ipu.rssi.ru

are devised to address the previously mentioned drawbacks, i.e. conservatism of the solutions based on Lagrangian relaxations, and practical intractability due to the dimension of the resulting problem. They also provide approximately feasible solutions, in case no robustly feasible solution exists. Also, a feature of the presented theory is the absence of assumptions on the way the uncertainty enters into $F(x, \Delta)$. This framework may therefore accommodate the cases in which the uncertainty cannot be described using the LFT formalism. The price to pay for the fore-mentioned enhancements is that the proposed algorithms are based on randomization in the space of the uncertainty, therefore their solution and convergence properties can only be assessed in a probabilistic sense.

The algorithms proposed here are based on an adaptation of stochastic gradient methods. The first algorithm is aimed to the solution of unfeasible inequalities, while the second one is appropriate for the feasible case. A deterministic counterpart of the second method can be traced back to Kaczmarz method for solving linear equations [11], and to Agmon-Motzkin-Shoenberg method for solving linear inequalities, [1], [12]. For non-linear convex inequalities, it has been proposed in [15]. A version with finite convergence for linear inequalities was used by V. Yakubovich in the middle of the sixties for solving adaptive control problems (see [5] and references therein) and extended to nonlinear inequalities by V. Fomin [10]. The method has some common features with the method of alternating projections [17], but an iteration of our method is *not* a projection on the corresponding inequality. The randomized counterpart of these algorithms, and their adaptation to robust LMIs are new, to the best of the authors' knowledge.

Related algorithms, oriented to robust LQR design problems, are proposed in [16], while applications of stochastic gradient algorithms to set-membership identification problems have been recently proposed in [8].

2 Preliminaries

We will assume that the uncertainty Δ is a random matrix, with given probability distribution f_Δ over its support set $\mathbf{\Delta}$, and that it is possible to generate samples of Δ according to this probability distribution. We will not discuss here issues related to the choice of the probability distribution f_Δ. A natural choice (which also has theoretical rationale behind, see [2]) is to assume uniform distribution in the space of Δ. In this case, efficient algorithms for the generation of samples on $\mathbf{\Delta}$ are discussed in depth in [7].

We introduce a scalar function $\varphi(x, \Delta)$, defined as

$$\varphi(x, \Delta) = ||F_+(x, \Delta)||, \tag{6}$$

where the notation A_+ indicates the projection of the matrix A onto the cone of positive semidefinite matrices, see more details below. This function will be called a *feasibility indicator function* (FIF) in the sequel. It follows from the definition that $\varphi(x, \Delta) > 0$ if and only if $F(x, \Delta) \not\preceq 0$, and it is zero otherwise. With these premises, a vector $\tilde{x} \in \mathbb{R}^m$ is robustly feasible for (3) iff it satisfies $\varphi(\tilde{x}, \Delta) \leq 0$, for all $\Delta \in \mathbf{\Delta}$. If there exist no robustly feasible point, we say that $\tilde{x}$ is an *approximately feasible* solution for (5) if

$$\tilde{x} = \arg\min_{x \in \mathcal{X}} E\{\varphi(x, \Delta)\}, \tag{7}$$

where E denotes expectation with respect to Δ. In this case, the proposed solution takes into account uncertainty by minimizing the expectation over Δ of the constraint violation.

We introduce the notation $[\xi]_\mathcal{X}$ to denote the projection of element ξ onto $\mathcal{X}$, i.e.

$$||\xi - [\xi]_\mathcal{X}|| = \min_{y \in \mathcal{X}} ||\xi - y||.$$

Let $\mathcal{S}^n$ denote the Hilbert space of $n \times n$ real symmetric matrices, equipped with the Frobenius norm and the scalar product $< A, B >= \mathbf{Tr}\, AB$. Let $\mathcal{C}$ be the cone of positive semidefinite matrices, $\mathcal{C} = \{A \in \mathcal{S}^n : \ A \succeq 0\}$. The following lemma will be used in the sequel.

Lemma 1 *Let A_+ be the projection of A onto the cone $\mathcal{C}$, then the following conditions are equivalent*

1. *$A_+ = [A]_\mathcal{C}$, i.e. A_+ is a projection of A onto $\mathcal{C}$;*
2. *$A = A_+ + A_-$, where $A_+ \in \mathcal{C}$, $A_- \in -\mathcal{C}$, and $< A_+, A_- >= 0$;*
3. *$A_+ = RD_+R^T$, where R and $D = \mathbf{diag}\,(d_1, \ldots, d_n)$ are the eigenvector and eigenvalues matrices of A, respectively, that is $A = RDR^T$, and $D_+ = \mathbf{diag}\,(d_1^+, \ldots, d_n^+)$, being $d_i^+ = \max(d_i, 0)$.*

A proof of this lemma may be found in [14]. We remark that the assertion 3. provides an efficient method for explicitly computing the projection A_+.

We recall that a function $f : \mathbb{R}^m \to \mathcal{S}^n$ is convex (in matrix sense) if for any $x_1, x_2 \in \mathbb{R}^m$,

$$f(tx_1 + (1-t)x_2) \preceq tf(x_1) + (1-t)f(x_2), \quad \text{for } 0 \leq t \leq 1.$$

The following lemmas, whose simple proofs are omitted for brevity, hold.

Lemma 2 *The function $F_+(x, \Delta)$ is convex in x, in the above sense.*

Lemma 3 *If a function $f : \mathbb{R}^m \to \mathcal{C}$ is convex, then the function $g : \mathcal{C} \to \mathbb{R}$, $g(x) = \|f(x)\|$ is also convex.*

As an immediate corollary, we have

Lemma 4 *The function* $\varphi(x,\Delta) = \|F_+(x,\Delta)\|$, *where* $F(x,\Delta)$ *is defined in (4), is convex in* x.

The subgradient of this function can be found as follows.

Lemma 5 *Let* $\varphi(x,\Delta)$ *be defined as in (6), where* $F(x,\Delta)$ *is defined in (4), and let*

$$\nabla\varphi(x,\Delta) \doteq \frac{1}{\varphi(x,\Delta)} \left[\begin{array}{c} \mathbf{Tr} F_1 F_+(x,\Delta) \\ \vdots \\ \mathbf{Tr} F_m F_+(x,\Delta) \end{array} \right],$$

then

$$\partial_x \varphi(x,\Delta) = \left\{ \begin{array}{ll} \nabla\varphi(x,\Delta) & \text{if } \varphi(x,\Delta) \neq 0; \\ 0 & \text{otherwise} \end{array} \right.$$

is a subgradient of $\varphi(x,\Delta)$ *at* x.

3 Algorithm for Approximate Feasibility

In this section, we propose an iterative randomized algorithm for determining an approximately feasible solution for (5). The main result is presented in the following theorem.

Theorem 1 *Consider the robust feasibility problem (5). Let* Δ *be a random matrix with given probability distribution* f_Δ *over the support* $\mathbf{\Delta}$. *Let* $\varphi(x,\Delta)$ *be a feasibility indicator function, and let*

$$f(x) \doteq E\{\varphi(x,\Delta)\},$$

where E *denotes expectation with respect to* Δ. *Suppose further that* $\mathcal{X}$ *is a convex and closed set,* $f(x)$ *has a minimum point* $\tilde{x}$ *on* $\mathcal{X}$, *and* $\|\partial_x\{\varphi(x,\Delta)\}\| \leq \mu$, *for all* $x \in \mathcal{X}$, $\Delta \in \mathbf{\Delta}$.

Given an initial vector x_0, *consider the recursion*

$$x_{k+1} = [x_k - \lambda_k \partial_x\{\varphi(x_k,\Delta^k)\}]_{\mathcal{X}}, \qquad (8)$$

where ∂_x *denotes a subgradient with respect to* x, *and* Δ^k, $k = 0,1,\ldots$ *are i.i.d. random samples drawn from* f_Δ. *Let*

$$\bar{x}_k = \frac{m_{k-1}}{m_k}\bar{x}_{k-1} + \frac{\lambda_k}{m_k}x_k, \qquad (9)$$

for $\bar{x}_0 = 0, m_0 = 0, m_k = m_{k-1} + \lambda_k, \lambda_k > 0$.

Then

$$Ef(\bar{x}_k) - f(\tilde{x}) \leq C(k), \qquad (10)$$

where $C(k) = \dfrac{\|x_0 - \tilde{x}\|^2 + \mu^2 \sum_{i=0}^{k-1} \lambda_i^2}{2\sum_{i=0}^{k-1}\lambda_i}$. *Moreover, if it holds that* $\lambda_i \to 0$, $\sum_{i=0}^{\infty} \lambda_i = \infty$, *then*

$$\lim_{k\to\infty} Ef(\bar{x}_k) = f(\tilde{x}).$$

Proof. Consider the distance from the current point x_{k+1} to the optimal point $\tilde{x}$. By the definition of projection, we have that $\|[x]_{\mathcal{X}} - x^*\| \leq \|x - x^*\|$, for any x, and for any $x^* \in \mathcal{X}$, therefore

$$\begin{array}{rcl} \|x_{k+1} - \tilde{x}\|^2 & \leq & \|x_k - \tilde{x}\|^2 \\ - 2\lambda_k(x_k - \tilde{x})^T \partial\varphi(x_k,\Delta^k) + \lambda_k^2 \|\partial\varphi(x_k,\Delta^k)\|^2. \end{array} \qquad (11)$$

Now, for a convex function $g(x)$, for any x, x^* it holds that $(x - x^*)^T \partial g(x) \geq g(x) - g(x^*)$, hence $(x_k - \tilde{x})^T \partial\varphi(x_k,\Delta^k) \geq \varphi(x_k,\Delta^k) - \varphi(\tilde{x},\Delta^k)$. On the other hand, from the boundedness condition on the subgradients we have that $\|\partial\varphi(x_k,\Delta^k)\|^2 \leq \mu^2$, therefore (11) writes

$$\begin{array}{rcl} \|x_{k+1} - \tilde{x}\|^2 & \leq & \|x_k - \tilde{x}\|^2 \\ & & - 2\lambda_k(\varphi(x_k,\Delta^k) - \varphi(\tilde{x},\Delta^k)) + \lambda_k^2\mu^2. \end{array} \qquad (12)$$

Denoting $u_k = E\|x_k - \tilde{x}\|^2$, and taking expectation of both sides of (12), we get

$$u_{k+1} \leq u_k - 2\lambda_k\left(Ef(x_k) - f(\tilde{x})\right) + \lambda_k^2\mu^2,$$

and $u_k \leq u_0 - 2\sum_{i=0}^{k-1} \lambda_i \left(Ef(x_i) - f(\tilde{x})\right) + \mu^2 \sum_{i=0}^{k-1} \lambda_i^2$, therefore

$$\sum_{i=0}^{k-1} \lambda_i \left(Ef(x_i) - f(\tilde{x})\right) \leq \frac{1}{2}(u_0 + \mu^2 \sum_{i=0}^{k-1} \lambda_i^2). \qquad (13)$$

It is obvious that (9) provides a recurrent version of a Cesaro mean, therefore $\bar{x}_k$ is an averaged version of the x_k's: $\bar{x}_k = \frac{\sum_{i=0}^k \lambda_i x_i}{\sum_{i=0}^k \lambda_i}$. From Jensen inequality for convex functions, we then have

$$f(\bar{x}_k) = f\left(\frac{\sum_{i=0}^k \lambda_i x_i}{\sum_{i=0}^k \lambda_i}\right) \leq \frac{\sum_{i=0}^k \lambda_i f(x_i)}{\sum_{i=0}^k \lambda_i},$$

therefore

$$Ef(\bar{x}_k) - f(\tilde{x}) \leq \frac{\sum_{i=0}^k \lambda_i (Ef(x_i) - f(\tilde{x}))}{\sum_{i=0}^k \lambda_i},$$

and, by (13)

$$Ef(\bar{x}_k) - f(\tilde{x}) \leq \frac{u_0 + \mu^2 \sum_{i=0}^k \lambda_i^2}{2\sum_{i=0}^k \lambda_i},$$

which proves (10). With the further assumptions on the step sizes $\sum_{i=0}^{\infty} \lambda_i = \infty$; $\lambda_i \to 0$, it immediately follows that $Ef(\bar{x}_k) \to f(\tilde{x})$, for $k \to \infty$. □

Remarks. The above algorithm can be considered as a modification of a general algorithm by Nemirovskii and Yudin [13], adapted for the problem at hand.

The previous theorem gives an estimate of accuracy for a finite sample. In particular, optimizing $C(k)$ over λ_i, we get $\lambda_i = \lambda^* = \frac{\|x_0 - \tilde{x}\|}{\mu} \frac{1}{\sqrt{k}}$, $i = 0,\ldots,k-1$. If the number of iterations is fixed in advance to k, then the best choice are constant step sizes $\lambda^* = \frac{\|x_0 - \tilde{x}\|}{\mu\sqrt{k}}$, which yields

$$Ef(\bar{x}_k) - f(\tilde{x}) \leq 2\frac{\|x_0 - \tilde{x}\|\mu}{\sqrt{k}}.$$

If we do not fix a-priori the number of iterations, a good choice for the steps is $\lambda_k = \frac{c}{\sqrt{k}}$, which provides asymptotically the same estimate $Ef(\bar{x}_k) - f(\tilde{x}) = O\left(\frac{1}{\sqrt{k}}\right)$. In the next section, we provide a similar algorithm (but with different stepsize rule) which guarantees that a robustly feasible solution will be found in a finite number of steps with probability one.

4 Algorithm for Feasible Case

We now consider the case when a robustly feasible solution exists. In particular, we assume that a *strong feasibility* condition holds: there exist $x^* \in \mathcal{X}, \varepsilon > 0$ such that

$$\varphi(x, \Delta) \leq 0, \quad \forall x \in \mathcal{X} : ||x - x^*|| \leq \varepsilon, \forall \Delta \in \mathbf{\Delta}. \quad (14)$$

Consider the following recursion

$$x_{k+1} = [x_k - \lambda_k \partial_x \{\varphi(x_k, \Delta^k)\}]_{\mathcal{X}}, \quad (15)$$

where ∂_x denotes a subgradient with respect to x, and Δ^k, $k = 0, 1, \ldots$ are i.i.d. random samples drawn from f_Δ. Define the stepsizes λ_k as

$$\lambda_k = \begin{cases} \eta \dfrac{\varphi(x_k, \Delta^k) + \varepsilon ||\partial_x\{\varphi(x_k, \Delta^k)\}||}{||\partial_x\{\varphi(x_k, \Delta^k)\}||^2} & \text{if } \varphi(x_k, \Delta^k) \neq 0; \\ 0 & \text{otherwise,} \end{cases} \quad (16)$$

where $0 < \eta < 2$ is a parameter of the algorithm. We shall also need the following technical assumption, that guarantees non-zero probability to distinguish if a vector x is a feasible solution or not.

Assumption 1 *If x is not a robustly feasible solution for (3), then it must hold that*

$$\Pr(F(x, \Delta) \not\preceq 0) > 0.$$

This means that the measure of the set of "bad" Δ's with respect to the probability density f_Δ must be non-zero.

The convergence result for the recursion is stated in the following theorem.

Theorem 2 *If Assumption 1 and the strong feasibility condition hold, then for any $x_0 \in \mathcal{X}$, the recursion (15) finds a robustly feasible solution of (3) in a finite number of iterations with probability one.*

A related result for quadratic matrix inequalities, has been developed in [16] in the context of a robust controller design problem. The proof of Theorem 2 has similarities to the one in [16], and is given in the Appendix.

Remarks. In stepsize rule (16) we have assumed that the value of ε (the radius of a ball contained in the feasibility set) is known. If it is not the case, we can replace ε in (16) by $\varepsilon_{s(k)}$, where $s(k)$ is the number of correction steps performed before k-th iteration and $\varepsilon_s > 0, \varepsilon_s \to 0, \sum_{s=0}^{\infty} \varepsilon_s = \infty$. Then, the last inequality in the proof of the theorem (see the Appendix) writes

$$||x_{k+1} - x^*||^2 \leq ||x_k - x^*||^2 - \eta(2 - \eta) \sum_{s=0}^{s(k)} \varepsilon_s^2.$$

From this inequality, we can again conclude that the modified algorithm terminates in a finite number of steps with probability one.

5 Applications

In this section, we present numerical examples of application of the proposed theory. In particular, we will consider an example involving robust feasibility of uncertain linear algebraic inequalities, and problems related to quadratic stability of interval plants.

5.1 Uncertain Linear Inequalities

Consider a robust feasibility problem for a set of linear algebraic inequalities in the form

$$A(\Delta_A)x \leq b(\Delta_b), \quad (17)$$

where $A \in \mathbb{R}^{n,m}$, $x \in \mathbb{R}^m$, Δ_A, Δ_b are matrices that account for the structured uncertainty acting on A and b, and the $\leq$ sign is to be intended element-wise. In particular, we will consider a randomly generated example with additive, independent uncertainty acting on each entry of the data A, b:

$$A(\Delta_A) = A_0 + \Delta_A; \quad b(\Delta_b) = b_0 + \Delta_b,$$

where

$$A_0 = \begin{bmatrix} -12.8819 & 13.6427 & -8.1623 \\ -9.5296 & 4.8204 & 20.9407 \\ 7.7817 & -7.8707 & 0.8015 \\ -0.0633 & 7.5200 & -9.3730 \\ 5.2449 & -1.6689 & 6.3574 \end{bmatrix}; \quad b_0 = \begin{bmatrix} 1.6820 \\ 0.5936 \\ 0.7902 \\ 0.1053 \\ -0.1586 \end{bmatrix};$$

$$\Delta \doteq [\Delta_A \ \ \Delta_b]; \quad ||\Delta||_\infty \leq r,$$

where $||.||_\infty$ denotes the ℓ_∞ norm (each element of the matrix is independently bounded in magnitude). It is clear that (17) can be rewritten as a robust LMI problem (see for instance [18]), therefore the algorithms proposed above may be applied to determine a feasible or an approximately feasible solution. However, we can in this case dispense with the general matrix notation, and repeat a similar reasoning for the cone of non-negative vectors. A feasibility indicator function is in this case simply given by $\varphi(x, \Delta) = ||(Ax - b)_+||$, where the suffix $+$ is now to be intended as projection onto the cone of non-negative vectors. A subgradient is easily computed following Lemma 5 as

$$\partial_x \varphi(x,\Delta) = \begin{cases} A^T(Ax-b)_+/\|(Ax-b)_+\| & \text{if } \varphi(x,\Delta) \neq 0; \\ 0 & \text{otherwise.} \end{cases}$$

We first set $r = 0.65$, and run $N = 250$ iterations of the stochastic gradient algorithm proposed in Section 3, starting with $x_0 = 0$, and found the solution $x_N = [-0.2724 \quad -0.2526 \quad -0.0882]^T$. We then performed a Monte-Carlo test using x_N, to estimate the empirical probability of robust feasibility. Using $100,000$ uniformly generated samples of the uncertainty, we obtained $\hat{p}_{feas} = 0.9989$.[1] Then, we set $r = 0.55$, and applied the algorithm of Section 4 with $\eta = 1.8$, $x_0 = 0$. This algorithm converged in less than $N = 30$ iterations to the solution $x_{N,f} = [\ -0.1697 \quad -0.1719 \quad -0.0565\ ]^T$. Checking $100,000$ randomly generated systems of inequalities (17), we verified all of them to be satisfied. It is interesting to note that the point x_N also satisfies all inequalities with $r = 0.55$.

5.2 Quadratic Stability of Interval Plant

We here consider the problem of assessing quadratic stability for a linear system described by a matrix whose elements belong to independent intervals, that is

$$A(\Delta) = A_0 + \Delta, \ \ |\Delta_{ij}| \leq rS_{ij}, \ \ r > 0. \qquad (18)$$

We will first use a 3×3 example system taken from [3], where

$$A_0 = \begin{bmatrix} -2 & -2 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & -2 \end{bmatrix}; \quad S = \begin{bmatrix} .1651 & .9394 & .5691 \\ .2451 & .4727 & .1457 \\ .7004 & .4014 & .3141 \end{bmatrix}.$$

It is well-known (see for instance [6]) that the system (18) is quadratically stable if and only if there exists a matrix $P \succ 0$ that simultaneously satisfies $N_v = 2^{n^2} = 512$ Lyapunov equations involving the vertex matrices

$$A^T(\Delta_v^k)P + PA(\Delta_v^k) \preceq 0, \ \ k = 1, \ldots, N_v,$$

where Δ_v^k represents the k-th vertex of the polyhedron described by $|\Delta_{ij}| \leq rS_{ij}$.

We first set $r = 0.5$. In this case we know from [3] that there exist a common solution for all the Lyapunov inequalities involving the vertices. The algorithm proposed in Section 4, drawing one uniform random vertex samples of the uncertainty at each iteration, converges in less than $N = 50$ iterations to the solution

$$P_{N,f} = \begin{bmatrix} 1.2487 & 0.8155 & 0.3177 \\ 0.8155 & 2.0443 & 0.2425 \\ 0.3177 & 0.2425 & 0.5371 \end{bmatrix}.$$

[1] Levels of probability cannot be estimated exactly using Monte-Carlo techniques. In general, they can be estimated within a given *accuracy* and with a probabilistic *confidence* level on the result, depending on how large is the number of samples used, [7]. Using the result of Chernoff, for instance, $100,000$ samples guarantee the result up to accuracy 0.005, with 99% confidence.

This solution may be checked to actually satisfy simultaneously all the required Lyapunov inequalities.

We then set $r = 1$. In this case it may be proved that no common solution exists for the Lyapunov inequalities. We computed an approximately feasible solution using the algorithm in Section 3. After $N = 250$ iterations we obtained the solution

$$P_N = \begin{bmatrix} 1.2042 & 0.9899 & -0.2649 \\ 0.9899 & 1.7455 & -0.0967 \\ -0.2649 & -0.0967 & 0.5577 \end{bmatrix}.$$

For this solution, we computed the empirical probability of satisfaction of robust feasibility, using $100,000$ uniform random samples of the uncertainty, obtaining $\hat{p}_{feas} = 0.996$. It should be remarked that the above solutions are computed in about one second on a standard workstation. Also, as already mentioned, the exact simultaneous solution of the above Lyapunov equations goes beyond the capabilities of most of the existing LMI solvers, for systems of order greater than four. As a more challenging example, we considered a system of order $n = 10$, with nominal matrix A_0 available at `http://www.polito.it/~calafior/data.htm`. The considered system was obtained by integer truncation, and it is nominally stable. We want to determine a common Lyapunov matrix $P \succ 0$ that proves quadratic stability, for independent element-wise uncertainty on the entries of A_0, with $r = 0.5$. Using standard tools, this would require the simultaneous solution of $2^{100} \simeq 1.26\,10^{30}$ Lyapunov inequalities.

We used the algorithm of Section 4, starting from an initial point P_0 that solves the Lyapunov equation for the nominal plant, $A_0^T P_0 + P_0 A_0 = -I$. The algorithm converged in about $N = 2000$ iterations to the solution P_N which, for space reasons, is reported at the above web address. For this solution, we estimated the empirical probability of robust feasibility on $100,000$ randomly selected vertices, obtaining $\hat{p}_{feas} = 1.0$.

6 Conclusions

Two fast randomized algorithms for determining feasible or approximately feasible solutions to robust LMI problems have been discussed in the paper. These techniques proved to be useful for problems which are intractable by means of standard exact LMI methods. In all cases, the solutions provided by the randomized approach (which came at very low computational cost) may serve as a good initial guess for a deterministic algorithm.

The first algorithm is general purpose, and can be applied in the case when we do not know in advance if a robustly feasible solution exists. The second algorithm converges faster that the first one, if a robustly feasible solution exists, and in this case guarantees termination in a finite number of steps. This latter algorithm may also be applied to unfeasible problems: unfeasibility is in this case revealed by lack of convergence of the algorithm, which can be stopped after a pre-defined number of iterations. The "goodness" of resulting solution may then be checked via Monte-Carlo randomization, as discussed in the examples.

It is expected that these techniques will be applied advantageously to the solution of various problems related to robust control design and identification, [16], [8]. Further research is needed in the direction of developing similar algorithms for the optimization of a functional under robust constraints, which is the subject of current study.

Appendix: Proof of Theorem 2

Define

$$\bar{x} = x^* + \frac{\varepsilon}{||\partial_x\{\varphi(x_k,\Delta^k)\}||}\partial_x\{\varphi(x_k,\Delta^k)\},$$

where x^* is a robustly feasible solution. Then, due to (14), $\bar{x}$ is a feasible solution of (3) and, in particular, $\varphi(\bar{x},\Delta^k) \le 0$ for all k. Now, due to the properties of a projection

$$\begin{aligned}&||x_{k+1}-x^*||^2 \le ||x_k - x^* - \lambda_k\partial_x\{\varphi(x_k,\Delta^k)\}||^2 = \\ &||x_k-x^*||^2 + \lambda_k^2||\partial_x\{\varphi(x_k,\Delta^k)\}||^2 \\ &-2\lambda_k(x_k-\bar{x})^T\partial_x\{\varphi(x_k,\Delta^k)\} - 2\lambda_k(\bar{x}-x^*)^T\partial_x\{\varphi(x_k,\Delta^k)\}.\end{aligned}$$

We now consider the last two terms in the inequality above. Due to convexity of $\varphi(x,\Delta)$ and to the feasibility of $\bar{x}$, we obtain

$$(x_k-\bar{x})^T\partial_x\{\varphi(x_k,\Delta^k)\} \ge \varphi(x_k,\Delta^k) - \varphi(\bar{x},\Delta^k) \ge \varphi(x_k,\Delta^k),$$

while, due to definition of $\bar{x}$

$$(\bar{x}-x^*)^T\partial_x\{\varphi(x^k,\Delta^k)\} = \varepsilon||\partial_x\{\varphi(x_k,\Delta^k)\}||.$$

Thus, we write

$$\begin{aligned}&||x_{k+1}-x^*||^2 \le ||x_k-x^*||^2 + \lambda_k^2||\partial_x\{\varphi(x_k,\Delta^k)\}||^2 \\ &-2\lambda_k\left(\varphi(x_k,\Delta^k) + \varepsilon||\partial_x\{\varphi(x_k,\Delta^k)\}||\right).\end{aligned}$$

Now, if $\varphi(x_k,\Delta^k) > 0$, substituting the value of λ_k (16), we get

$$\begin{aligned}&||x_{k+1}-x^*||^2 \le \\ &||x_k-x^*||^2 - \frac{\eta(2-\eta)(\varphi(x_k,\Delta^k)+\varepsilon||\partial_x\{\varphi(x_k,\Delta^k)\}||)^2}{||\partial_x\{\varphi(x_k,\Delta^k)\}||^2} \le \\ &||x_k-x^*||^2 - \varepsilon^2\eta(2-\eta).\end{aligned}$$

Therefore, if $\varphi(x_k,\Delta^k) > 0$, then we obtain

$$||x_{k+1}-x^*||^2 \le ||x_k-x^*||^2 - \varepsilon^2\eta(2-\eta).$$

From this formula, we conclude that no more than $M = ||x_0-x^*||^2/(\varepsilon^2\eta(2-\eta))$ correction steps can be executed. On the other hand, if x_k is unfeasible, than, due to Assumption 1, there is a non-zero probability to make a correction step. Thus, with probability one, the method can not terminate at an unfeasible point. We therefore conclude that the algorithm must terminate after a finite number of iterations at a feasible solution.

References

[1] S. Agmon, "The Relaxation Method for Linear Inequalities," *Canad. J. Math.,* vol. 6, pp. 382-393, 1954.

[2] E.W. Bai, R. Tempo, and M. Fu, "Worst Case Properties of the Uniform Distribution and Randomized Algorithms for Robustness Analysis," *Mathematics of Control, Signals, and Systems,* vol. 11, pp. 183-196, 1998.

[3] B.R. Barmish and P.S. Scherbakov, "Linear Matrix Inequalities with Uncertain Parameters: the Approximate Feasibility Concept," *CISS Conference,* J. Hopkins, Princeton, 1999.

[4] A. Ben-Tal, L. El Ghaoui, and A. Nemirovskii, "Robust Semidefinite Programming," in R. Saigal, L. Vandenberghe, H. Wolkowicz, editors, *Handbook of Semidefinite Programming,* Kluwer Academic Publishers, Waterloo, Canada, 2000.

[5] V. Bondarko, V.A. Yakubovich, "The Method of Recursive Aim Inequalities in Adaptive Control Theory," *Intern. J. on Adapt. Contr. and Sign. Proc.*, vol. 6, pp. 141-160, 1992.

[6] S. P. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in Systems and Control Theory.* Philadelphia: SIAM Publications, 1994.

[7] G. Calafiore, F. Dabbene, and R. Tempo, "Randomized Algorithms for Probabilistic Robustness with Real and Complex Structured Uncertainty," to appear on *IEEE Transactions on Automatic Control*, Feb. 2001.

[8] F. Dabbene, P. Gay and B.T. Polyak, "Inner Ellipsoidal Approximation of Membership Set: a Fast Recursive Algorithm," in Proceedings of *39*th *CDC*, Sidney, Dec. 2000.

[9] L. El Ghaoui, F. Oustry, and H. Lebret, "Robust Solutions to Uncertain Semidefinite Programs," *SIAM J. Optimization,* vol. 9, no. 1, 1998.

[10] V.N. Fomin, *Mathematical Theory of Learning Recognizing Systems.* Leningrad: LGU Publications, 1976 (in Russian).

[11] S. Kaczmarz, "Angenäherte Aufslösung von Systemen Linearer Gleichungen," *Bull. Intern. Acad. Polon. Sci., Lett. A,* pp. 355-357, 1937. — English translation: "Approximate Solution of Systems of Linear Equations," *Intern. J. Control*, vol. 57, no. 6, pp. 1269-1271, 1993.

[12] T.S. Motzkin, I. Shoenberg, "The Relaxation Method for Linear Inequalities," *Canad. J. Math.,* vol. 6, pp. 393-404, 1954.

[13] A. Nemirovskii, D.B. Yudin, "On Cesaro's Convergence of the Gradient Descent Method for Finding Saddle Points of Convex-Concave Functions," *Doklady AN SSSR*, vol. 239, no. 4, 1978.

[14] B. Polyak, "Gradient Methods for Solving Equations and Inequalities," *USSR Comp. Math. Math. Phys.*, vol. 4, no. 4, pp. 17-32, 1964.

[15] B. Polyak, "Minimization of Nonsmooth Functionals," *USSR Comp. Math. Math. Phys.*, vol. 9, pp. 14–29, 1969.

[16] B.T. Polyak and R. Tempo, "Probabilistic Robust Design with Linear Quadratic Regulators," in Proceedings of *39*th *CDC*, Sidney, Dec. 2000.

[17] R.E. Skelton, T. Iwasaki, K. Grigoriadis, *A Unified Algebraic Approach to Linear Control Design.* London: Taylor and Francis, 1997.

[18] L. Vandenberghe and S. Boyd, "Semidefinite Programming," *SIAM Review*, vol. 38, n. 1, pp. 49-95, March 1996.