

Sufficient Conditions for Invariance Control of a Class of Nonlinear Systems

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Abstract

This paper addresses control of nonlinear, control input affine systems coupled with a controllable linear subsystem via state and control variables. The proposed method of Invariance Control achieves positive invariance of a state space region and exponential stability of the linear subsystem. A design method for a switching controller and sufficient conditions for positive invariance are presented. The proposed theory is illustrated by simulations of a non-minimum phase system with partly stable zero-dynamics.

1 Introduction

This paper focuses on nonlinear systems with reduced relative degree which can be transformed into a controllable linear subsystem and a nonlinear subsystem forming the internal dynamics. For this class of systems we consider the problem of exponentially stabilizing the linear subsystem to the origin while keeping a state space region invariant; in this paper *invariance* is always meant in the sense of *positive invariance*. A controller that is able to achieve these two control goals is an *invariance controller* [1] with the following benefits:

- pre-specified work space areas may be taken into account,
- stability in the sense of Lyapunov of the internal dynamics can be achieved if the invariance region contains only stable trajectories of the internal dynamics,
- robustness with respect to system parameter perturbations [2] and control input limitations can be achieved, and
- application to a broad class of nonlinear control affine systems is possible.

Amongst existing nonlinear control methods some are suitable to control systems with reduced relative degree, e.g. passivity based control [3, 4], backstepping [5], etc. Output regulation has been addressed by [6, 7, 8]. Discontinuous feedback control algorithms have been proposed by [9, 10]. An unsolved problem is to establish invariance properties for given state space regions.

A solution to the invariance problem is obvious if a control Lyapunov function (clf) for the overall system is known; the level set of the clf corresponding to the initial conditions determines the boundary of an invariance region. This requires the knowledge of a clf, difficult to derive due to a decreasing condition in the whole interior (except the origin) of the invariance region. We weaken this condition by introducing an *invariance function* which requires the decreasing condition on the boundary of the invariance region only. Sufficient conditions (some of them are also necessary) for the existence of an invariance controller with piecewise linear structure are derived. An explicit design method for the switched linear controller and sufficient conditions for a suitable invariance region are presented.

The major advantage and innovation of *Invariance Control* lies in the fact that for nonlinear systems, besides common stabilization tasks of a linear (or feedback linearized) subsystem, invariance of a state space region can be guaranteed. Moreover, it also has been shown that *Invariance Control* performs well in experiments: It has been applied to stabilize the experimental underactuated robot R2D1 [11] and for roll over avoidance of vehicles with an elevated center of mass [12]. Both systems belong to the class of nonlinear non-minimum phase systems.

The paper is organized as follows: Section 2 defines the system class, the control goal and the control structure; sufficient conditions for Invariance Control are given. After this main result we present one realization and design procedure for an invariance controller together with sufficient conditions for an invariance region in Section 3. Simulations illustrate the proposed theory in Section 4, followed by conclusions in Section 5.

2 Invariance Control

We consider nonlinear systems of the form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u \quad (1-a)$$

$$\dot{\mathbf{z}} = \mathbf{f}(\mathbf{x}, \mathbf{z}) + \mathbf{g}(\mathbf{x}, \mathbf{z})u \quad (1-b)$$

with the system states $\mathbf{z} \in \mathbb{R}^{n-r}$, $\mathbf{x} \in \mathbb{R}^r$, the smooth drift $\mathbf{f} \in \mathcal{C}^1$ and control input vector field $\mathbf{g} \in \mathcal{C}^1$ with

$\mathbf{f}, \mathbf{g} : (\mathbb{R}^r, \mathbb{R}^{n-r}) \rightarrow \mathbb{R}^{n-r}$ and the control input $u \in \mathbb{R}$. The linear subsystem (\mathbf{A}, \mathbf{b}) is assumed to be controllable and without further loss of generality to be in Brunovsky canonical form.

The control design goal is as follows: Given an initial state $(\mathbf{x}(t_0), \mathbf{z}(t_0))$, find a state space region $\mathcal{G} \subseteq \mathbb{R}^n$ and a controller $u(\mathbf{x}, \mathbf{z})$ such that $\mathbf{x} = \mathbf{0}$ is exponentially stable and \mathcal{G} is invariant, i.e.

$$\|\mathbf{x}(t)\| \leq a\|\mathbf{x}(t_0)\|e^{-b(t-t_0)}, \quad a, b > 0 \quad (2-a)$$

$$(\mathbf{z}(t), \mathbf{x}(t)) \in \mathcal{G} \quad \forall t \in [t_0, \infty). \quad (2-b)$$

A control method that is able to fulfill both control goals is given by *Invariance Control* [1]; the region \mathcal{G} is called an *invariance region*. Due to (2-a) the subsystem (1-a) are *controlled dynamics* and (1-b) *internal dynamics*. Inserting $\mathbf{x}(t) = \mathbf{0}, u = 0$ into the internal dynamics yields the *zero-dynamics* of the overall system (1-a)-(1-b) as

$$\dot{\mathbf{z}} = \mathbf{f}(\mathbf{0}, \mathbf{z}) = \mathbf{f}_{zd}(\mathbf{z}). \quad (3)$$

In this paper *(un)stable* stands for *(un)stable in the sense of Lyapunov*. Similar to [13] the system (1-a)-(1-b) is called *minimum phase* if the origin $\mathbf{z} = \mathbf{0}$ of the zero-dynamics (3) is asymptotically stable. It is *weakly minimum phase* if there exists a smooth function $W(\mathbf{z})$ such that $L_W \mathbf{f}_{zd} \leq 0$ in a neighborhood of $\mathbf{z} = \mathbf{0}$ and *non-minimum phase* elsewhere.

2.1 Invariance Condition

The following, intuitively obvious theorem is the key result of a necessary and sufficient condition for a region \mathcal{G} to be invariant. Here $\nabla a(\mathbf{x})$ denotes the gradient of the function $a(\mathbf{x})$ w.r.t. \mathbf{x} .

Theorem 1 *Let $\Phi(\mathbf{x}, \mathbf{z}, t) \in C^1$ define a state space region $\mathcal{G} = \{(\mathbf{x}, \mathbf{z}, t) \mid \Phi(\mathbf{x}, \mathbf{z}, t) \leq 0\}$ with the boundary $\partial\mathcal{G} = \{(\mathbf{x}, \mathbf{z}, t) \mid \Phi(\mathbf{x}, \mathbf{z}, t) = 0\}$. Furthermore let 0 be a regular value of Φ for all $(\mathbf{x}, \mathbf{z}, t) \in \partial\mathcal{G}$. Then \mathcal{G} is positively invariant if and only if for all $(\mathbf{x}, \mathbf{z}, t) \in \partial\mathcal{G}$ the invariance condition*

$$\dot{\Phi} = \frac{\partial\Phi}{\partial\mathbf{z}}(\mathbf{f} + \mathbf{g}u) + \frac{\partial\Phi}{\partial\mathbf{x}}(\mathbf{A}\mathbf{x} + \mathbf{b}u) + \frac{\partial\Phi}{\partial t} \leq 0 \quad (4)$$

is fulfilled.

Refer to [14], Theorem 16.9 or another text book on ordinary differential equations for a formal proof. In the following a function Φ satisfying the implicit invariance condition (4) is called an *invariance function*. Evidently, Theorem 1 is closely related to the notion of Lyapunov functions. However it is important to emphasize that the invariance condition (4) is a less restrictive result since it involves a condition which must be satisfied on the boundary $\partial\mathcal{G}$ only; in contrast to

this a Lyapunov function must satisfy a decreasing condition in the whole interior of \mathcal{G} rather than on the boundary only. There are various possible strategies for Invariance Control. In this paper we restrict ourselves to the case of time invariant invariance regions, i.e. $\partial\Phi/\partial t = 0$.

2.2 Switching Control

Amongst a great variety of possible controller structures we focus on the case that the invariance controller is of the linear time varying form

$$u(\mathbf{x}, \mathbf{k}(t)) = \mathbf{k}(t)^T \mathbf{x} \quad (5)$$

where $\mathbf{k}(t) = [k_i(t)] \in \mathcal{K}, i \in \{1, \dots, r\}$ is a switched control parameter vector from the set \mathcal{K} of admissible control parameter vectors defined later such that the controlled, switched linear subsystem (1-a) is exponentially stable under arbitrary switching of $\mathbf{k}(t)$.

A switching strategy that keeps $\mathbf{k}(t)$ constant in the interior of \mathcal{G} is used. Let $t_i, i \in \mathcal{T} = \{1, 2, \dots\}$ denote the times when the state $(\mathbf{x}(t_i), \mathbf{z}(t_i))$ hits the boundary $\partial\mathcal{G}$. (Note that \mathcal{T} need not be finite; however, we will show later that Invariance Control leads to a finite number of switching times.) At these switching times the actual control parameter vector \mathbf{k}_i is switched to another control parameter vector \mathbf{k}_{i+1} which is determined such that both control goals (2-a) and (2-b) hold. Hence for all switching times we have

$$\mathbf{k}(t) = \kappa((\mathbf{x}, \mathbf{z}), \mathbf{k}(t^-)) = \mathbf{k}_i, \quad t_i \leq t < t_{i+1} \quad (6)$$

where $\kappa : (\mathbb{R}^r, \mathbb{R}^{n-r}) \times \mathcal{K} \rightarrow \mathcal{K}$, and t^- denotes that $\mathbf{k}(t)$ is piecewise constant from the right. Note that in general the switching times depend on all states, hence κ is a function of (\mathbf{x}, \mathbf{z}) and the foregoing vector $\mathbf{k}(t^-)$. Inserting (5) into (1-a)-(1-b) we obtain together with (6) the controlled system as

$$\dot{\mathbf{x}} = \mathbf{A}_{\mathbf{k}} \mathbf{x} \quad (7-a)$$

$$\dot{\mathbf{z}} = \mathbf{f}(\mathbf{x}, \mathbf{z}) + \mathbf{g}(\mathbf{x}, \mathbf{z}) \mathbf{k}(t)^T \mathbf{x} \quad (7-b)$$

$$\mathbf{k}(t) = \kappa((\mathbf{x}, \mathbf{z}), \mathbf{k}(t^-)) \quad (7-c)$$

where $\mathbf{A}_{\mathbf{k}} = \mathbf{A} + \mathbf{b}\mathbf{k}^T(t)$ and thus (7-a) is a family of linear systems, depending on \mathbf{k} .

Clearly, not every state space region is suitable to be controlled invariant as the existence of such a suitable control parameter vector requires certain restrictive conditions to the geometric shape of the invariance region. Moreover, as the times when the state hits the boundary are not known in advance it is clear that a control law which assures exponential stability of the linear subsystem in case of *arbitrary* switching is desirable. Therefore we next give conditions to \mathcal{G} and the switched controller under which both control goals (2-a)-(2-b) can be fulfilled.

2.3 Existence-Theorem for Invariance Control

In the following we define $\text{sign}[a] = \pm 1$ for positive (negative) value of a and $\text{sign}[0] = 0$.

Proposition 1 *There exists a positive definite function $c(\|\mathbf{x}\|)$ and a set \mathcal{K} of control parameter vectors such that the following two propositions hold:*

i) The family of linear systems (7-a) is globally uniformly exponentially stable, implying the existence of a common, quadratic Lyapunov function $V(\mathbf{x})$ for the family (7-a) of linear systems.

ii) Given any state $\mathbf{x}^ \in \mathbb{R}^r \neq \mathbf{0}$. For $\nabla V(\mathbf{x}^*)^T \mathbf{b} \neq 0$ there exists a suitable control parameter vector $\mathbf{k} \in \mathcal{K}$ such that*

$$\text{sign}[\nabla V(\mathbf{x}^*)^T \mathbf{b}] \mathbf{k}^T \mathbf{x}^* \leq -c(\|\mathbf{x}^*\|). \quad (8)$$

Proof: *i)* By [15] there exists a common quadratic Lyapunov function V for the family of systems (7-a) if it is uniformly exponentially stable for each element of \mathcal{K} . Hence global uniform exponential stability is given. (Note that *uniform* is meant with respect to the switching signal $\mathbf{k}(t)$.)

ii) Given any positive constant $w > 0$. There always exists a vector \mathbf{k}^* such that $\dot{V}(\mathbf{x}^*, \mathbf{k}^*) \leq -w$. To see this consider the quadratic Lyapunov function $V = \mathbf{x}^T \mathbf{P} \mathbf{x}$ and the corresponding time derivative

$$\dot{V} = \mathbf{x}^T (\mathbf{P}^T \mathbf{A} + \mathbf{A} \mathbf{P}) \mathbf{x} + 2 \mathbf{x}^T \mathbf{P} \mathbf{b} u$$

along system (1-a). Inserting the control input $u = -\nu \mathbf{x}^T \mathbf{P} \mathbf{b} = \nu \mathbf{k}^T \mathbf{x}$ with a constant $\nu > 0$ yields

$$\dot{V} = \mathbf{x}^T (\mathbf{P}^T \mathbf{A} + \mathbf{A} \mathbf{P}) \mathbf{x} - 2\nu (\mathbf{k}^T \mathbf{x})^2.$$

Hence choosing ν sufficiently large \dot{V} can be made arbitrarily large negative at one given state \mathbf{x}^* .

The time derivative of V at the point \mathbf{x}^* and for \mathbf{k}^* can be written as

$$\dot{V}(\mathbf{x}^*, \mathbf{k}^*) = \nabla V^T \mathbf{A} \mathbf{x}^* + |\nabla V^T \mathbf{b}| \text{sign}[\nabla V^T \mathbf{b}] \mathbf{k}^{*T} \mathbf{x}^*.$$

From this we can conclude that (8) must hold in order to be able to make \dot{V} arbitrarily large negative. ■

Assumption 1 *i) The invariance region is chosen such that no escape points exist, i.e. for all $(\mathbf{x}, \mathbf{z}) \in \partial \mathcal{G}$*

$$[\mathbf{b}^T \mathbf{g}^T] \nabla \Phi u = 0 \Rightarrow [(\mathbf{A} \mathbf{x})^T \mathbf{f}^T] \nabla \Phi \leq 0. \quad (9)$$

ii) The set \mathcal{K} fulfills i) and ii) of Proposition 1.

iii) For all states $(\mathbf{x}, \mathbf{z}) \in \partial \mathcal{G}$ the sign matching condition

$$\text{sign}[[\mathbf{b}^T \mathbf{g}^T] \nabla \Phi] = \text{sign}[\nabla V^T \mathbf{b}] \quad (10)$$

between the invariance function Φ and the Lyapunov function V holds.

Some remarks are in order:

- State space points on the boundary $\partial \mathcal{G}$ where the drift points to the exterior of \mathcal{G} and the control input vector field tangents the boundary $\partial \mathcal{G}$, i.e. $[(\mathbf{A} \mathbf{x})^T \mathbf{f}^T] \nabla \Phi > 0 \wedge [\mathbf{b}^T \mathbf{g}^T] \nabla \Phi = 0$, are called *escape points* [1] as at these points no control input exists to keep \mathcal{G} invariant.
- The *sign matching condition* assures that both control goals: *invariance of \mathcal{G}* and *exponential stability of the linear subsystem* do not contradict each other.

Theorem 2 *If Assumption 1 holds and the initial values $(\mathbf{x}(t_0), \mathbf{z}(t_0)) \in \mathcal{G}$, then there exists an invariance controller of the form (5) such that both control goals (2-a)-(2-b) are achieved and that the number of control parameter switches is finite.*

Proof: *Case $[\mathbf{b}^T \mathbf{g}^T] \nabla \Phi = 0$:* $\dot{\Phi} = [(\mathbf{A} \mathbf{x})^T \mathbf{f}^T] \nabla \Phi \leq 0$ by assumption of (9). Therefore invariance of \mathcal{G} follows independently of $\mathbf{k}(t)$.

Case $[\mathbf{b}^T \mathbf{g}^T] \nabla \Phi \neq 0$: From the sign matching condition (10) it follows $\nabla V^T \mathbf{b} \neq 0$. Hence (8) holds. Inserting (8) and (10) into (4) yields with $\partial \Phi / \partial t = 0$ for the invariance condition

$$\dot{\Phi} = \nabla \Phi^T \begin{bmatrix} \mathbf{A} \mathbf{x} \\ \mathbf{f} \end{bmatrix} - c(\|\mathbf{x}\|) \left| \nabla \Phi^T \begin{bmatrix} \mathbf{b} \\ \mathbf{g} \end{bmatrix} \right|, \quad (11)$$

being negative for an appropriate large value of the function $c(\|\mathbf{x}\|)$. Due to (8) there exists for all times a control parameter vector $\mathbf{k}(t)$, such that $-c(\|\mathbf{x}\|)$ becomes negative enough to make (11) negative, hence the invariance condition (4) is fulfilled.

One necessary condition for sliding along $\partial \mathcal{G}$ is $\dot{\Phi} = 0 \wedge \Phi = 0$. By requiring $\dot{\Phi} \leq -\epsilon(\|\mathbf{x}\|)$ (instead of $\dot{\Phi} \leq 0$) with a positive definite function $\epsilon(\|\mathbf{x}\|)$ we thus can avoid sliding. Moreover the number of switching times is finite as the controlled system state \mathbf{x} exponentially approaches the origin $\mathbf{x} = \mathbf{0}$; thus there exists a time t^* when $u(t^*)$ is small enough such that *i)* of Assumption 1 holds for all future times $t \geq t^*$, i.e. no more switching occurs. ■

Remarks

- Note that *i)* of Assumption 1 is *necessary* for invariance as depending on the initial values, the control parameters, and the desired value y^d , the state may hit $\partial \mathcal{G}$. If *i)* of Assumption 1 did not hold the invariance condition (4) would be violated.
- If in addition to Assumption 1 the projection of \mathcal{G} onto the subspace of internal dynamics is a bounded set, then the internal dynamics (1-b) are stable in the sense of Lyapunov. The stability is semiglobal in the minimum phase case and local within \mathcal{G} in the weakly minimum phase case.

- The initial control parameter vector $\mathbf{k}(t_0)$ may sometimes be chosen a priori such that the system state never hits the boundary $\partial\mathcal{G}$.

With Theorem 2 our main theorem has been stated for Invariance Control of system (1-b)-(1-b) and a controller of the form (5). In the next section one special design method to find a suitable switching control parameter vector $\mathbf{k}(t)$ for the switched linear controller is presented. Moreover, for the resulting switched linear controller we give sufficient conditions to an invariance region \mathcal{G} .

3 A Specific Design Method

In order to fulfill (8) the control parameter vector $\mathbf{k}(t) \in \mathcal{K}$ is switched when the state hits the boundary $\partial\mathcal{G}$. A design method for such a controller is presented in the following.

3.1 Exponential Stabilizing Switching Control

Now a switched linear control law is derived from a clf to stabilize the family of linear subsystems (7-a) under arbitrary switchings of $\mathbf{k}(t) \in \mathcal{K}$. Following [16] we construct a clf by a backstepping design. An invertible coordinate transformation $\boldsymbol{\kappa} = [\kappa_i] = \mathbf{S}\mathbf{x}$, $\boldsymbol{\kappa} \in \mathbb{R}^r$ is given by

$$\mathbf{S} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -s_1 & 1 & 0 & \cdots & 0 \\ s_1 s_2 & -s_2 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \pm s_1 \cdots s_{r-1} & \mp s_2 \cdots s_{r-1} & \cdots & -s_{r-1} & 1 \end{bmatrix}$$

where the negative (positive) sign of the last row is valid if r is even (odd). Inserting the control law

$$u(\mathbf{z}) = \kappa_r(\mathbf{z})s_r \quad (12)$$

into (1-a) and inserting the inverse coordinate transformation $\mathbf{x} = \mathbf{S}^{-1}\boldsymbol{\kappa}$, $\mathbf{S} = [S_{i,j}]$, $i \neq j \wedge i \neq j+1 : S_{i,j} = 0$, $S_{i,i} = 1$, $S_{i+1,i} = s_i$, $i \in \{1, \dots, r-1\}$ yields

$$\dot{\mathbf{x}} = \begin{bmatrix} s_1 & 1 & 0 & \cdots & 0 \\ 0 & s_2 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & s_{r-1} & 1 \\ 0 & \cdots & 0 & 0 & s_r \end{bmatrix} \boldsymbol{\kappa} = \mathbf{W}\boldsymbol{\kappa} \quad ,$$

and (1-a) is given in $\boldsymbol{\kappa}$ -coordinates by

$$\dot{\boldsymbol{\kappa}} = \mathbf{S}\dot{\mathbf{x}} = \mathbf{S}\mathbf{W}\boldsymbol{\kappa} = \mathbf{P}\boldsymbol{\kappa} \quad (13)$$

with

$$\mathbf{P} = \begin{bmatrix} s_1 & 1 & 0 & \cdots & 0 \\ -s_1^2 & -s_1 + s_2 & 1 & \ddots & \vdots \\ s_1^2 s_2 & s_1 s_2 - s_2^2 & -s_2 + s_3 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ & & & & -s_{r-1} + s_r \end{bmatrix} .$$

Let $\mathbf{Q} = [q_{ij}] = \mathbf{P}^T + \mathbf{P}$. Then the time derivative of

$$V = \boldsymbol{\kappa}^T \boldsymbol{\kappa} \quad (14)$$

is given by

$$\dot{V} = \boldsymbol{\kappa}^T \mathbf{Q} \boldsymbol{\kappa} \quad (15)$$

and can always be made negative definite with a suitable choice of $\mathbf{Q} < 0$, hence (14) is a clf. With $D_k = \det [q_{i,j}]$, $i, j \in \{1, \dots, k\}$ the test for negative definiteness of \mathbf{Q} is given by Sylvester's Criterion

$$(-1)D_1(s_1) = -2s_1 > 0 \quad (16-a)$$

$$(-1)^2 D_2(s_1, s_2) = -1 - 2s_1^2 - s_1^4 + 4s_1 s_2 > 0 \quad (16-b)$$

...

$$(-1)^r D_r(s_1, \dots, s_r) = \det \mathbf{Q} > 0 . \quad (16-c)$$

Starting with an appropriate choice of $s_1 < 0$ in (16-a), one can calculate the $r-1$ control parameters s_i backwards.

Lemma 1 Consider the linear subsystem (1-a) in $\boldsymbol{\kappa}$ -coordinates controlled by (12) with $s_r(t)$ as the only switched control parameter. There exists a constant s_r^* such that with $s_r(t) \leq s_r^*$ a common quadratic Lyapunov function for the resulting family (7-a) of linear subsystems is given by (15).

Proof: The time derivative of (15) can be expressed by $\dot{V} = \sigma(\mathbf{z}, s_1 \dots s_{r-1}) + (s_r - s_{r-1}) \kappa_r^2$ with

$$\sigma(\mathbf{z}, s_1, \dots, s_{r-1}) = \boldsymbol{\kappa}_r^T \mathbf{Q}_{r,r} \boldsymbol{\kappa}_r + 2\kappa_r \sum_{i=1}^{r-1} q_{r,i} \kappa_i ,$$

where $\boldsymbol{\kappa}_r = [\kappa_1, \dots, \kappa_{r-1}]^T$ and $\mathbf{Q}_{r,r} \in \mathbb{R}^{(r-1) \times (r-1)}$ equals \mathbf{Q} , reduced by the r -th row and column. Let s_r^* be such that (16-c) holds. Then for $\mathbf{x} \neq \mathbf{0}$ and $s_r = s_r^*$ we have $\dot{V} < 0$. This and the fact that σ does not depend on s_r allows to conclude $\dot{V} < 0$ for all $s_r \leq s_r^*$ and $\mathbf{x} \neq \mathbf{0}$. As V is a quadratic positive definite function the proposed condition holds for system (13). Applying the linear coordinate transformation we get the same result for the family of linear subsystems (7-a). ■

Due to the assumed Brunovsky canonical form we obtain

$$\nabla V(\mathbf{x})^T \mathbf{b} = \frac{\partial V(\boldsymbol{\kappa})}{\partial \boldsymbol{\kappa}} \left(\frac{\partial \boldsymbol{\kappa}}{\partial x_r} \right)^T = 2\kappa_r , \quad (17)$$

hence (8) holds as its left hand side evaluates to $2 \text{sign}[\kappa_r] \kappa_r s_r(t) = 2 |\kappa_r| s_r(t)$ where $s_r(t) < 0$.

3.2 Construction of \mathcal{G}

Using a switching controller (12), sufficient conditions for an invariance function $\Phi(\mathbf{x}, \mathbf{z})$ can be stated. Let $\Phi_0(\mathbf{x}, \mathbf{z}) \in \mathcal{C}^1$ be such that for $(\mathbf{x}, \mathbf{z}) \in \partial\mathcal{G} \wedge \kappa_r(\mathbf{x}) = 0$ we have

$$[(\mathbf{A}\mathbf{x})^T \mathbf{f}(\mathbf{x}, \mathbf{z})^T] \nabla \Phi_0(\mathbf{x}, \mathbf{z}) \leq 0 \quad (18\text{-a})$$

and there exists a $\gamma > 0$ for all $(\mathbf{x}, \mathbf{z}) \in \partial\mathcal{G}$ such that

$$\text{sign} \left[[\mathbf{b}^T \mathbf{g}(\mathbf{x}, \mathbf{z})^T] \nabla \Phi_0(\mathbf{x}, \mathbf{z}) + 2\gamma \kappa_r(\mathbf{x}) \right] = \text{sign}[\kappa_r(\mathbf{x})] . \quad (18\text{-b})$$

Then a suitable invariance function is given by

$$\Phi = \Phi_0(\mathbf{x}, \mathbf{z}) + \gamma \kappa_r(\mathbf{x})^2 \quad (19)$$

as (17) together with (18-b) assures the *sign matching condition* (10); from the definition of the sign-function and (18-b) we conclude

$$[\mathbf{b}^T \mathbf{g}^T] \nabla \Phi(\mathbf{x}, \mathbf{z}) = 0 \Leftrightarrow \kappa_r(\mathbf{x}) = 0 ,$$

hence condition (18-a) guarantees no *escape points* on $\partial\mathcal{G}$.

Remark: Note that if (1-a)-(1-b) is in cascade form, i.e. $\mathbf{g} = \mathbf{0}$ and if we can choose Φ_0 as a function of \mathbf{z} only such that (18-a) simplifies to

$$\nabla \Phi_0(\mathbf{z})^T \mathbf{f}(\mathbf{x}, \mathbf{z}) \leq 0 \quad \forall (\mathbf{x}, \mathbf{z}) \in \partial\mathcal{G} \wedge \kappa_r(\mathbf{x}) = 0$$

then (18-b) is universally valid. Thus in this special case a first integral of the zero-dynamics of (1-a)-(1-b) yields the function Φ_0 .

4 Numerical Example

We consider a simple model with the (partially unstable) zero-dynamics of the underactuated SCARA type robot R2D1 [11] as the internal dynamics (1-b) and a simple integrator as the linear subsystem (1-a):

$$\begin{aligned} \dot{x} &= u \\ \dot{z}_1 &= z_2 \\ \dot{z}_2 &= \frac{-Bz_2^2 \sin x + D \sin \alpha \sin(x + z_1)}{m_{22} + B \cos x} \end{aligned}$$

The constants from the experimental setup are $B = 0.16, m_{22} = 0.18, \alpha = -\pi/6$ and $D = 5.2$. This system has unstable zero-dynamics almost everywhere except in a small region around the equilibrium points. The control goal is to steer x from its initial position $x(0) = -2$ to $x(\infty) = \pi/4$ such that the internal dynamics keep stable. Therefore we use an invariance function that restricts the system state to a local minimum phase region around the equilibrium point. A function $\Phi_0(\mathbf{z})$ fulfilling (18-a) is given by

$$\begin{aligned} \Phi_0(\mathbf{z}) &= z_2^2 - c_1 e^{-c_3(z_1 + \pi/2)} - \\ & c_2 (c_3 \sin(z_1 + \pi/2) - \cos(z_1 + \pi/2)) \end{aligned}$$

with $c_1 = 0.02, c_2 = -6.92, c_3 = 1.80$. The switching controller follows to $u = s_1(t)(x - \pi/4)$, hence the invariance function is $\Phi(\mathbf{x}, \mathbf{z}) = \Phi_0(\mathbf{z}) + \gamma(x - \pi/4)^2$. Note that with $\{(x, \mathbf{z}) \mid \Phi_0(\mathbf{z}) \leq 0\}$ containing only stable trajectories the same holds for \mathcal{G} as $\gamma(x - \pi/4)^2 \geq 0$. For the switching controller we set $\epsilon = 5, \gamma = 1$, and $s_1(0) = -2$; the initial state of the internal dynamics were $z_1(0) = -2, z_2(0) = 0$. As shown in Fig. 1 both control goals are fulfilled; the controller switches three times. In Fig. 2 the effect of switching can be seen: each time the state hits $\partial\mathcal{G}$, the parameter $s_1(t)$ is set to a larger negative value such that $\dot{\Phi} = -\epsilon$. In Fig. 3 one can see one part of the region \mathcal{G} together with the stable trajectory. Performing the simulation with a fixed parameter $s_1 = -2$ results in unstable internal dynamics, indicated by a dash-dotted line in Fig. 4; the dotted lines are phase curves of the zero-dynamics.

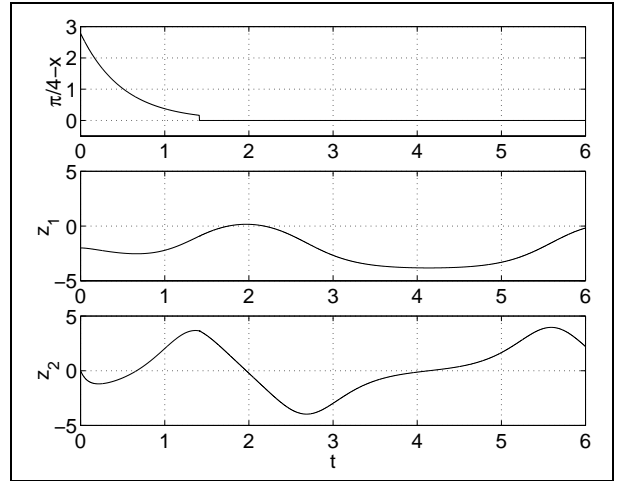


Figure 1: Trajectory with Invariance Control.

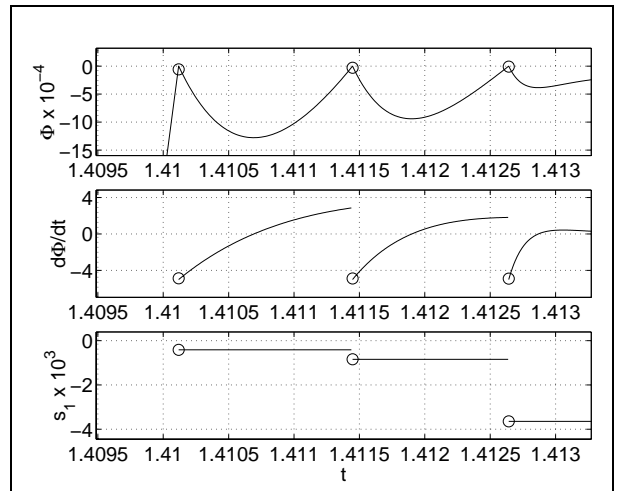


Figure 2: Variables of the switching controller.

5 Conclusions

This paper presented sufficient conditions for invariance control of a general class of nonlinear systems.

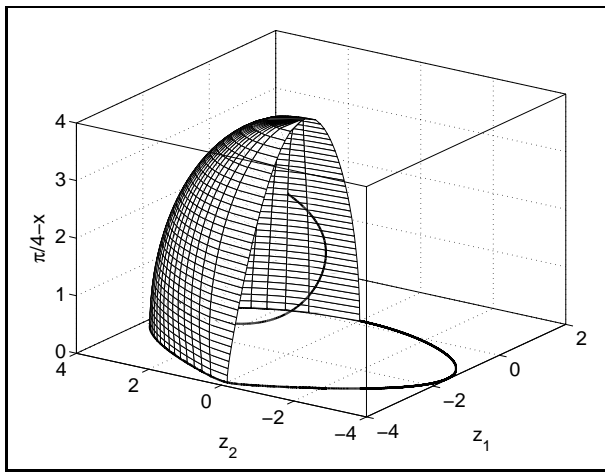


Figure 3: Parts of \mathcal{G} with the trajectory.

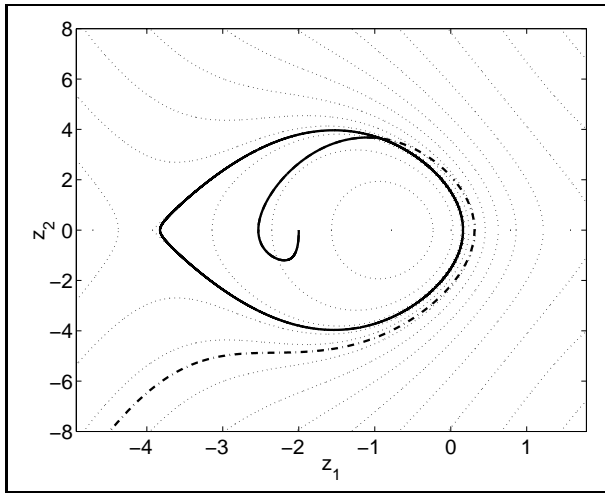


Figure 4: Dotted lines: zero-dynamics; thick solid line: invariance controlled case; dash-dotted line: case of constant $s_1 = -2$.

These enable the design of an invariance controller in two steps: First, find a switching controller which assures stability of the linear (or feedback linearized) subsystem. Second, determine a state space region that fulfills the sign matching condition and does not have escape points. In addition, specific restrictions to the invariance region may be taken into account during this step of the controller design.

For the first step we have shown that a backstepping design leading to clfs is suitable, while for the second step we stated sufficient conditions for a region to be an invariance region. The fact that these conditions must hold on the boundary of the invariance region simplifies the search for a suitable invariance function enormously. However, the need for an analytic solution of an invariance function is still an unsolved problem in generic situations. Therefore future research is to address the *numerical* calculation of invariance regions.

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