One Optimum Design Method of Classical Control System

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Abstract The classical control method depends on the experiences of project designers and is based on classical design of servo-system. One optimum method which is designed to solve the problem of classical control system design is proposed. This paper presents an optimum performance index function about order and parameters of lead-link and also approves the existence of solution. Finally, an application is given to certify the effectiveness of the method.

Key words Servo-system optimum design lead compensation

0 Introduction

Classical control theory is not only perfect in theory but also effective in the application of practical engineering after more than 50 years of its development. But when project designers design a system by the use of classical control thery, they often depend on their experience and seldom consider problems from the optimum viewpoint. There are few references about optimum design of classical control system. [1] discussed optimum design of memoryless adjuster based on amplitude-margin and phase-margin in delayed time system, it extended from infinite dimensional adjuster to finite dimensional adjuster on the basis of [2], introduced static nonlinear disturbance into the input to get the conditions of robust stabilization, introduced dynamic linear disturbance to get the conditions of relative stabilization. [3] gave an optimum design of formulating quadratic feedback theory based on gain-bandwidth to solve non-parametric uncertainty in the control system. It studied the optimum design about PID in [4] and [5]. [6]applied Routh approach to make model deflation optimum design, it optimized servo-system by the use of PI state feedback design.

When we design system by the use of classical control theory, and use lead-link as (1) to compensate the phase of system to add the phase margin, the phase φ_0 need

$$G_c(s)=\frac{1+aTs}{1+Ts}\quad (a>1) \qquad (1)$$

to be compensated could be an angle which belongs to the assemble $(\frac{\pi}{2},\pi)$, so we must serialize multi lead-links. And then how to decide the forms of those lead-links to achieve the best compensation effect is the problem we must solve. For this case, this paper will give an optimum design method. This problem exists not only when $\varphi_0 \in (\frac{\pi}{2},\pi)$ but also when φ_0 equals other value.

1 Mathematical Description

We can calculate that $\varphi_m = \arcsin\frac{a-1}{a+1}$ and

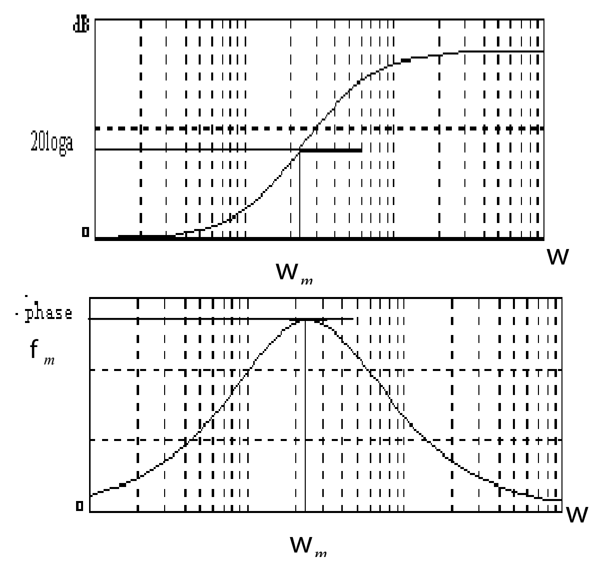


Fig.1 Bode graph of lead-link

$\omega_m = \dfrac{1}{\sqrt{aT}}$. Its Bode graph is showed in Fig.1. Where ω_m is the geometrical center frequency of lead-link, and $\varphi_m \in (0, \frac{\pi}{2})$ is the compensation phase which is compensated by $G_c(S)$ at the frequency of ω_m. The phase is the largest at its geometrical center frequency, so we set the frequency need to be compensated as its center frequency, that is $\omega_c = \omega_m$. If we assume that enlargement factor of the ith lead-link is a_i, then its transfer function is

$$G_i(s) = \frac{1 + \dfrac{\sqrt{a_i}}{\omega_m} s}{1 + \dfrac{1}{\sqrt{a_i}\omega_m} s} \tag{2}$$

$i = \underline{n}$. So, the transfer function produced by serializing n lead-links can be expressed as $G(s) = \prod_{i=1}^{n} G_i(s), i = \underline{n}$.

2 Optimum Design of System

The problem we must solve is how to decide on the values of n and a_i to compensate phase of φ_0 at the frequency of ω_m so as to get optimum compensation effectiveness. Now, we will give the definition of optimum design and two theorems related.

Definition 1: At the precondition that one lead-link compensates φ_0 at the frequency of ω_m, the raised amplitude of the lead-link is the smallest at the frequency of ω_0 （$\omega_0 > \omega_m$） by select its parameters n and $a_i (i = \underline{n})$, then, this lead-link is defined as optimum lead-link.

Theorem 1: If we are given n lead-links $G_1(s), G_2(s), ..., G_n(s)$ to achieve the compensation of φ_0 at the frequency of ω_m, we can obtain optimum compensation effectiveness by the use of the same n lead-links, that is, $a_1 = a_2 = ... = a_n$.

Proof: We assume that the phase compensated by the ith lead-link at the frequency of ω_m is φ_i, the enlargement factor is a_i, then its transfer function is expressed as(2), so it follows

$$A_i(\omega_0) = 20\log \frac{\sqrt{1 + \left(\sqrt{a_i}\omega_0 / \omega_m\right)^2}}{\sqrt{1 + \left(\omega_0 / \sqrt{a_i}\omega_m\right)^2}}$$

$$A(\omega_0) = \sum_{i=1}^{n} A_i(\omega_0), \quad \sum_{i=1}^{n} \varphi_i = \varphi_0$$

where $a_i = \dfrac{1 + \sin\varphi_i}{1 - \sin\varphi_i}$, $\varphi_i \in (0, \frac{\pi}{2})$.

Let $c = (\frac{\omega_0}{\omega_m})^2$, then

$$A(\omega_0) = 20\log \sqrt{\frac{1 + a_1 c}{1 + \frac{c}{a_1}} \cdot \frac{1 + a_2 c}{1 + \frac{c}{a_2}} \cdot \ldots \cdot \frac{1 + a_n}{1 + \frac{c}{a_n}}}$$

Let $f(x) = \dfrac{1 + c \cdot \dfrac{1 + \sin x}{1 - \sin x}}{1 + c \cdot \dfrac{1 - \sin x}{1 + \sin x}}$

$y = f(\varphi_1) \cdot f(\varphi_2) \cdot \ldots \cdot f(\varphi_n)$

$F = y + \lambda (\sum_{i=1}^{n} \varphi_i - \varphi_0)$, and equate the partial differentiation of F with respect to φ_i and λ to zero,

$$\begin{cases} f'(\varphi_1) \cdot y + \lambda \cdot f(\varphi_1) = 0 \\ f'(\varphi_2) \cdot y + \lambda \cdot f(\varphi_2) = 0 \\ \quad \cdot \\ \quad \cdot \\ \quad \cdot \\ f'(\varphi_n) \cdot y + \lambda \cdot f(\varphi_n) = 0 \\ \sum_{i=1}^{n} \varphi_i - \varphi_0 = 0 \end{cases} \tag{3}$$

$f(\varphi_i) \neq 0$, $y \neq 0$, so $f'(\varphi_i) \neq 0$.Then, from (3) we can get

$$\frac{-\lambda}{y} = \frac{f'(\varphi_1)}{f(\varphi_1)} = \frac{f'(\varphi_2)}{f(\varphi_2)} = \ldots = \frac{f'(\varphi_n)}{f(\varphi_n)} \tag{4}$$

Because

$$\frac{f'(\varphi_i)}{f(\varphi_i)} = \frac{4c(1+\sin^2\varphi_i + c\cos^2\varphi_i)}{(c-1)^2\cos^3\varphi_i + 4c\cos\varphi_i} \tag{5}$$

If we consider (4) and (5), then

$$\frac{4c(1+\sin^2\varphi_i + c\cos^2\varphi_i)}{(c-1)^2\cos^3\varphi_i + 4c\cos\varphi_i} = \frac{4c(1+\sin^2\varphi_j + c\cos^2\varphi_j)}{(c-1)^2\cos^3\varphi_j + 4c\cos\varphi_j} \tag{6}$$

$\forall i, j \in \underline{n}$, so

$$(\cos\varphi_j - \cos\varphi_i)[2(c-1)^2(\cos^2\varphi_i + \cos^2\varphi_j + \cos\varphi_j\cos\varphi_i) + 8c + (c-1)^3\cos^2\varphi_i \cos^2\varphi_j + (4c-4c^2)\cos\varphi_i\cos\varphi_j] = 0 \tag{7}$$

let $w(x) = 2(c-1)^2(\cos^2\varphi_i + \cos^2\varphi_j + \cos\varphi_j \cos\varphi_i) + 8c + (4c-4c^2)\cos\varphi_i\cos\varphi_j$,

it can be known that $w(x) > 0$, then $w(x) + (c-1)^3\cos^2\varphi_i\cos^2\varphi_j > 0$, so equation (7) has only one solution $\varphi_i = \varphi_j$. The solution of equation (6) is $\varphi_i = \varphi_j$ for the random of i, j, so y has only one extreme point when $\varphi_1 = \varphi_2 = \ldots = \varphi_n = \frac{\varphi_0}{n}$. y is minimum at the condition of $\varphi_1 = \varphi_2 = \ldots = \varphi_n = \frac{\varphi_0}{n}$, because $f(x)$ approaches ∞ as $\varphi_i \to \frac{\pi}{2}$. That is, $A(\omega_0)$ is minimum when

$$a_1 = a_2 = \ldots = a_n = \frac{1+\sin\frac{\varphi_0}{n}}{1-\sin\frac{\varphi_0}{n}}.$$

Theorem 2: When we serialize the same n lead-links to compensate φ_0 at ω_m, the larger n is, the better compensation effectiveness is.

Proof: As shown in Fig.1, the amplitude frequency response characteristic is monotonic increasing function of ω. For simplifying the problem, we consider the condition of $\omega = \omega_m$, it is also true when $\omega \neq \omega_m$. The amplitude of serializing the same n lead-links at ω_m is

$$A(j\omega_m) = 20\log\left(\sqrt{\frac{1+\sin\frac{\varphi_0}{n}}{1-\sin\frac{\varphi_0}{n}}}\right)^n \tag{8}$$

The theorem is proved if we can show that $A(j\omega_m) = 10n\log\left(\frac{1+\sin\frac{\varphi_0}{n}}{1-\sin\frac{\varphi_0}{n}}\right)$ is a strictly monotonic decreasing function of n. We consider the function

$$g(x) = x \cdot \ln\frac{1+\sin\frac{\varphi_0}{x}}{1-\sin\frac{\varphi_0}{x}} \tag{9}$$

and by differentiation with respect to x ,

$$\frac{dg}{dx} = \ln\frac{1+\sin\frac{\varphi_0}{x}}{1-\sin\frac{\varphi_0}{x}} - \frac{2\varphi_0}{x\cdot\cos\frac{\varphi_0}{x}}$$

$\frac{d^2g}{dx^2} = \frac{2\varphi_0^2}{x^3\cos^2\frac{\varphi_0}{x}}\cdot\sin\frac{\varphi_0}{x}$, we find that the second derivative of $g(x)$ is greater than zero, that is, the first derivative of $g(x)$ is a strictly monotonic increasing function. Because $\lim_{x\to\infty}\frac{dg}{dx} = 0$, the first derivative of $g(x)$ is less than zero, so $g(x)$ is a strictly monotonic decreasing function, and the same for $g(n)$. So $A(j\omega_m)$ is a strictly monotonic decreasing function.

We can, therefore, conclude from the two theorems that to get optimum compensation effect we should serialize the same enough lead-links. But another problem is risen, that is, as the number of lead-link increased, project implementation will be difficult, and when n is added to some value, the amplitude at the frequency, of which we are care is decreased little. It would be very difficult to serialize n lead-links in practical project, so it is necessary to optimize n and achieve optimum solution between decreasing n and getting enough little amplitude.

According to the analysis above, we give one optimum index function about n

$$J(n,\omega) = 20\log\sqrt{\frac{a\omega_m^2 + a^2\omega^2}{a\omega_m^2 + \omega^2}} + 1.5\times n \qquad (10)$$

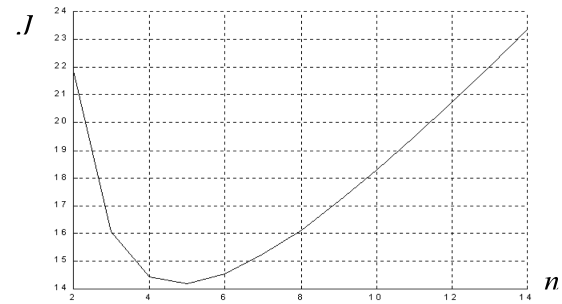
where n is the order of lead-link, ω is the frequency care for, ω_m is the geometric center frequency of lead-link, φ_0 is the phase which need to be compensated at ω_m. If ω is given, n_0 which make equation (10) be minimum is the optimum order.

Now, we discuss the existence of minimum of equation (10). Equation (10) consists of two parts. The first part is amplitude of single lead-link which is a monotonic decreasing function of n. The second part is a monotonic increasing function of n. Because $\lim\limits_{\frac{\varphi_0}{n}\to\frac{\pi}{2}} J(n,\omega) = \infty$ and $\lim\limits_{n\to\infty} J(n,\omega) = \infty$, the minimum of equation (10) must exist, that is, the solution of optimum design of lead-link must exist.

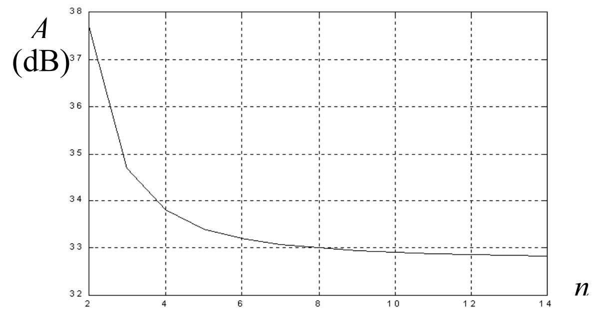
The index function of equation (10) is related to the frequency ω concerned. If there is a harmonic peak in system's amplitude frequency response characteristic, we should set ω as concerned frequency at which there is a harmonic peak. If there is no obvious harmonic peak, we set $\omega = 3\omega_m$. It can be found that the first part of equation (10) is the amplitude of single lead-link at frequency of ω, and it takes little effect on performance index when $\omega \geq 3\omega_m$, so we take $\omega = 3\omega_m$. Also, a signal takes little effect on a system when the frequency is three times higher than the cross frequency.

3 Numerical Calculation and Research Application

To demonstrate the effectiveness of the conclusion above, we take emulation calculation and research application. Those are relationship curves of $J(\omega_0, n)$ and $A(\omega_0)$ at different associations of ω_m, φ_0 and ω_0 in Fig2. We know J is the minimal when $n = 5$ according to Fig2.a. According to Fig2.b, the amplitude decreased is less than 0.3 dB by increasing n by



a: Relationship curve of J and n



b: Relationship curve of A and n

Fig 2 Relationship curve of J, A and n When ω_m=40Hz, ω_0=120Hz, φ_0=120°

one order when $n \geq 5$, it has no sense for compensation of system. It is not suitable to enhance the order of system to get very small amplitude margin in real-time control system. The quantization error becomes bigger by increasing the order because of the finite of computer word length.

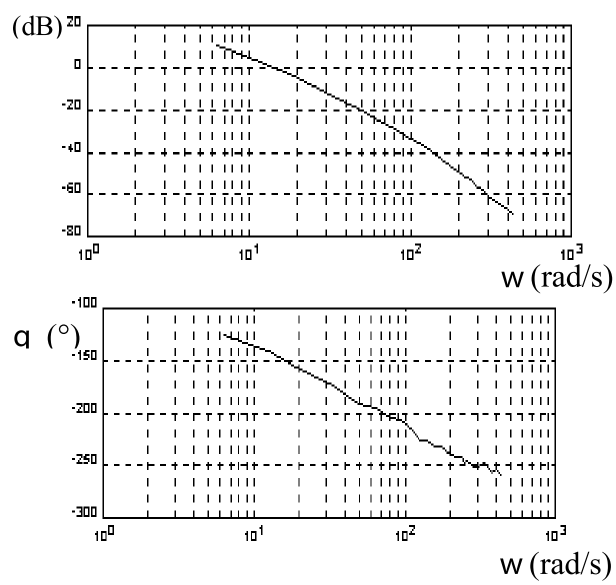


Fig3. Bode graph of external ring of table

This paper introduced the lead-link optimum design method into design of external ring control system of Three-axis emulation table. The object characteristic curve of external ring is shown in Fig3.

According to the requirement of double-ten performance of $6HZ$ and object characteristic, we set the cross frequency as $\omega_m = 40HZ$, as shown in Fig3,to get phase margin of 45° at ω_m,we must design lead-link to let $\varphi_0 = 108^\circ$.There is no obvious resonance according to Fig3, so we let $\omega = 3\omega_0$.Then, using optimum design method given in this paper, $n = 5$ can be achieved by equation(10). Compensation link as equation (11) can be got.We close the system, the response curves are shown in Fig4 and Fig5.

$$G_c(s) = \prod_{i=1}^{5} G_i(s)\,, G_i(s) = \frac{1+0.0057s}{1+0.0027s} \qquad (11)$$

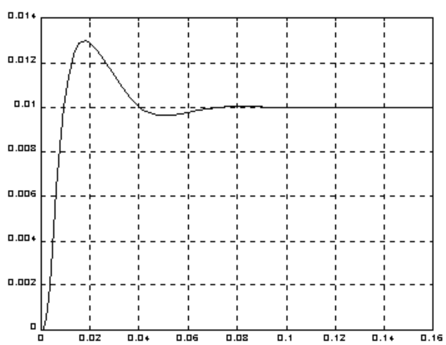


Fig4 Response curve of 0.01° step

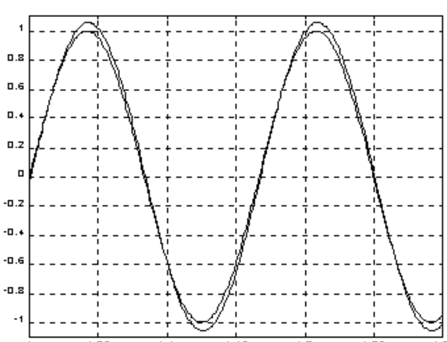


Fig5 Response curve of 1°、6Hz

According to the practice response curve, performance of this system meets index requirement well, so the optimum design method given in this paper is feasible for application of practical project.

4 Conclusion

The analysis and practice application of the optimum design method based on classical control theory given in this paper show that the method is feasible. It avoids using the rule-of-thumb and the method of trial and error when designing control system. This is important in practical project. The conclusion of this paper is right for any φ_0, and usually $\varphi_0 \in (0,\pi)$. What this paper argued is lead-link such as equation (1) not any other type of expression. The synthesis optimum design of lead-link and lag-link needs to be researched on further.

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