A Unified Characterization and Solution of Input-to-State Stabilization via State-Dependent Scaling

Hiroshi Ito
Dept. of Control Engineering and Science, Kyushu Institute of Technology
680-4 Kawazu, Iizuka, Fukuoka 820-8502, Japan. `hiroshi@ces.kyutech.ac.jp`

Abstract

The author presents solutions to input-to-state stabilization and integral input-to-state stabilization problems for nonlinear systems based on the novel concept of state-dependent scaling design. Both state-feedback and output-feedback controllers are constructed in a unified way. The method provides global solutions whenever the system is in the strict-feedback or output-feedback form. The paper encompasses input-to-state stabilization and integral input-to-state stabilization in the presence of structured, static and dynamic uncertainties.

1 Introduction

The notion of input-to-state stability(ISS) has played an important role in recent development of nonlinear control theory[11], which was originally introduced in [13]. The ISS has already found wide applicability such as nonlinear stabilization and backstepping design[11], inverse optimal control[3, 10], small-gain theorem[9], stability and performance of interconnected systems[16].

The concept of ISS is a natural answer to the situation where boundedness of operator norms('finite linear gains' in other words) is far too strong a requirement for general nonlinear systems. The ISS replaces the finite linear gains with nonlinear gains instead of focusing only on local properties[4]. ISS is a global property which takes into account not only initial states in a manner fully compatible with Lyapunov stability, but also the effect of input perturbations. The idea of nonlinear gain was extended by the integral input-to-state stability(iISS) in which the size of inputs is measured by integral norms[14]. For linear systems, both ISS and iISS are equivalent to asymptotic stability. For general nonlinear systems, the iISS is strictly weaker than ISS, and ISS implies iISS. A nonlinear system is iISS if and only if there is some output function which makes the system smoothly dissipative and weakly zero-detectable[1]. This equivalence describes an important connection between iISS and another popular concept 'dissipation' which has guided developments of nonlinear $\mathcal{H}_\infty$ control and related robust control techniques.

This paper addresses the problem of designing input-to-state and integral input-to-state stabilizing control laws. The concept of state-dependent(SD) scaling design is employed and it leads to an explicit construction of state feedback and output feedback control laws. The SD scaling design is a new technique which thoroughly utilize the SD scaling and diffeomorphism to design nonlinear control systems[6, 8]. This paper does not repeat the concept and details of the SD scaling design framework which has been already presented in the previous papers and references therein. In [6, 7], the SD scaling design method has succeeded in directly solving robust nonlinear global stabilization and inverse optimal control problems without resort to ISS, by contrast with other previous methods based on ISS. Since abovementioned papers bypassed the ISS, it was not clear how to solve an important class of nonlinear control problems by using the SD scaling design approach when the problems are characterized directly in terms of ISS and iISS. This paper presents new characterizations of ISS and iISS problems through the SD scaling design and introduces some necessary nontrivial modifications to the scaling, Lyapunov functions and recursive design of feedback gains and observers presented in [6, 8]. The stabilizing control laws are systematically generated by selecting SD scaling and parameters of the coordinate change recursively.

The paper presents both state-feedback and output-feedback global stabilization of nonlinear systems in the strict-feedback form. Input-to-state and integral input-to-state stabilization are considered for uncertain systems as well as known systems. The uncertainties are allowed to be either static or dynamic. The existence of solutions are demonstrated and the controller designs of all problems are done within a single unified framework. The recursive design procedure is written by scalar-valued simple inequalities in terms of design parameters, which is amenable to efficient numerical computation. Proofs can be found in [5].

2 State Feedback Stabilization

Consider the uncertain nonlinear system Σ described by

$$\Sigma : \left\{ \begin{array}{l} \dot{x} = A(x)x + B(x)w + B_\delta(x)w_\delta + G(x)u \\ z_\delta = C_\delta(x)x + D_\delta(x)w_\delta + H_\delta(x)u \end{array} \right. . \qquad (1)$$

where $x(t) \in \mathbf{R}^n$ is the state, $u(t) \in \mathbf{R}$ is the control input, $w(t) \in \mathbf{R}^p$ is the disturbance input, and $w_\delta(t), z_\delta(t) \in \mathbf{R}^q$ are channels through which the uncertain components affect the system. Functions $A(x)$, $B(x)$, $G(x)$ $B_\delta(x)$, $C_\delta(x)$, $H_\delta(x)$ and $D_\delta(x)$ are $\mathcal{C}^0$. The two signals z_δ and w_δ

$$z_\delta = \begin{bmatrix} z_{\delta_1} \\ z_{\delta_2} \\ \vdots \\ z_{\delta_m} \end{bmatrix}, \quad w_\delta = \begin{bmatrix} w_{\delta_1} \\ w_{\delta_2} \\ \vdots \\ w_{\delta_m} \end{bmatrix}, \qquad \begin{array}{c} w_{\delta_i}(t) \in \mathbf{R}^{q_i} \\ z_{\delta_i}(t) \in \mathbf{R}^{q_i} \\ q_i \geq 0, \quad q = \sum_{i=1}^{m} q_i \end{array}$$

are connected by an uncertain system $\Sigma_\triangle$ which is represented by a causal static nonlinear mapping $\Delta : z_\delta \mapsto w_\delta$.

$$\Sigma_\triangle : \ \Delta = \text{block-diag}[\Delta_1, \Delta_2, \cdots, \Delta_m], \qquad (2)$$

Some of the mappings $\Delta_i : z_i \mapsto w_i$, $i = 1, 2, \ldots, m$ can be zero in vector size q_i. Each mapping Δ_i is defined as

$$\Delta_i \ : \ w_{\delta_i} = h_{\delta_i}(z_{\delta_i}, t), \qquad (3)$$

where h_{δ_i} is a vector-valued function satisfying $h_{\triangle_{\delta_i}}(0,t) = 0$ for all $t \geq 0$. We assume that Δ_i are square in size of input and output vectors, which does not cause any loss of generality. The uncertainty $\Sigma_\triangle$ is said to be admissible if Δ_i satisfies

$$\|z_{\delta_i}(t)\| \geq \|w_{\delta_i}(t)\|, \qquad \forall t \in [0, \infty) \ . \qquad (4)$$

Uncertainty components having super-linear growth in x can be included by a judicious choice of $B_\delta(x)$, $C_\delta(x)$, $D_\delta(x)$ and $H_\delta(x)$. Indeed, these matrices specify the "nonlinear size"(including magnitude, nonlinearity, location and structure) of uncertainties. Consider the state-feedback control:

$$u = K(x)x \tag{5}$$

where K is a $\mathcal{C}^0$ function. We use a global diffeomorphism

$$\chi = S(x)x \tag{6}$$

between $x \in \mathbf{R}^n$ and $\chi \in \mathbf{R}^n$. The time-derivative of χ is

$$\dot{\chi} = \left[\frac{\partial S}{\partial x_1}x, \frac{\partial S}{\partial x_2}x, \cdots, \frac{\partial S}{\partial x_n}x\right]\dot{x} + S(x)\dot{x} = T(x)\dot{x}\ ,$$

where $T(x)$ is a $\mathcal{C}^0$ function. Then, the closed-loop system consisting of (1) and the feedback law (5) is obtained as

$$\Sigma_{cl} : \begin{cases} \dot{\chi} = T\left(\hat{A}\hat{S}\chi + Bw + B_\delta w_\delta\right) \\ z_\delta = \hat{C}_\delta \hat{S}\chi + D_\delta w_\delta \end{cases}$$

$$\hat{S} = \begin{bmatrix} S^{-1} \\ KS^{-1} \end{bmatrix},\ \hat{A} = [A\ G],\ \hat{C}_\delta = [C_\delta\ H_\delta].$$

This paper employs the idea of state-dependent scaling to achieve input-to-state stabilization of the uncertain nonlinear system. Define the following set of scaling matrices

$$\mathbf{L} = \left\{ \Lambda = \operatorname*{block-diag}_{i=1}^{m} \Lambda_i : \Lambda_i = \lambda_i(x) I_{q_i},\ \lambda_i(x) > 0\ \forall x \in \mathbf{R}^n \right\} \tag{7}$$

Here, I_{q_i} denotes an identity matrix which is compatible in size with z_{δ_i}. The scaling matrices are functions of the state variable. The state-dependent scaling is useful for estimating the worst case value of the time-derivative of Lyapunov functions[6]. As in [6], another type of SD scaling matrices for repeated uncertainties can be incorporated in the set of scaling matrices straightforwardly. For brevity, they are not included in this paper. The following provides new characterization of the ISS property in the state-feedback case.

Theorem 1 *If there exist a positive definite matrix P, positive real numbers ν, ξ and a scaling function matrix $\Lambda \in \mathbf{L}$ such that*

$$M^{sf}(x) = \begin{bmatrix} \begin{pmatrix} \hat{S}^T\hat{A}^T T^T P + \\ PT\hat{A}\hat{S} + \nu P \end{pmatrix} & PTB & PTB_\delta & \hat{S}^T\hat{C}_\delta^T\Lambda \\ B^T T^T P & -\xi I & 0 & 0 \\ B_\delta^T T^T P & 0 & -\Lambda & D_\delta^T\Lambda \\ \Lambda\hat{C}_\delta\hat{S} & 0 & \Lambda_\delta D & -\Lambda \end{bmatrix} < 0 \tag{8}$$

is satisfied for all $x \in \mathbf{R}^n$, the state-feedback law (5) renders the nonlinear system Σ input-to-state stable for all admissible uncertainties $\Sigma_\triangle$.

The characterization in the above theorem is addressed by a strict inequality. It can be replaced with a non-strict inequality $M^{sf} \le 0$. A control law satisfying the non-strict inequality assures the existence of appropriate ν, ξ, Λ for which $M^{sf} < 0$ is satisfied if Σ is well-posed for all admissible uncertainties.

When $q = 0$, the system Σ involves no uncertainty. In such a case, the above theorem reduces to the standard ISS with respect to the mapping between the disturbance w and x.

Corollary 1 *Suppose $q = 0$ holds. If there exist a positive definite matrix P and positive real numbers ν and ξ such that*

$$N^{sf}(x) = \begin{bmatrix} \hat{S}^T\hat{A}^T T^T P + PT\hat{A}\hat{S} + \nu P & PTB \\ B^T T^T P & -\xi I \end{bmatrix} < 0 \tag{9}$$

is satisfied for all $x \in \mathbf{R}^n$, the state-feedback law (5) renders the nonlinear system Σ input-to-state stable.

For linear systems, it is verified that the condition in Corollary 1 is satisfied if and only if there exist $\nu > 0$, $\xi > 0$, $\epsilon > 0$ and $P > 0$ such that

$$\left(A + GK + \frac{\nu}{2}I\right)^T P + P\left(A + GK + \frac{\nu}{2}I\right) + \xi^{-1}PBB^T P + \epsilon I = 0$$

By virtue of the theory of Riccati equations, the existence of the parameters (ν, ξ, ϵ, P) and K is guaranteed if and only if the pair (A, G) is stabilizable. This property is precisely the same as the fact that a linear closed-loop system is ISS if and only if $(A + GK)$ is a Hurwitz matrix[14].

Now, we focus on the existence of the state-feedback law and the construction of the controller solving the condition in Theorem 1. We shall prove the existence for the nonlinear system Σ satisfying the following structural assumptions.

$$A(x) = \begin{bmatrix} a_{11} & a_{12} & 0 & \cdots & \cdots & 0 \\ a_{21} & a_{22} & a_{23} & 0 & & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ & & & & \ddots & 0 \\ a_{n-1,1} & a_{n-1,2} & \cdots & \cdots & & a_{n-1,n} \\ a_{n1} & a_{n2} & \cdots & \cdots & & a_{nn} \end{bmatrix}, G(x) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ a_{n,n+1} \end{bmatrix} \tag{10}$$

$$B(x) = \begin{bmatrix} B_{11} & 0 & \cdots & 0 \\ B_{21} & B_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ B_{n1} & \cdots & B_{n,n-1} & B_{nn} \end{bmatrix} \tag{11}$$

$$a_{ij}(x) = a_{ij}(x_1, x_2, \cdots, x_i), \quad 1 \le i \le n,\ 1 \le j \le i+1 \tag{12}$$

$$a_{i,i+1}(x_1, x_2, \cdots, x_i) \ne 0, \quad 1 \le i \le n, \quad \forall x \in \mathbf{R}^n \tag{13}$$

$$B_{ij}(x) = B_{ij}(x_1, x_2, \cdots, x_i), \quad 1 \le i \le n,\ 1 \le j \le i \tag{14}$$

In addition, the system Σ is supposed to satisfy $m = 2n$ and

$$B_\delta(x) = \begin{bmatrix} B_{\delta,11} & U_{L1} & 0 & 0 & \cdots & 0 & 0 \\ B_{\delta,21} & U_{21} & B_{\delta,22} & U_{L2} & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ B_{\delta,n1} & U_{n1} & B_{\delta,n2} & U_{n2} & \cdots & B_{\delta,nn} & U_{Ln} \end{bmatrix} \tag{15}$$

$$C_\delta(x) = \begin{bmatrix} C_{\delta,11} & 0 & 0 & \cdots & 0 & 0 \\ 0 & U_{R1} & 0 & \cdots & 0 & 0 \\ C_{\delta,21} & C_{\delta,22} & 0 & \ddots & 0 & 0 \\ 0 & 0 & U_{R2} & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ C_{\delta,n-1,1} & C_{\delta,n-1,2} & \cdots & \cdots & C_{\delta,n-1,n-1} & 0 \\ 0 & 0 & \cdots & \cdots & 0 & U_{R,n-1} \\ C_{\delta,n1} & C_{\delta,n2} & \cdots & \cdots & C_{\delta,n,n-1} & C_{\delta,nn} \\ 0 & 0 & \cdots & \cdots & 0 & 0 \end{bmatrix} \tag{16}$$

$$D_\delta(x) = \begin{bmatrix} D_{\delta,1} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & D_{\delta,2} & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & D_{\delta,n} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}, H_\delta(x) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ U_{Rn} \end{bmatrix} \tag{17}$$

where $B_{\delta,ij} \in \mathbf{R}^{1\times q(2i-1)}$, $C_{\delta,ij} \in \mathbf{R}^{q(2i-1)\times 1}$, $D_{\delta,i} \in \mathbf{R}^{q(2i-1)\times q(2i-1)}$, $U_{L,i} \in \mathbf{R}^{1\times q2i}$ and $U_{R,i} \in \mathbf{R}^{q2i\times 1}$ satisfy

$$B_{\delta,ij}(x) = B_{\delta,ij}(x_{[i]}),\ C_{\delta,ij}(x) = C_{\delta,ij}(x_{[i]}) \tag{18}$$

$$U_{Li}(x) = U_{Li}(x_{[i]}),\ U_{Ri}(x) = U_{Ri}(x_{[i]}),\ U_{ij}(x) = U_{ij}(x_{[i]}) \tag{19}$$

$$D_{\delta,i}(x) = D_{\delta,i}(x_{[i]}),\ I - D_{\delta,i}(x_{[i]})D_{\delta,i}^T(x_{[i]}) > 0,\ \forall x \in \mathbf{R}^n \tag{20}$$

for $1 \le i \le n$ and $1 \le j \le i$. Let $x_{[k]}$ denote the first k components of the state:

$$x_{[k]} = [x_1, x_2, \cdots, x_k]^T\ .$$

We also make a standard assumption

$$a_{i,i+1}^2(x) > U_{Ri}(x)^T U_{Ri}(x) U_{Li}(x) U_{Li}^T(x), \quad \forall x \in \mathbf{R}^n \tag{21}$$

for $i = 1, 2, \ldots, n$ so that coefficients of virtual and actual inputs of Σ cannot be made zero by uncertainties[3]. The structure of Σ defined with (10) through (21) is called the robust strict-feedback form[6]. For the diffeomorphism between x and χ, we take

$$S(x) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -s_1 & 1 & 0 & \cdots & 0 \\ s_1 s_2 & -s_2 & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ (-1)^{n-1} s_1 \cdots s_{n-1} & \cdots & s_{n-2}s_{n-1} & -s_{n-1} & 1 \end{bmatrix} \quad (22)$$

Let the state-feedback be in the following form.

$$u = s_n(x)\chi_n \quad (23)$$

The smooth scalar functions $s_1(x_{[1]})$, $s_2(x_{[2]})$, $\cdots$, $s_n(x_{[n]})$ are to be designed from s_1 through s_n in a recursive manner. The state-dependent scaling is chosen as

$$\mathbf{L} = \left\{ \Lambda = \operatorname*{block-diag}_{i=1}^{2n} \Lambda_i \; : \; \begin{array}{cc} \Lambda_i = \lambda_i(x_{[(i+1)/2]}) I_{q_i} & \text{for odd } i \\ \Lambda_i = \lambda_i(x_{[i/2]}) I_{q_i} & \text{for even } i \\ \multicolumn{2}{c}{\lambda_i(x) > 0, \; \forall x \in \mathbf{R}^n} \end{array} \right\} \quad (24)$$

The following demonstrates that the solutions $\{s_1, \cdots, s_n\}$, $\{\lambda_1, \cdots, \lambda_{2n}\}$ and P of (8) always exist for any $\nu, \xi > 0$.

Theorem 2 *The system Σ in the robust strict-feedback form can be input-to-state stabilized by the state-feedback law (23) for all admissible uncertainties Σ_Δ.*

Due to the triangular structure of Σ, recursive construction of $\{s_k, \lambda_{2k-1}, \lambda_{2k}\}$ from $k = 1$ through $k = n$ is always feasible based on an idea which is similar to [6]. By using Schur Complements Formula, it follows from $\xi > 0$ that $M^{sf} < 0$ is identical with

$$\bar{M}^{sf} = \begin{bmatrix} \begin{pmatrix} \hat{S}^T \hat{A}^T T^T P + PT\hat{A}\hat{S} + \\ \nu P + \xi^{-1} PTBB^T T^T P \end{pmatrix} & PTB_\delta & \hat{S}^T \hat{C}_\delta^T \Lambda \\ B_\delta^T T^T P & -\Lambda & D_\delta^T \Lambda \\ \Lambda \hat{C}_\delta \hat{S} & \Lambda D_\delta & -\Lambda \end{bmatrix} < 0$$

Restricting P to being diagonal, we can pick n matrices $\bar{M}^{sf}_{[k]}$, $k = 1, \ldots, n$ of principal minors from the matrix $\bar{M}^{sf}$, which satisfy the following properties

(i) $\bar{M}^{sf}_{[k]}$ is independent of $\{x_{k+1}, x_{k+2}, \cdots, x_n\}$.

(ii) $\bar{M}^{sf}_{[k]}$ does not include $\{s_{k+1}, \cdots, s_{n-1}, s_n, \lambda_{2k+1}, \lambda_{2k+2}, \cdots, \lambda_{2n}\}$.

(iii) $\bar{M}^{sf}_{[k]} < 0$ implies $\bar{M}^{sf}_{[k-1]} < 0$.

(iv) $\bar{M}^{sf}_{[n]} = \bar{M}^{sf}$

On the assumption of $\bar{M}^{sf}_{[k-1]} < 0$, Schur Complements Formula reduces the problem of solving $\bar{M}^{sf}_{[k]} < 0$ for $\{s_k$, λ_{2k-1}, $\lambda_{2k}\}$ into simple inequalities equivalently.

$$\lambda_{2k-1} > 0, \quad \lambda_{2k} > 0 \quad (25)$$

$$-\lambda_{2k-1} \bar{F}_{k3} < (I - D_{\delta,k} D_{\delta,k}^T) \quad (26)$$

$$U_{Rk}^T U_{Rk} \lambda_{2k} s_k^2 + 2P_k a_{k,k+1} s_k + U_{Lk} U_{Lk}^T P_k^2 \lambda_{2k}^{-1} + P_k^2 \alpha < 0 \quad (27)$$

The $\mathcal{C}^0$ function $\bar{F}_{k3}(x_{[k]})$ which is semi-negative definite for all $x_{[k]}$ is independent of $\{s_k, \lambda_{2k-1}, \lambda_{2k}\}$. The $\mathcal{C}^0$ function $\alpha(x_{[k]})$ is independent of $\{s_k, \lambda_{2k}\}$. The set of inequalities(25-27) can be solved globally for smooth functions $\{s_k, \lambda_{2k-1}, \lambda_{2k}\}$ easily, so that the existence of the state-feedback law solving $M^{sf} < 0$ for all $x \in \mathbf{R}^n$ is proved. The computation of solving $\bar{M}^{sf}_{[k]} < 0$ directly is amenable to efficient algorithms of numerical optimization due to affine properties of $\bar{M}^{sf}_{[k]} < 0$ with respect to the decision parameters.

3 Output Feedback Stabilization

Consider another uncertain nonlinear system Σ described by

$$\Sigma : \begin{cases} \dot{x} = A(y)x + B(y)w + B_\delta(y)w_\delta + G(y)u \\ z_\delta = C_\delta(y)x \\ y = C_y x \end{cases} . \quad (28)$$

where C_y is a constant row vector, and $y(t) \in \mathbf{R}^1$ is the measurement output. Suppose that the state variable x cannot be measured. The uncertain system Σ_Δ is defined by (2) and (3). The uncertainty Σ_Δ is said to be admissible if (4) is satisfied for all $i = 1, \ldots, m$. We employ the following observer to estimate the state.

$$\begin{cases} \dot{\hat{x}} = A(y)\hat{x} + Y(y, \hat{x})(y - \hat{y}) + G(y)u \\ \hat{y} = C_y \hat{x} \end{cases} \quad (29)$$

This section seeks the output feedback control consisting of (29) and

$$u = K(y, \hat{x})\hat{x} \; . \quad (30)$$

Functions Y and K are $\mathcal{C}^0$ functions which have yet to be determined. For the output-feedback case, state-dependent scaling matrices are chosen as

$$\mathbf{L} = \left\{ \Lambda = \operatorname*{block-diag}_{i=1}^{m} \Lambda_i \; : \; \begin{array}{c} \Lambda_i = \lambda_i(y, \hat{x}) I_{q_i} \\ \lambda_i(y, \hat{x}) > 0, \; \forall (y, \hat{x}) \in \mathbf{R}^{n+1} \end{array} \right\} \quad (31)$$

Consider a global diffeomorphism between $[\hat{x}^T, \hat{x}^T - x^T]^T \in \mathbf{R}^{2n}$ and $[\hat{\chi}^T, \eta]^T \in \mathbf{R}^{2n}$ as follows:

$$\begin{bmatrix} \hat{\chi} \\ \eta \end{bmatrix} = \begin{bmatrix} S(y, \hat{x}) & 0 \\ 0 & W \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{x} - x \end{bmatrix} \quad (32)$$

The time-derivative of $\hat{\chi}$ is obtained as

$$\begin{aligned} \dot{\hat{\chi}} = & \left[\frac{\partial S}{\partial y_1}\hat{x}, \frac{\partial S}{\partial y_2}\hat{x}, \cdots, \frac{\partial S}{\partial y_n}\hat{x} \right] C_y \dot{x} + \left[\frac{\partial S}{\partial \hat{x}_1}\hat{x}, \frac{\partial S}{\partial \hat{x}_2}\hat{x}, \cdots, \frac{\partial S}{\partial \hat{x}_n}\hat{x} \right] \dot{\hat{x}} \\ & + S(y, \hat{x})\dot{\hat{x}} = X(y, \hat{x})\dot{x} + T(y, \hat{x})\dot{\hat{x}} \; . \end{aligned}$$

The square matrix W is constant and non-singular. The closed-loop system consisting of (28) and the output-feedback law (29-30) is represented on the new coordinate $(\hat{\chi}, \eta)$ by

$$\begin{aligned} \begin{bmatrix} \dot{\hat{\chi}} \\ \dot{\eta} \end{bmatrix} = & \begin{bmatrix} (X+T)\hat{A}\hat{S} & -(XA+TYC_y)W^{-1} \\ 0 & \hat{W}^T \bar{A} W^{-1} \end{bmatrix} \begin{bmatrix} \hat{\chi} \\ \eta \end{bmatrix} + \begin{bmatrix} XB \\ -WB \end{bmatrix} w \\ & + \begin{bmatrix} XB_\delta \\ -WB_\delta \end{bmatrix} w_\delta, \quad z_\delta = C_\delta \begin{bmatrix} S^{-1} & -W^{-1} \end{bmatrix} \begin{bmatrix} \hat{\chi} \\ \eta \end{bmatrix} \end{aligned}$$

$$\bar{A} = \begin{bmatrix} C_y^T & A^T \end{bmatrix}, \hat{W} = \begin{bmatrix} -Y^T W^T \\ W^T \end{bmatrix}, \hat{S} = \begin{bmatrix} S^{-1} \\ KS^{-1} \end{bmatrix}, \hat{A} = [A \quad G]$$

Theorem 3 *If there exist positive definite matrices P and $\tilde{P}$, positive real numbers ν, ξ, $\tilde{\nu}$ and a scaling function matrix $\Lambda \in \mathbf{L}$ such that*

$$M^{of}(y, \hat{x}) = \left[\begin{array}{ccccc} \begin{pmatrix} \hat{S}^T \hat{A}^T (X+T)^T P + \\ P(X+T)\hat{A}\hat{S} + \nu P \end{pmatrix} & PXB & PXB_\delta & S^{-T} C_\delta^T \Lambda & -P(XA+TYC_y)W^{-1} \\ B^T X^T P & -\xi I & 0 & 0 & -B^T W^T \tilde{P} \\ B_\delta^T X^T P & 0 & -\Lambda & 0 & -B_\delta^T W^T \tilde{P} \\ \Lambda C_\delta S^{-1} & 0 & 0 & -\Lambda & -\Lambda C_\delta W^{-1} \\ -W^{-T}(XA+TYC_y)^T P & -\tilde{P}WB & -\tilde{P}WB_\delta & -W^{-T} C_\delta^T \Lambda & \begin{pmatrix} W^{-T} \bar{A} \hat{W} \tilde{P} + \\ \tilde{P} \hat{W}^T \bar{A}^T W^{-1} + \tilde{\nu} \tilde{P} \end{pmatrix} \end{array} \right] < 0 \quad (33)$$

is satisfied for all $(y, \hat{x}) \in \mathbf{R}^{n+1}$, the output-feedback law (29-30) renders the nonlinear system Σ input-to-state stable for all admissible uncertainties Σ_Δ.

It can be verified that the strict inequality characterization in Theorem 3 can be rewritten by the non-strict inequality $M^{of} \leq 0$.

The block-component situated at the bottom right corner of M^{of} reveals that ISS requires the observer error dynamics by itself to have a certain level of robustness even if the system is free from uncertainties Δ_i. This situation contrasts with nominal asymptotic stabilization[11, 8]. Namely, conventional observer backstepping[11] based on cancellation of nonlinearity in error dynamics and linear observer design is not sufficient to assure ISS since its observer only assures global asymptotic stability of the error dynamics. If the nonlinear system involves uncertainties, the observer should be robust and we need the concept of robust observer[8].

Now we suppose that the system Σ satisfies the following triangular structure.

$$A(y)=\begin{bmatrix} a_{11} & a_{12} & 0 & \cdots\cdots & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots\ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ a_{n-1,1} & a_{n-1,2} & \cdots & \cdots & a_{n-1,n} \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{bmatrix}, \quad G(y)=\begin{bmatrix} 0 \\ \vdots \\ 0 \\ a_{n,n+1} \end{bmatrix} \tag{34}$$

$$a_{i,i+1}(y) \neq 0, \quad 1 \leq i \leq n, \quad \forall y \in \mathbf{R} \tag{35}$$

$$B(y) = \begin{bmatrix} B_{11} & 0 & \cdots & 0 \\ B_{21} & B_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ B_{n1} & \cdots & B_{n,n-1} & B_{nn} \end{bmatrix} \tag{36}$$

$$B_\delta(y) = \begin{bmatrix} B_{\delta,11} & 0 & \cdots & 0 \\ B_{\delta,21} & B_{\delta,22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ B_{\delta,n1} & \cdots & B_{\delta,n,n-1} & B_{\delta,nn} \end{bmatrix} \tag{37}$$

$$C_\delta(y) = \begin{bmatrix} C_{\delta,11} & 0 & \cdots & 0 \\ C_{\delta,21} & C_{\delta,22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ C_{\delta,n1} & \cdots & C_{\delta,n,n-1} & C_{\delta,nn} \end{bmatrix} \tag{38}$$

where $B_{\delta,ij}(y) \in \mathbf{R}^{1\times q_{(2i-1)}}$, $C_{\delta,ij}(y) \in \mathbf{R}^{q_{(2i-1)}\times 1}$ and $m = n$. The above matrices are dependent only on the output y so that this paper calls the structure of Σ the robust output-feedback form. Note that the class is more general than a standard output-feedback form[11] in which the nonlinearity is restricted to $A(y)x = A_0 x + \psi(y)$. We assume that the output equation of Σ is given by

$$y = x_1$$

or equivalently $C_y = [1\ 0 \cdots 0]$. This case is sometimes called output feedback in the nonlinear control literature[11]. We define $S(x_1, \hat{x})$ and the feedback gain as follows:

$$S^{-1}(x_1, \hat{x}_{[n-2]}) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ s_1 & 1 & 0 & \cdots & 0 \\ 0 & s_2 & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & s_{n-1} & 1 \end{bmatrix} \tag{39}$$

$$u = s_n(x_1, \hat{x}_{[n-1]})\hat{\chi}_n \tag{40}$$

The parameters $s_1(x_1)$, $s_2(x_1, \hat{x}_1)$, ..., $s_n(x_1, \hat{x}_{[n-1]})$ are smooth scalar-valued functions which are to be determined in a recursive manner from s_1 through s_n. Let W be

$$W = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ w_2 & 1 & 0 & \cdots & 0 \\ 0 & w_3 & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & w_n & 1 \end{bmatrix} \tag{41}$$

whose entries w_i for $2 \leq i \leq n$ are constant. Define the observer gain by

$$Y(x_1) = -W^{-1}\begin{bmatrix} w_1(x_1) \\ 0 \end{bmatrix} = -\begin{bmatrix} w_1 \\ -w_1 w_2 \\ \vdots \\ (-1)^{n-1} w_1 w_2 \cdots w_n \end{bmatrix} \tag{42}$$

where w_1 is a $\mathcal{C}^0$ function of x_1. The parameters $w_1, \cdots, w_n$ have yet to be determined recursively from $k = n$ through $k = 1$. The state-dependent scaling for the output-feedback problem is chosen as

$$\mathbf{L} = \left\{ \Lambda = \operatorname*{block-diag}_{i=1}^{n} \Lambda_i : \begin{array}{l} \Lambda_i = \lambda_i(y, \hat{x}_{[i-2]}) I_{q_i} > 0 \\ \forall (y, \hat{x}_{[i-2]}) \in \mathbf{R} \times \mathbf{R}^{i-2} \end{array} \right\} \tag{43}$$

We restrict our attention to the following class of systems.

Assumption 1 *The function $A(x_1)x$ satisfies*

$$A(x_1)x = A_0 x + \psi(x_1) + \phi(x_1)x_2 \tag{44}$$

with a constant matrix A_0 and $\mathcal{C}^0$ functions ψ and ϕ. There exist constants $\alpha_i > 0$ such that

$$\left| a_{i2}^2(x_1)/a_{12}(x_1) \right| \leq \alpha_i, \quad i = 2, 3, \ldots, n \tag{45}$$

hold for all $x_1 \in \mathbf{R}$. The matrix B satisfies

$$B(x_1) = \begin{bmatrix} B_{11}(x_1) & 0 & \cdots & 0 \\ 0 & B_{22}(x_1) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & B_{nn}(x_1) \end{bmatrix}, \; B_{ii}(x_1) \in \mathbf{R}^{1\times p_i} \tag{46}$$

and there exist constants $\beta_i > 0$ such that

$$\frac{B_{11}(x_1)B_{11}^T(x_1)}{\sqrt{a_{12}^2(x_1)}} \leq \beta_0, \quad \frac{B_{22}(x_1)B_{22}^T(x_1)}{\sqrt{a_{12}^2(x_1)}} \leq \beta_1, \quad \begin{array}{c} B_{ii}(x_1)B_{ii}^T(x_1) \leq \beta_i \\ i = 2, 3, \ldots, n \end{array} \tag{47}$$

The matrices B_δ and C_δ satisfy

$$B_\delta(x_1) = \begin{bmatrix} B_{\delta,11}(x_1) \\ B_{\delta,21}(x_1) \\ \vdots \\ B_{\delta,n1}(x_1) \end{bmatrix}, \quad C_\delta(x_1) = [\, C_{\delta,11}(x_1)\ 0 \cdots 0 \,] \tag{48}$$

where $B_{\delta,i1}(x_1) \in \mathbf{R}^{1\times q_1}$, $C_{\delta,11}(x_1) \in \mathbf{R}^{q_1\times 1}$ and $q_1 = q$.

This assumption is the same as that in [8]. It should be noted that the diagonal restriction (46) imposed on B does not cause any loss of generality. Indeed, an ISS problem with a triangular B can be recasted as another ISS problem with a diagonal B. We are now in a position to state the following theorem.

Theorem 4 *Under the Assumption 1, the system Σ in the output-feedback form can be input-to-state stabilized for all admissible uncertainties Σ_Δ by the output-feedback law (29-30) with (40) and (42).*

The proof of Theorem 4 needs some nontrivial modifications in the recursive procedure for observer-gain design established in [8] in addition to the feedback-gain design. The subproblems of feedback-gain design and observer-gain design are derived from the application of Schur Complements Formula to (33). We first determine the parameters w_k of the observer gain from $k = n$ down to $k = 1$. Then, the parameters $\{s_k, \lambda_k\}$ of feedback gain design solving $M^{of} < 0$ are determined from $k = 1$ up to $k = n$ in a recursive manner. It is only required to solve simple scalar-valued inequalities which are affine in w_k or $\{s_k, \lambda_k\}$ in each step of the recursive design.

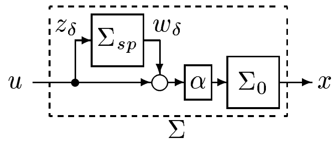

Figure 1: Nonlinear plant with input unmodeled dynamics

4 Integral Input-to-State Stabilization

Let $\nu = 0$ and $\tilde{\nu} = 0$ in the characterization $M^{sf} < 0$ and $M^{of} < 0$ of previous sections. Then, the time-derivative of the quadratic Lyapunov functions satisfies

$$\frac{d}{dt} V(x_{cl}) \leq \xi w^T w \tag{49}$$

where x_{cl} denotes the state of the entire closed-loop system. The inequality (49) implies that the closed-loop system is zero-output smoothly dissipative[1]. The closed-loop system is also proved to be 0-GAS since

$$\frac{d}{dt} V(x_{cl}) \leq x_{cl}^T \mathcal{M}(x_{cl}) x_{cl} \tag{50}$$

holds for $w \equiv 0$, and $\mathcal{M}(x_{cl}) < 0$ holds for all x_{cl}. Owing to the result of [1, 12], Theorem 1 and 3 guarantee iISS of the closed-loop systems when $\nu = 0$ and $\tilde{\nu} = 0$. Note that every input-to-state stable system is necessarily integral input-to-state stable but the converse is not true[12]. For linear systems, it is obvious that there exists $\nu > 0$ such that $M^{sf} < 0$ is satisfied if and only if $M^{sf} < 0$ is satisfied with $\nu = 0$. Similarly, the existence of $\nu > 0$ and $\tilde{\nu} > 0$ satisfying $M^{of} < 0$ also implies and is implied by $M^{of} < 0$ of $\nu = \tilde{\nu} = 0$. This fact explains exactly the equivalence between ISS and iISS property for linear systems[14].

5 Robustness for Passive Uncertainty

This section addresses the problem of designing controllers which remain input-to-state stabilizing in the presence of a certain class of dynamic uncertainties. The following is the definition of strict passivity[2].

Definition 1 *The system*

$$\Sigma_{sp} : \begin{cases} \dot{x}_\delta = f_\delta(x_\delta) + g_\delta(x_\delta) z_\delta \\ w_\delta = h_\delta(x_\delta), \qquad x_\delta(t) \in \mathbf{R}^{n_\delta} \end{cases} \tag{51}$$

is said to be strictly passive if there exist a $\mathcal{C}^1$ positive definite radially unbounded function $V_\delta(x_\delta)$ and a class $\mathcal{K}_\infty$ function $\psi(\cdot)$ such that

$$\int_0^t w_\delta^T z_\delta d\tau \geq V_\delta(x_\delta(t)) - V_\delta(x_\delta(0)) + \int_0^t \psi(\|x_\delta(\tau)\|) d\tau \tag{52}$$

for all $z_\delta \in \mathcal{C}^0$, $x_\delta(0) \in \mathbf{R}^n$ and $t \geq 0$.

Consider the uncertain system Σ shown in Fig1 in which $\Sigma_{sp} : z_\delta \mapsto w_\delta$ is a dynamic uncertainty which is assumed to be strictly passive. The system Σ is described by

$$\Sigma : \begin{cases} \dot{x} = A(x)x + B(x)w + G(x)\alpha(w_\delta + u) \\ z_\delta = u \end{cases} . \tag{53}$$

where α is a real number and $\alpha > 0$. We consider the following state-feedback control

$$u = K(x)x \tag{54}$$

and define the following functions.

$$\hat{S} = \begin{bmatrix} S^{-1} \\ KS^{-1} \end{bmatrix}, \quad \hat{A} = [A \ \alpha G]$$

We now introduce a new class of scaling matrices as follows:

$$\mathbf{L}_d = \left\{ \Lambda(s) = \lambda(s) I : \lambda \in \mathcal{C}^0,\ 0 < \lambda(s) \leq \bar{\lambda}, \forall s \in [0, \infty) \right\} \tag{55}$$

where $\bar{\lambda}$ is an arbitrary finite number. In particular, we are interested in $\Lambda(s)$ whose independent variable s is a quadratic function of χ. This new class of scaling is different from state-dependent scaling for static uncertainties in that it is uniformly bounded. This new class of scaling enables us to establish the input-to-state stabilization in the presence of the dynamic input uncertainty.

Theorem 5 *Given any $\alpha_l > 0$, the uncertain system consisting of (53) and (51) is input-to-state stabilized by a state feedback law (54) for all $\alpha \in [\alpha_l, \infty)$ if there exist a positive definite matrix P and positive real numbers ν, ξ and a scaling function $\Lambda \in \mathbf{L}_d$ such that*

$$M^{sp}(x) = \begin{bmatrix} \hat{S}^T \hat{A}^T T^T P + PT\hat{A}\hat{S} + \nu P & PTB \\ B^T T^T P & -\xi I \end{bmatrix} \leq 0 \tag{56}$$

$$PTG + S^{-T} K^T \Lambda = 0 \tag{57}$$

are satisfied with $s = x^T S^T PSx$ for all $x \in \mathbf{R}^n$ and all $\alpha \in [\alpha_l, \infty)$.

The theorem is proved by employing the Lyapunov function

$$V(x_{cl}) = \int_0^{V_0(x)} \frac{1}{\lambda(s)} ds + 2\alpha V_\delta(x_\delta), \quad V_0(x) = \chi^T P \chi \tag{58}$$

where $x_{cl} = [x^T, x_\delta^T]^T$ and P is a positive definite matrix.

Next, we show that a controller which fulfills (56) and (57) can be always constructed if Σ is in the strict-feedback form. Suppose that the matrices $A(x)$, $B(x)$ and $G(x)$ are given as (10-14). Let the state-feedback law be (23) and P is a diagonal matrix. Then, the equation (57) reduces to

$$\begin{bmatrix} 0 \\ P_n a_{n,n+1} \end{bmatrix} + \begin{bmatrix} 0 \\ s_n \lambda \end{bmatrix} = 0$$

Thus, for the feedback gain, we pick

$$s_n = -\frac{P_n a_{n,n+1}}{\lambda} \tag{59}$$

By virtue of the development in [6], the condition $M^{sp} < 0$ is satisfied if

$$\begin{array}{ll} 2P_k a_{k,k+1} s_k - \beta_k(x_{[k]}) < 0, & \text{for } k = 1, 2, \ldots, n-1 \\ 2P_k \alpha a_{k,k+1} s_k - \beta_k(x_{[k]}) < 0, & \text{for } k = n \end{array} \tag{60}$$

are achieved by finding $s_k(x_{[k]})$ recursively from $k = 1$ through $k = n$. The function $\beta_k(x)$ is an appropriate $\mathcal{C}^0$ function which is independent of $\{s_k, \cdots, s_n\}$. Since $a_{k,k+1}(x_{[k]})$ are positive and $P_k, \alpha > 0$, there always exist $\{s_1(x_{[1]}), \cdots, s_{n-1}(x_{[n-1]})\}$ satisfying (60). As for $k = n$, substituting (59) into (60), we obtain

$$2\alpha P_n^2 a_{n,n+1}^2 > \lambda \beta_n(x), \quad \forall x \in \mathbf{R}^n \tag{61}$$

It is seen that there exits a $\mathcal{C}^0$ function $\lambda(\chi^T P \chi)$ such that

$$2\alpha_l P_n^2 a_{n,n+1}^2 > \lambda \beta_n(x), \quad \forall x \in \mathbf{R}^n \tag{62}$$

$$0 < \lambda(\chi^T P \chi) < \bar{\lambda}, \quad \forall x \in \mathbf{R}^n \tag{63}$$

are satisfied with a finite number $\bar{\lambda}$. It should be noted that s_n and λ are independent of α. To summarize the above discussion, we state the following theorem.

Theorem 6 *Suppose that the system (53) is in the strict-feedback form. Given any $\alpha_l > 0$, the uncertain system consisting of (53) and (51) can be always input-to-state stabilized by a state feedback law (59) for all $\alpha \in [\alpha_l, \infty)$.*

An important point of the above theorem is that the ISS can be achieved robustly by using the state-dependent scaling and the Schur complements formula recursively. This feature is quite different from, for example, the development[10] where the Legendre-Fenchel transform and Young's Inequality are employed to prove ISS in the presence of the passive uncertainty. It is also interesting that the state-dependent scaling approach is able to construct an inverse optimal controller without referring to the Sontag-type controller[7].

According to Theorem 6, by letting $\alpha_l \to 0$, we can make the stability margin extremely large, which means the gain margin tends to $(0, \infty)$ and the phase margin tends to 90°. However, we should be careful that the gain of the control law can be harmfully very high, according to (59) and (62). For output-feedback control, it is generally known that the state-feedback/observer design reduces stability margins. It is possible to characterize the reduced margins in the case of the output-feedback by restricting the set of uncertain dynamics and uncertain parameters accordingly.

The introduction of the new type of scaling (55) is crucial for establishing the input-to-state stability in the presence of input unmodeled dynamics. If the scaling is replaced by the unbounded one (7), the ISS is not guaranteed in the presence of dynamic uncertainties. If the scaling is replaced by constant scaling, in general, the condition (61) cannot be met globally for nonlinear systems. Thus, the new scaling (55) and the creation of a new type of Lyapunov functions (58) from the scaling are significant.

6 Concluding Remarks

In this paper, the input-to-state stabilization and the integral input-to-state stabilization have been characterized by using the state-dependent scaling and diffeomorphism exclusively. The recursive design procedure presented is based on recursive application of the Schur complements formula to the characterization. This paper use neither Young's formula nor completing the squares which are usually conservative than the Schur complements formula[8]. All developments in this paper only use the state-dependent scaling, the diffeomorphism and the Schur complements, and combination of them has been found useful in dealing with ISS and iISS problems. The systems are allowed to have uncertain parameters and dynamics. For input unmodeled dynamics, a new class of state-dependent scaling has been introduced to create Lyapunov functions of a new type in the SD scaling design.

The ISS has been also achieved by output feedback. In contrast to asymptotic stabilization of nominal systems[11, 8], the ISS requires the observer to have a certain level of robustness. Conventional observer backstepping[11] based on cancellation of nonlinearity in error dynamics and linear observer design is not sufficient. In order to construct such a robust observer, this paper has employed a recursive procedure whose order is reverse of backstepping for feedback-gain design.

Corollary 1 and Theorem 3 of this paper can be regarded as improved versions of the input-to-state stabilization results presented in [7, 8]. The key difference is that this paper does not introduce unnecessary fictitious output functions which previous papers[7, 8] used as free parameters. The characteristic matrix N^{sf} in Corollary 1 does not have any fictitious output and scaling matrices, while the previous papers augment the characteristic matrix by including fictitious output channels and corresponding scaling matrices. Using the Schur complements formula, it can be seen that the indirect design in [7, 8] tends to require more effort of control than the method of this paper to make the characteristic matrix negative definite. In addition, the characterization presented in this paper offers more flexibility to deal with advanced problems such as ISS problems of uncertain systems addressed in this paper.

References

[1] D. Angeli, E.D. Sontag and Y. Wang, "A remark on integral input to state stability," *Proc. Amer. Contr. Conf.*, June 1998, pp.2491-2496.

[2] C.I. Byrnes, A. Isidori and J. C. Willems, "Passivity, feedback equivalence, and the global stabilization of minimum phase nonlinear systems," *IEEE Trans. Automat. Contr.*, vol 36, pp.1228–1240, 1991.

[3] R. A. Freeman and P. V. Kokotović, *Robust nonlinear control design: State-space and Lyapunov techniques*, Birkhäuser, Boston, 1996.

[4] H. Ito, "Local stability and performance robustness of nonlinear systems with structured uncertainty," *IEEE Trans. Automat. Contr.*, vol 44, pp.1250-1254, 1999.

[5] H. Ito, "Solutions and characterizations of input-to-state stabilization via state-dependent scaling design," *Tech. Report in Computer Sci. and Syst. Eng.*, CSSE-6, ISSN 1344-8803, Kyushu Inst. of Tech., Nov. 10, 1999.

[6] H. Ito and R.A. Freeman, "A new look at robust backstepping through state-dependent scaling design," *Proc. 1999 Amer. Contr. Conf.*, June, 1999, pp.11-16.

[7] H. Ito and R.A. Freeman, "Generalized state-dependent scaling: Backstepping for local optimality, global inverse optimality, and global robust stability," *Proc. 1999 European Contr. Conf.*, August, 1999, pp.AP-8.

[8] H. Ito and M. Krstić, "Recursive scaling design for robust global nonlinear stabilization via output feedback," *Int. J. Robust and Nonlinear Contr.*, vol.10, pp.821-848, 2000.

[9] Z.-P. Jiang and A.R. Teel and L. Praly, "Small-gain theorem for ISS systems and applications," *Mathe. Contr. Signals and Syst.*, vol 7, pp.95-120, 1994.

[10] M. Krstić and H. Deng, *Stabilization of nonlinear uncertain systems*, Springer-Verlag, New York, 1998.

[11] M. Krstić, I. Kanellakopoulos and P. V. Kokotović, *Nonlinear and adaptive control design*, John Wiley & Sons, New York, 1995.

[12] D. Liberzon, E.D. Sontag and Y. Wang, "On integral-input-to-state stabilization," *Proc. Amer. Contr. Conf.*, June 1999, pp.1598-1602.

[13] E.D. Sontag, "Smooth stabilization implies coprime factrization," *IEEE Trans. Automat. Contr.*, vol. 34, pp.435-443, 1989.

[14] E.D. Sontag, "Comments on integral variants of ISS," *Systems & Control Letters*, vol. 34, pp.93-100, 1998.

[15] E.D. Sontag and Y. Wang, , "On characterization of the input-to-state stability property," *Systems & Control Letters*, vol. 24, pp.351-359, 1995.

[16] A.R. Teel, "A nonlinear small gain theorem for the analysis of control systems with saturation," *IEEE Trans. Automat. Contr.*, vol 41, pp.1256-1270, 1996.