Strictly Positive Real Lemma for Discrete-time Descriptor Systems [1]

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Abstract

In this paper, the strictly positive real (SPR) lemma for discrete-time descriptor systems is addressed. The relationship between strict positive realness and bounded realness (BR) of transfer matrices for such systems is developed. Based on this development, an LMI-based necessary and sufficient condition for a descriptor system being admissible and simultaneously SPR is established.

1 Introduction

Since the (strictly) positive real lemma has many important applications in the control of continuous-time and discrete-time state-space systems [1] and the descriptor system model is a natural mathematical representation for many practical dynamic systems due to its ability to describe the dynamic as well as the algebraic relationships between the chosen descriptor variables [2], the development of the (strictly) positive real lemma for descriptor systems becomes an essential and attractive topic. The continuous-time case is studied in [3], where a necessary and sufficient condition in a generalized algebraic Riccati equation has been derived for the *extended* strict positive realness, which a stronger sense of the strict positive realness. However, to our understanding, no any result about the (strictly) positive real condition for the *discrete*-time descriptor systems has been appeared in the literature yet.

In this paper, the strictly positive real lemma for discrete-time descriptor systems is derived. We study connection between the strictly positive real lemma and the bounded real lemma for the considered descriptor systems. Based on the generalized bounded real lemma obtained in [4] and the definition of strict positive realness, we propose a necessary and sufficient condition in LMIs to such a problem.

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2 Preliminaries

We consider the following discrete-time descriptor system:

$$\begin{array}{rcl} Ex(k+1) & = & Ax(k) + Bw(k) \\ y(k) & = & Cx(k) + Dw(k) \end{array} \qquad (1)$$

where $x \in \mathbb{R}^n$ is the descriptor variable, $w \in \mathbb{R}^{n_w}$ is the exogenous input, and $y \in \mathbb{R}^{n_y}$ is the measured output. The matrix $E \in \mathbb{R}^{n\times n}$ has $rank E \leq n$, thus may be singular. The other matrices have appropriate sizes. For brevity, we will use (E, A, B, C, D) to denote such a descriptor system. If we consider the state properties only, this notation can be simplified to be (E, A).

Some important features concerning the study of descriptor systems are recalled from [2, 5]. When a pair (E, A) is called *regular*, it meets the requirement that, for any specified initial condition, solution to (1) exists and is unique. We say a pair (E, A) is *impulse-free* if the impulsive behavior of the solution possibly appearing at the initial time due to inconsistent initial conditions is avoided. Finally, a pair (E, A) is called *stable* if all finite generalized eigenvalues of the pair lie in the stable region, i.e. the interior of unit disk. For short, we call a pair (E, A) *admissible* if it is both regular, impulse-free, and stable.

If the descriptor system is regular, the transfer matrix from w to y can be uniquely defined as

$$T_{yw} := C(zE - A)^{-1}B + D.$$

In the following, we give the definitions about strict positive realness and bounded realness of real rational transfer matrices.

Definition 1 [6] *Let $H(z)$ be a square real rational transfer matrix in z. $H(z)$ is said to be strictly positive real (SPR) if it is analytic in $|z| \geq 1$ and $H(e^{j\theta}) + H^T(e^{-j\theta}) > 0$ for all $\theta \in [0, 2\pi]$.*

Definition 2 *The real rational transfer matrix $G(z)$ is called bounded real if all elements of $G(z)$ are analytic* *in $|z| \geq 1$ and $I - G(e^{j\theta})G^T(e^{-j\theta}) > 0$ for all $\theta \in [0, 2\pi]$. Equivalently, $G(z)$ is bounded real if it is stable with H_∞-norm strictly bounded by 1.*

3 Main results

In this section, a version of SPR lemma for discrete-time descriptor systems is formulated as an LMI feasibility problem. The proving strategy is similar to that used in [7], where continuous-time state-space systems are considered. We begin with a result that indicates a bilateral relationship between SPR matrices and BR matrices via the bilinear transformation.

Lemma 1 *Let $G(z)$ and $H(z)$ be two square transfer matrices of the same dimension related by*

$$G(z) := (H(z) - I)(H(z) + I)^{-1} \tag{2}$$

or equivalently

$$H(z) := (I - G(z))^{-1}(I + G(z)) \tag{3}$$

provided that the two inverses exist. Then $G(z)$ is bounded real if and only if $H(z)$ is SPR.

Based on this lemma, one theorem is stated below.

Theorem 1 *Let $H(z)$ denote the transfer matrix of the descriptor system (1), with $D + D^T \geq 0$, and let $G(z)$ be defined by (2). If the pair (E, A) is admissible and $H(z)$ is SPR, then $G(z)$ has a realization $(E, \hat{A}, \hat{B}, \hat{C}, \hat{D})$ where*

$$\hat{A} = A - B(D+I)^{-1}C \qquad \hat{B} = \sqrt{2}B(D+I)^{-1}$$
$$\hat{C} = \sqrt{2}(D+I)^{-1}C \qquad \hat{D} = (D-I)(D+I)^{-1}$$

and the pair $(E, \hat{A})$ is admissible.

In terms of the generalized bounded real lemma obtained in [4] and Theorem 1, the strictly positive real lemma for discrete-time descriptor systems (1) is characterized by a set of LMIs in the next theorem. It should be noted that the pair (E, A) is not assumed to be regular and impulse-free a priori.

Theorem 2 *Assume that $D + D^T \geq 0$. The following statements are equivalent*

(I) (E, A) is admissible and T_{yw} is SPR.

(II) There exists a matrix $P = P^T \in \mathbb{R}^{n \times n}$ satisfying the following LMIs

$$\begin{bmatrix} A^TPA - E^TPE & A^TPB - C^T \\ B^TPA - C & -D - D^T + B^TPB \end{bmatrix} < 0,$$
$$E^TPE \geq 0.$$

When setting $E = I$ in Theorem 2, the SPR lemma for discrete-time *state-space* systems is given below.

Corollary 1 *The transfer matrix $H(z) := C(zI - A)^{-1}B + D$ is SPR and matrix A is stable if and only if there exists a matrix $P = P^T > 0$ such that*

$$\begin{bmatrix} A^TPA - P & A^TPB - C^T \\ B^TPA - C & -D - D^T + B^TPB \end{bmatrix} < 0.$$

4 Conclusion

In this paper, based on the derived relationship between SPR and BR transfer matrices, the SPR lemma for discrete-time descriptor systems is formulated as an LMI feasibility problem. Therefore, the admissibility and the SPR property of a discrete-time descriptor system can be easily tested by using currently available software packages for solving problems in LMIs. The result also recovers the SPR lemma for discrete-time *state-space* systems by setting $E = I$. This research builds a foundation of LMI-based controller design in many control problems for discrete-time descriptor systems.

References

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