# A projector to design a passive-based feedback

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**Abstract**

This paper deals with the decomposition of the drift term of nonlinear multivariable regular systems. A transverse and tangent decomposition of the vector field is presented, then a feedback neutralizing the transverse part is studied. Zero dynamics stability is considered and sufficient conditions to obtain a global passivity of the system are given. An other decomposition based on the workless field; attracting field; and rejecting field is also studied. Some illustrative examples are given all along the paper.

**Key Words:** Structural analysis, Control Design, Stability, Passivity, Nonlinear multivariable systems.

## 1 Introduction

In this paper, our main motivation is to design a very simple feedback in order to “stabilize” a regular square input-affine system (transform the system into a passive one). For this an adequate projector allows us to analyze the original structural dissipative properties of the system and after design a feedback.

As it is well known, the notion of passivity was first introduced in the electrical network theory literature ([4, 7]) and also in the area of robotics[5, 6]. A more general form of passivity is called "dissipativity"[14], which is closely related to a rather intuitive phenomena of energy loss of a physical system. It has been shown that the feedback interconnection of two passive systems is stable irrespective of physical parametric uncertainties in the two systems. A geometric approach to feedback equivalence of nonlinear passive systems was contributed in [1]. Unfortunately, there are many systems which are not naturally passive and this stability result can not be extended directly to these systems. One possible way to overcome this problem is to design a suitable pre-feedback which make passive these systems.

In [8], in the case of mono-variable systems, a projector was introduced to obtain a canonical decomposition of the drift vector field as : the *workless* field; the *attracting* field; and the *rejecting* field. This approach has found application in the robotics and mechanical systems. Moreover, in [9, 10] one finds a generalization of this projector to multivariable systems but with a single output which is a storage function. Here we generalize it by considering multi outputs instead of single output. Other decomposition of the drift vector field is to rewrite it into the *tangent part* and the *transverse part* ([8]). This is done, in our case, in order to design a pre-feedback which transforms the system in a passive one.

The outline of the paper is as follows: in section 2, firstly, a transverse and tangent decomposition of the vector field is presented, leading to the design of a feedback which neutralizes the transverse part. Secondly, from the structure of the tangent part of the drift vector field, we analyzes the stabilization of the zero dynamics and give sufficient conditions in order to obtain a global passivity of the system after a static feedback. We give in section 4 an other decomposition of this vector into a *workless*; *attracting*; and *rejecting* field. Some illustrative examples are given along the paper.

## 2 A Tangent and Transverse decomposition

### 2.1 Notations and assumptions

Consider the regular square input-affine system :

$$\begin{aligned} \dot{x} &= f(x) + \sum_{i=1}^{m} g_i(x) u_i \qquad (1)\\ y_i &= \sigma_i(x) \qquad \text{for } 1 \leq i \leq m \leq n \end{aligned}$$

where $x \in \mathcal{X} \subset \Re^n$; $u \in \mathcal{U} \subset \Re^m$, $y \in \mathcal{Y} \subset \Re^m$, $g_i$ and $f$ are vector fields functions on $\mathcal{X} \in \Re^n$, and $\sigma_i(x)$ are functions on $\mathcal{X} \in \Re^n$. All the functions are assumed to be sufficiently smooth. We denote by $S$ the working surface

$$S = \bigcap_{i=1}^{m} \sigma_i^{-1}(0) = \{x \in \mathcal{X}\,;\, \sigma_i(x) = 0; \forall\, i = 1, .., m\} \qquad (2)$$

and its tangent space at a point $x$ by $T_xS = \bigcap_{i=1}^{m} \ker d\sigma_i(x)$.

Note $G(x) = [g_1(x), ..., g_m(x)]$ a $n \times m$ matrix, $A(x) = \left[L_{g_j}\sigma_i(x)\right]_{1\leq i,j\leq m}$ (the so called decoupling matrix) and $\nabla\sigma = [\nabla\sigma_1(x), ..., \nabla\sigma_m(x)]$ a $n \times m$ matrix where $\nabla\sigma_i(x) = \left(\dfrac{\partial\sigma_i}{\partial x_1}, ....., \dfrac{\partial\sigma_i}{\partial x_m}\right)^T$ is the gradient of $\sigma_i$. It is easy to verify that $A(x) = [\nabla\sigma(x)]^T G(x)$. We denote $e_k = [0, ...., 0, 1, 0, ..., 0]^T \in \Re^m$ (the $k^{th}$ component of the vector $e_k$ is equal to 1).

We suppose, throughout the paper, that the following hypothesis holds :

**Hypothesis 1**

1) *The vector fields $g_i$ are linearly independent.*

2) *The functions $\sigma_i$ are linearly independent such that*

$$\forall\, v \neq 0 \in span\,\{g_1(x), ..., g_m(x)\} : v \notin T_xS \qquad (3)$$

*which is called the "Transversality Condition".*

The point 2) means that $span\{g_1,...,g_m\}$ is transverse to the level surface $S$ which is of dimension $n-m$.

**Lemma 1** *The following conditions are equivalent*

*1) $span\{g_1,...,g_m\}$ is transverse to the level surface $S$*

*2) the matrix $A(x) = [L_{g_i}\sigma_k(x)]_{1\le k,i\le m}$ is invertible*

*3) there exists a regular static state feedback*

$$u = \alpha(x) + \beta(x)v$$

*such that, $d\sigma_i(\widetilde{g}_i) = 1$ and $d\sigma_i(\widetilde{g}_j) = 0$ for $i \neq j$, which means : $\widetilde{A}(x) = [L_{\widetilde{g}_i}\sigma_k(x)]_{1\le k,i\le m} = I_m$, where*

$$\widetilde{f}(x) = f(x) + \sum_{j=1}^{m}\alpha_j(x)g_j(x); \quad \widetilde{g}_i(x) = \sum_{j=1}^{m}\beta_{ji}(x)g_j(x)$$

*4) the modified system is input-output decoupled, that means : the derivative $\dot{\sigma}_i = L_f\sigma_i + L_{\widetilde{g}_i}\sigma_i v_i$ doesn't involve the input $v_j$ for $j \neq i$.*

It is well known that point 2) is equivalent to 3) and equivalent to 4) (see for example [3]). The point 1) is a geometric interpretation for decoupling technique. This is proved in the next:

**Proof:** 1) $A(x)$ is singular if and only if there exist $i \le m$ and $a_1.....,a_{i-1},a_{i+1},....,a_m \in R$, such that for each $k = 1:m$

$$d\sigma_k(g_i - \sum_{j\neq i}^{m}a_jg_j) = 0 \Leftrightarrow \left(g_i - \sum_{j\neq i}^{m}a_jg_j\right) \in \bigcap_{i=1}^{m}\ker d\sigma_i$$

$\Leftrightarrow$ $span\{g_1,...,g_m\}$ is not transverse to $S$. ∎

**Remark 1** *The regular static state feedback in 3) of lemma (1) is*

$$\begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} = -\left(A(x)\right)^{-1}\left(\begin{bmatrix} L_fy_1 \\ \vdots \\ L_fy_m \end{bmatrix} - \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix}\right)$$

*leading to : $\widetilde{f} = f(x) - (A(x))^{-1}L_f\nabla\sigma^T$ and $\widetilde{G}(x) = G(x)A(x)^{-1}$ then $\widetilde{A}(x) = \nabla\sigma^TG(x)A(x)^{-1} = I_m$.*

### 2.2 Construction of the projector

Thanks to the hypothesis 1, the following proposition points out the properties of the projector $M(x)$ which will be used to obtain the desired decomposition :

**Proposition 1** *Consider the $n\times n$ matrix*

$$M(x) = I_n - G(x)\left[A(x)\right]^{-1}\nabla\sigma(x)^T \quad (4)$$

*then $M(x)$ is a projector along the $span\{g_1(x),...,g_m(x)\}$ on the tangent space $T_xS$ of $S$, which is equivalent to*

*i)- $M(x)\left[I_n - M(x)\right] = 0$*

*ii)- $M(x)g_k = 0$ for all $1\le k\le m$*

*iii)- $M(x)^T\nabla\sigma_k = 0$ for all $1\le k\le m$*

**Proof:** *i)* as $\nabla\sigma^TG(x) = A(x)$ we have

$$\begin{aligned} & M(x)\left[I_n - M(x)\right] \\ = & \left[I_n - G(x)A(x)^{-1}\nabla\sigma^T\right]G(x)A(x)^{-1}\nabla\sigma^T \\ = & G(x)A(x)^{-1}\nabla\sigma^T - G(x)A(x)^{-1}\nabla\sigma^T = 0 \end{aligned}$$

*ii)* as $\nabla\sigma_k(x) = \nabla\sigma(x)e_k$ and $g_k(x) = G(x)e_k$,

we obtain

$$\begin{aligned} & M(x)g_k(x) \\ = & \left[G(x) - G(x)A(x)^{-1}\nabla\sigma^TG(x)\right]e_k \\ = & 0 \qquad 1\le k\le m \end{aligned}$$

*iii)* as $G(x)^T\nabla\sigma = A(x)^T$, we get

$$\begin{aligned} & M(x)^T\nabla\sigma_k(x) \\ = & \left[I_n - \nabla\sigma\left[A(x)^{-1}\right]^TG(x)^T\right]\nabla\sigma(x)e_k \\ = & \left[\nabla\sigma(x) - \nabla\sigma(x)\left(A(x)^{-1}\right)^TG(x)^T\nabla\sigma(x)\right]e_k \\ = & 0 \qquad 1\le k\le m \end{aligned}$$

∎

Thanks to the projector $M(x)$ given in (4), $\forall\, x\in\mathcal{X}$,

$$\begin{aligned} T_x\mathcal{X} &= Ker[M(x)] \oplus Ker[I_n - M(x)] \\ &= span\{g_1,...,g_m\} \oplus T_xS \end{aligned}$$

using this decomposition we obtain the following corollary

**Corollary 1** *The drift vector field $f(x)$ is decomposed into tangent part and transverse part with respect to $S$ as*

$$f(x) = \underbrace{M(x)f(x)}_{tangent\ part} + \underbrace{(I_n - M(x))f(x)}_{transverse\ part} \quad (5)$$

*i.e. $M(x)f(x) \in T_xS$ and $(I_n - M(x))f(x) \in span\{g_1,...,g_m\}$. The system (1) can be rewritten as*

$$\dot{x} = M(x)f(x) + G(x)[A(x)^{-1}\nabla\sigma^Tf(x) + u] \quad (6)$$

### 2.3 Neutralization of the transverse part

The following proposition gives a feedback which neutralize the transverse part of the drift vector field $f(x)$ in 6.

**Proposition 2** *The regular static state feedback*

$$u = A(x)^{-1}\left[v - (\nabla\sigma(x))^T f(x)\right] \tag{7}$$

*transforms the system (1) into*

$$\dot{x} = M(x)f(x) + G(x)A(x)^{-1}v \tag{8}$$

**Proof:** From (6) and the feedback (7) we have

$$\begin{aligned}\dot{x} &= M(x)f(x) + G(x)A(x)^{-1}\nabla\sigma^T f(x)\\ &\quad + G(x)A(x)^{-1}\left[v - \nabla\sigma^T f(x)\right]\\ &= M(x)f(x) + G(x)A(x)^{-1}v\end{aligned}$$

■

Let us consider the "degenerate" storage function $V = \frac{1}{2}\sum_{i=1}^{m}\sigma_i^2$ (V is a semi definite positive function). Using the feedback (7) and property iii) of proposition 1, one obtain a property of passivity (see definition 10.4 page 439 in [2]).

**Corollary 2** *Let* $V = \frac{1}{2}\sum_{i=1}^{m}\sigma_i^2$ *then* $\dot{V} = \sum_{i=1}^{m}\sigma_i v_i$

**Proof:** Tacking derivative by time of the degenerate storage function $V$ : $\dot{V} = \sum_{i=1}^{m}\sigma_i\nabla\sigma_i^T\left[M(x)f(x) + G(x)A(x)^{-1}v\right] = \sum_{i=1}^{m}\sigma_i\nabla\sigma_i^T G(x)A(x)^{-1}v$ because $\nabla\sigma_i^T M(x)f(x) = 0$, and as $\nabla\sigma_i = \nabla\sigma e_i$, this gives

$$\begin{aligned}\dot{V} &= \sum_{i=1}^{m}\sigma_i\nabla\sigma_i^T G(x)A(x)^{-1}v\\ &= \sum_{i=1}^{m}\sigma_i e_i^T\underbrace{\nabla\sigma^T G(x)}_{A(x)}A(x)^{-1}v = \sum_{i=1}^{m}\sigma_i v_i\end{aligned}$$

■

**Remarks 1**

*1) if $m = n$, then $T_xS = \{0\}$, we have lossless passivity property, with a positive definite storage function $V(x)$.*

*2) if $m < n$, the corollary is not sufficient to guarantee the global passivity because it doesn't take into account the global state behavior (the function $V(x)$ is only positive semi definite). Consequently, some peaking phenomena may appear (see [11]). In fact the problem occurs on the tangent space when the manifold $S$ is not yet reached.*

### 2.4 Illustrative examples

The basic philosophy of the proposed approach is explained by some examples.

**Example 1** ***Chemical reactor*** *: Let us consider the continuous exothermic chemical reactor example used in [10], and in which a first order and exothermic reaction $A \longrightarrow B$ occurs:*

$$\begin{aligned}\dot{x} &= f(x) + G(x)u\\ y_1 &= \sigma_1(x) = x_1\\ y_2 &= \sigma_2(x) = x_3\end{aligned} \tag{9}$$

$$\text{with}\quad f(x) = \begin{pmatrix} f_1\\ f_2\\ f_3\end{pmatrix} = \begin{pmatrix} -k(x_3)x_1 - \beta x_1\\ k(x_3)x_1 - \beta x_2\\ \alpha k(x_3)x_1 - qx_3\end{pmatrix}$$

$$\text{and}\ G(x) = [g_1, g_2] = \begin{pmatrix}\beta & 0\\ 0 & 0\\ 0 & 1\end{pmatrix}$$

*where $x = [x_1, x_2, x_3]^T$, where $x_1$ is the concentration of the reactant $A$ and $x_2$ is the concentration of the product $B$. The variable $x_3$ represents the reactor temperature. $\beta > 0$ is a constant associated with the dilution rate while $\alpha > 0$ is the exothermicity of the concentration of the reactant $A$ in the feed flow. And we have also the constant $q > 0$, the function $k(x_3) = k_0\exp(-\frac{k_1}{x_3})$. and the input $u = [u_1, u_2]^T$.*

*Now, we define the decoupling matrix*

$$A(x) = [\nabla\sigma(x)]^T G(x) = \begin{pmatrix}\beta & 0\\ 0 & 1\end{pmatrix}$$

*and*

$$\begin{aligned}M(x) &= I_3 - G(x)\left[A(x)\right]^{-1}\left[\nabla\sigma(x)\right]^T\\ &= \begin{pmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}\end{aligned}$$

*this represents the projection on the second variable $x_2$. Considering the feedback (7) the system (9) becomes*

$$\begin{aligned}\dot{x}_1 &= v_1\\ \dot{x}_2 &= k(x_3)x_1 - \beta x_2\\ \dot{x}_3 &= v_2\end{aligned}$$

*Considering the degenerate storage function $V = \frac{1}{2}\sum_{i=1}^{2}\sigma_i(x)^2 = \frac{1}{2}(x_1^2 + x_3^2)$ one obtains*

$$\begin{aligned}\dot{V} &= x_1v_1 + x_3v_2\\ &= y^T v\end{aligned}$$

**Example 2** ***Lagrangian system :*** *Let $x \in R^n$, $y \in R^n$ and consider the following system*

$$\begin{cases} \dot{x} = y \\ \dot{y} = f(x,y) + \tilde{G}(x)u \\ \sigma_i = y_i \end{cases}$$

*It is assumed that $\tilde{G}(x)$ is $n \times n$ symmetric definite positive matrix (in the case of a mechanical system $\tilde{G}(x)^{-1}$ is the inertia matrix). Then the decoupling matrix is $A(x) = \tilde{G}(x)$ which implies that the projector $M(x) = I_{2n} - G(x)A(x)^{-1}\nabla\sigma(x)^T = \begin{pmatrix} I_n & 0_n \\ 0_n & 0_n \end{pmatrix}$.*

*In fact $G(x) = \begin{pmatrix} 0_n \\ \tilde{G}(x) \end{pmatrix}$ where $0_n$ is the $n \times n$ zero matrix. Using the feedback (7), the system becomes :*

$$\begin{cases} \dot{x} = y \\ \dot{y} = v \\ \sigma_i = y_i \end{cases}$$

*note that this feedback is the well known feedback linearization ([3]). Now consider a storage function (the total energy) (see def. 10.4 pp. 439 in [2])*

$$V = \frac{1}{2}y^T\tilde{G}(x)^{-1}y - Q(x)$$

*with $\frac{1}{2}y^T\tilde{G}(x)^{-1}y$ is the Kinetics energy and $Q(x)$ is the potential energy, then*

$$\dot{V} = y^T\tilde{G}(x)^{-1}v + \sum_{i=1}^{n} y_i \left[\frac{1}{2}y^T\frac{\partial\tilde{G}(x)^{-1}}{\partial x_i}y - \frac{\partial Q(x)}{\partial x_i}\right]$$

*and by using the following feedback*

$$v = \tilde{G}(x)\left[w - \left[\frac{1}{2}y^T\frac{\partial\tilde{G}(x)^{-1}}{\partial x_i}y - \frac{\partial Q(x)}{\partial x_i}\right]_{1\le i\le n}\right]$$

*one obtains*

$$\dot{V} = y^T w = \sum_{i=1}^{n} y_i w_i$$

### 2.5 Zero dynamics stability and passivity

The following theorem gives a sufficient conditions such that the system has a stable zero dynamics :

**Theorem 1** *Assume that there exist $s_1, \ldots.., s_{n-m}$ such that*

*i) $dim\{d\sigma_1, \ldots..., d\sigma_m, ds_1, \ldots., ds_{n-m}\} = n$,*

*ii) $\nabla s_j^T g_i = 0$, for all $j = 1,..,m$, and $i = 1,..,n-m$*

*iii) $s\nabla s^T M(x)f(x) = -P(x) + \sum\limits_{i=1}^{m} y_i N_i(x)$*

*where $s\nabla s^T = [s_1\nabla s_1^T, \ldots.., s_{n-m}\nabla s_{n-m}^T]^T$ with $N_i(x)$ a defined and bounded functions for all $x$ and $P(x)$ a positive semi definite function, there exists a feedback such that the system (1) has a stable zero dynamics.*

Note that i) and ii) are equivalent to $span\{g_1, ..., g_m\}$ is involutive.

**Proof:** By i) the function

$$V = \frac{1}{2}\sum_{i=1}^{m}\sigma_i^2 + \frac{1}{2}\sum_{j=1}^{n-m} s_j^2$$

is a non degenerate storage function. The derivative of $V$ with respect to the system (8) gives

$$\dot{V} = \sum_{i=1}^{m} y_i v_i - P(x) + \sum_{i=1}^{m} y_i N_i(x)$$

then $\dot{V}\Big|_S = -P(x) \le 0$, which implies that the zero dynamics of the system is asymptotically stable. ■

**Example 3** *Let us consider again the system studied in example 1. In this system the zero dynamics is*

$$\dot{x}_2 = -\beta x_2$$

*which is stable (because $\beta > 0$). We obtain the same conclusion by using theorem 1. In fact, it is sufficient to take $s_1 = x_2$, which gives $V = \frac{1}{2}(x_1^2 + x_3^2 + x_2^2)$, $N_1 = x_2 k(x_3)$, $N_2 = 0$, and $P(x) = \beta x_2^2$, this gives*

$$\dot{V} = x_1 v_1 + x_3 v_2 - \beta x_2^2 + x_1 x_2 k(x_3)$$

*thus the restriction of $\dot{V}$ to $S$ gives $\dot{V}\Big|_S = -\beta x_2^2 \le 0$ which means that the system has a stable zero dynamics*

Hereafter,a restrictive sufficient conditions is given, with respect to the transverse and tangent dynamics, under which the system can be made passive, with positive definite storage function, after regular static feedback.

From the proof of theorem 1, one obtains $\dot{V} = \sum_{i=1}^{m} y_i v_i - P(x) + \sum_{i=1}^{m} y_i N_i(x)$, and setting $v_i = w_i - N_i(x)$, we have

$$\dot{V} = \sum_{i=1}^{m} y_i w_i - P(x)$$

which guarantees the passivity. We resume the previous result on the next corollary :

**Corollary 3** *Under hypothesis 1 and conditions i) ii) and iii) of theorem 1, there exists a regular static feedback which transform the system (1) into a passive one with positive definite storage function.*

**Example 4** *For the previous example 3, considering again* $V = \frac{1}{2}(x_1^2+x_3^2+x_2^2)$ *we have* $P(x) = \beta x_2^2$, $N_1 = x_2 k(x_3)$, $N_2 = 0$. *As* $\beta > 0$, *one obtain*

$$\dot{V} = x_1 v_1 + x_3 v_2 - \beta x_2^2 + x_1 x_2 k(x_3)$$

*and by using the feedback* $v_1 = w_1 - x_2 k(x_3)$ *and* $v_2 = w_2$, *one obtains:*

$$\begin{aligned} \dot{V} &= x_1 w_1 + x_3 w_2 - \beta x_2^2 \\ &= \sum_{i=1}^{2} y_i w_i - \beta x_2^2 \leq y^T w \end{aligned}$$

*which means that the system is passive with a storage function* $V$ *definite positive.*

**Example 5** *(Lagrangian system : Rigid Body) Consider the equations for the angular velocities of rigid body with two external torques aligned with two principal axes ([3])*

$$\begin{cases} \dot{\omega}_1 = I_{23}\omega_2\omega_3 \\ \dot{\omega}_2 = I_{23}\omega_1\omega_3 + c_1 u_1 \\ \dot{\omega}_3 = I_{23}\omega_1\omega_2 + c_2 u_2 \\ \sigma_1 = \omega_1 + \omega_2 \\ \sigma_2 = \omega_3 + \omega_1^2 \end{cases}$$

*with* $I_{23} = \frac{I_2 - I_3}{I_1} < 0$. *Using the feedback (7) the system becomes :*

$$\begin{cases} \dot{\omega}_1 = I_{23}\omega_2\omega_3 \\ \dot{\omega}_2 = -I_{23}\omega_2\omega_3 + v_1 \\ \dot{\omega}_3 = -2I_{23}\omega_1\omega_2\omega_3 + v_2 \end{cases}$$

*It is easy to show that the zero dynamics* $\dot{\omega}_1 = I_{23}\omega_1^3$ *is stable because* $I_{23} < 0$.

*Now consider the storage function* $V = \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + s_1^2)$ *with* $s_1 = \omega_1$ *then*

$(s_1 \nabla s_1)^T M(x) f(x) = I_{23}\omega_1\omega_2\omega_3 = I_{23}\omega_1^4 + I_{23}\omega_1 y_1 y_2 - I_{23}\omega_1^3 y_1 - I_{23}\omega_1^2 y_2$.

*The conditions of the corollary 3 are satisfied, consequently, we can find a feedback (for example* $v_1 = w_1 - I_{23} w_1 y_2 + I_{23} w_1^3$ *and* $v_2 = w_2 + I_{23} w_1^2$ *) which guarantees the passivity of the system with a storage function* $V$ *positive definite.*

## 3 W.A.R. form

For a better understanding of sliding mode based control for nonlinear systems, the corresponding method is developed in [12, 13, 8]. This method consists in a decomposition of the drift vector field as (W.A.R. form) : the *Workless* field; the *Attracting* field; and the *Rejecting* field for multi-variable systems. Based on the previous results for the nonlinear square multivariable system (1), one considers a sliding surface $s(x)$ satisfying the transversality condition (2). One can rewrite this system in the W.A.R. form:

$$\begin{aligned} \dot{x} &= \sum_{k=1}^{m} \left[\mathcal{J}_k(x) + \mathcal{S}_k(x)\right] \nabla\sigma_k(x) + G(x)u \quad (10) \\ y_i &= \sigma_i(x) \qquad i = 1,..,m \end{aligned}$$

where $\mathcal{J}_k$ is a skew-symmetric matrix and $\mathcal{S}_k$ can be nonuniquely decomposed as the sum of two symmetric matrix $\mathcal{S}_k = \mathcal{S}_k^+ + \mathcal{S}_k^-$ where $\mathcal{S}_k^+$ is a positive semi-definite and $\mathcal{S}_k^-$ is a negative semi-definite matrix. The proposition 1 gives immediately :

**Corollary 4** *Consider the decoupled system*

$$\begin{aligned} \dot{x} &= f(x) + \sum_{i=1}^{m} g_i(x) u_i \quad (11) \\ y_i &= \sigma_i \end{aligned}$$

*this means that* $L_{g_k}\sigma_k \neq 0$ *and* $L_{g_k}\sigma_i = 0$ *for* $i \neq k$ *for* $1 \leq k, i \leq m$. *Then the projector* $M(x)$ *on* $T_x S$ *along span* $\{g_1,..,g_m\}$ *has the following form*

$$M(x) = I_n - \sum_{k=1}^{m} \frac{1}{L_{g_k}\sigma_k} g_k \left(\nabla\sigma_k\right)^T$$

*For* $m = 1$, *the projector* $M(x)$ *is the one introduced in [8].*

**Proof:** From the proposition (1) and the definition of $M(x) = I_n - G(x)\left[A(x)\right]^{-1} \nabla\sigma^T(x)$, and as the system (11) is supposed to be decoupled

$$A(x)^{-1} = \left[\frac{1}{L_{g_k}\sigma_k}\right]_{1\leq k\leq m}$$

$$\begin{aligned} & G(x)\left[A(x)\right]^{-1} \nabla\sigma^T(x) \\ =\ & \left[g_1(x),...,g_m(x)\right] \left[\frac{1}{L_{g_k}\sigma_k}\right]_{1\leq k\leq m} \left[\nabla\sigma_1^T,....,\nabla\sigma_m^T\right]^T \\ =\ & \left[g_1(x),...,g_m(x)\right] \left[\frac{1}{L_{g_1}\sigma_1}\nabla\sigma_1^T,....,\frac{1}{L_{g_m}\sigma_m}\nabla\sigma_m^T\right]^T \\ =\ & \sum_{k=1}^{m} \frac{1}{L_{g_k}\sigma_k} g_k \left(\nabla\sigma_k\right)^T \end{aligned}$$

■

**Theorem 2** *Let the drift vector field$f(x)$ be a smooth vector field, then*

$$\begin{aligned} f(x) &= \sum_{k=1}^{m} \left[\mathcal{J}_k(x) + \mathcal{S}_k(x)\right] \nabla\sigma_k(x) \\ &= \sum_{k=1}^{m} (\underbrace{\mathcal{J}_k(x)\nabla\sigma_k}_{workless\ field} + \underbrace{\mathcal{S}_k^{+}(x)\nabla\sigma_k}_{rejecting\ field} + \underbrace{\mathcal{S}_k^{-}(x)\nabla\sigma_k}_{attracting\ field}) \end{aligned}$$

*with :* $\mathcal{J}_k(x) = \frac{1}{2m}\frac{1}{L_{g_k}\sigma_k}\left[fg_k^T - g_k f^T\right]$ *and* $\mathcal{S}_k(x) = \frac{1}{2m}\frac{1}{L_{g_k}\sigma_k}\left[fg_k^T + g_k f^T\right]$, *where* $\mathcal{J}_k(x)$ *is a skew-symmetric matrix and* $\mathcal{S}_k(x)$ *is a symmetric one.*

The *workless* field is represented by $\sum_{k=1}^{m}\left[\mathcal{J}_k(x)\right]\nabla\sigma_k(x)$, the *attracting* field by $\sum_{k=1}^{m}\left[\mathcal{S}_k^{-}(x)\right]\nabla\sigma_k(x)$ and the *rejecting* field by $\sum_{k=1}^{m}\left[\mathcal{S}_k^{+}(x)\right]\nabla\sigma_k(x)$.

**Proof:** from the decomposition (5), one obtains

$$\begin{aligned} M(x)f(x) &= f(x) - \sum_{k=1}^{m}\frac{1}{L_{g_k}\sigma_k} g_k \left(\nabla\sigma_k\right)^T f(x) \\ &= \sum_{k=1}^{m}\frac{1}{L_{g_k}\sigma_k}\left[\frac{1}{m}L_{g_k}\sigma_k f - g_k\left(\nabla\sigma_k\right)^T f\right] \end{aligned}$$

where $L_{g_k}\sigma_k = g_k^T\nabla\sigma_k$ and $\left(\nabla\sigma_k\right)^T f = f^T\nabla\sigma_k$, then

$$M(x)f(x) = \sum_{k=1}^{m}\frac{1}{L_{g_k}\sigma_k}\left[\frac{1}{m}fg_k^T - g_k f^T\right]\nabla\sigma_k$$

which is skew-symmetric if $m = 1$ (see [8] for the SISO case). Now, for $m > 1$, we have :

$$\begin{aligned} \frac{1}{m}fg_k^T - g_k f^T &= \frac{1+m}{2m}\left[fg_k^T - g_k f^T\right] \\ &\quad + \frac{1-m}{2m}\left[fg_k^T + g_k f^T\right] \end{aligned}$$

$$\begin{aligned} M(x)f(x) &= \frac{1+m}{2m}\sum_{k=1}^{m}\frac{1}{L_{g_k}\sigma_k}\left[fg_k^T - g_k f^T\right]\nabla\sigma_k \\ &\quad + \frac{1-m}{2m}\sum_{k=1}^{m}\frac{1}{L_{g_k}\sigma_k}\left[fg_k^T + g_k f^T\right]\nabla\sigma_k \end{aligned}$$

Now

$$\begin{aligned} \left[I_n - M(x)\right]f(x) &= \sum_{k=1}^{m}\frac{1}{L_{g_k}\sigma_k} g_k\left(\nabla\sigma_k\right)^T f(x) \\ &= \frac{1}{2}\sum_{k=1}^{m}\frac{1}{L_{g_k}\sigma_k}\left[g_k f^T - fg_k^T\right]\nabla\sigma_k \\ &\quad + \frac{1}{2}\sum_{k=1}^{m}\frac{1}{L_{g_k}\sigma_k}\left[g_k f^T + fg_k^T\right]\nabla\sigma_k \end{aligned}$$

then

$$f(x) = \sum_{k=1}^{m}\mathcal{J}_k(x)\nabla\sigma_k + \sum_{k=1}^{m}\mathcal{S}_k(x)\nabla\sigma_k$$

therefore

$$\begin{aligned} \mathcal{J}_k(x) &= \frac{1}{2m}\frac{1}{L_{g_k}\sigma_k}\left[fg_k^T - g_k f^T\right] \\ \mathcal{S}_k(x) &= \frac{1}{2m}\frac{1}{L_{g_k}\sigma_k}\left[fg_k^T + g_k f^T\right] \end{aligned}$$

■

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