Right Coprime Factorizations Using System Upper Hessenberg Forms — The Multi-input System case

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Abstract: Based on a method for right coprime factorizations of linear systems using matrix elementary transformations, it is shown that a very simple iteration formula exists for right coprime factorizations of multi-input linear systems in system upper Hessenberg forms. This formula gives directly the coefficient matrices of the pair of solutions to the right coprime factorization of the system Hessenberg form, and involves only manipulations of inverses of a few triangular matrices and some matrix productions and summations. Based on this formula, a simple, efficient procedure for determining a right coprime factorization of a multi-input linear system is proposed, which first converts a given linear system into its system Hessenberg form using some orthogonal similarity transformations and then applies the iteration formula to the converted system Hessenberg form. An example demonstrates the usage of the approach.

Key Words: Linear systems; right coprime factorization; system Hessenberg forms, iterative solution; orthogonal transformation.

I. INTRODUCTION

Consider the following controllable multi-input linear system:

$$\dot{x} = Ax + Bu \tag{1.1}$$

where $x \in R^n$ is the state vector, $u \in R^r$ is the input vector, A and B are known matrices of appropriate dimensions, and B is of full-column rank. This note is concerned with the solution of the right coprime factorisation of system (1.1), that is, to find a pair of right coprime polynomial matrices $N(s)$ and $D(s)$ of dimensions $n \times r$ and $r \times r$, respectively, satisfying

$$(sI - A)^{-1}B = N(s)D^{-1}(s) \tag{1.2}$$

Coprime factorisation for linear systems is a basic problem in control systems theory. It has applications in many problems and has been investigated by a number of researchers. Beelin [1] proposes a numerical computational algorithm for solution of a coprime factorisation of a transfer function. Bongers and Heuberger [2] develop a reliable algorithm to perform a normalised coprime factorisation of proper discrete-time finite dimensional linear time-invariant systems. Green [3] develops a coprime factorisation approach to the synthesis of H-infinity controllers. Armstrong [4] considers robust stabilisation using a coprime factorisation approach. Almuthairi and Bingulac [5] and Bingulac and Almuthairi [6] propose new computationally simple algorithms for calculating coprime matrix descriptions, and also consider the minimal state space realisation problems based on coprime factorisation. Ohishi *et al*. [7] proposes a new speed servo system for a wide speed range based on doubly coprime factorisation and instantaneous speed observer. Besides the above, Duan [8] has shown that a coprime factorisation can be used to parameterise all the solutions to a generalised-type of Sylvester matrix equations [8,9,10], and has important applications in eigenstructure assignment [8,9, 11-15], observer design [16, 17], robust pole assignment [18, 19] and robust fault detection [20,21]. Duan [8] gives a simple way for calculating a right coprime factorisation for a linear system using matrix elementary transformations. His approach is very efficient for low-order systems, but is not convenient to use when the system dimension is relatively large.

The purpose of this note is to present a simpler alternative approach for right coprime factorisation for linear systems. As is well known, any controllable linear system can be transformed by some orthogonal similarity transformation into a system Hessenberg form. It is also well agreed that this transformation procedure is numerically reliable. The basic idea in our approach is to first transform the original system into system Hessenberg form and then to find the solution to the right coprime factorization of the system Hessenberg form. The method in [8] is utilized to deduce the right coprime factorization of a system Hessenberg form. Due to the special structure of the system Hessenberg form, a very simple iteration formula for solution to the right coprime factorization of the system Hessenberg form is established, which gives directly the coefficient matrices of the pair of solutions to the right coprime factorization of the system Hessenberg form. This iteration formula involves only inverses of some triangular matrices and some matrix productions and summations, and thus possesses good numerical property and is convenient to use.

The next section introduces the system Hessenberg forms for controllable linear systems and states the method in [8] for solving right coprime factorizations. Section 3 presents the iteration formula for right coprime factorization of a system Hessenberg form. An illustrative example is given in Section 4. Concluding remarks follow in Section 5. The proof of the iteration formula is presented in the appendix.

II. PRELIMINARIES

For convenience, a linear system in the form of (1.1) will be also denoted in the following by the matrix pair $(A\ \ B)$.

Definition 2.1: A controllable system $(H_A\ \ H_B) \in R^{n\times n}\times R^{n\times r}$ is said to be in system upper Hessenberg form if

$$H_A=\begin{bmatrix} H_{\mu,\mu} & H_{\mu,\mu-1} & \cdots & H_{\mu,2} & H_{\mu,1} \\ H_{\mu-1} & H_{\mu-1,\mu-1} & \cdots & H_{\mu-1,2} & H_{\mu-1,1} \\ 0 & H_{\mu-2} & \cdots & H_{\mu-2,2} & H_{\mu-2,1} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & H_1 & H_{11} \end{bmatrix}, H_b=\begin{bmatrix} H_\mu \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{2.1a}$$

where

$$H_{i,j}\in R^{n_i\times n_j},\quad i,j=1,2,\cdots,\mu \tag{2.1b}$$

$$n_1\le n_2\le\cdots\le n_\mu=\mathrm{rank}(B)=r;\quad n_1+n_2+\cdots+n_\mu=n \tag{2.1c}$$

and $H_i\in R^{n_i\times n_{i+1}}$, $i=1,2,\cdots,\mu$, are a series of full-row rank upper triangular matrices.

Regarding the above system Henssenberg form, the following result holds [22].

Lemma 2.1: For each controllable system (1.1), there exists an orthogonal matrix T and a system $(H_A\ \ H_B)$ in system upper Hessenberg form such that

$$(T^{\mathrm T}AT\quad T^{\mathrm T}b)=(H_A\quad H_B) \tag{2.2}$$

The above lemma states that any controllable system $(A\ \ B)$ can be transformed into a system upper Hessenberg form. Moreover, this transformation process is also numerically stable because orthogonal similarity transformation is used. In view of this lemma, the following definition can be introduced for convenience.

Definition 2.2. Let $(H_A\ \ H_B)$ be a system upper Hessenberg form, and the relation (2.2) hold for some orthogonal matrix *T*. Then $(H_A\ \ H_B)$ is called a system upper Hessenberg form of the system $(A\ \ B)$ associated with the orthogonal matrix *T*.

Let $N_H(s)$ and $D_H(s)$ be a solution to the right coprime factorization of the system upper Hessenberg form $(H_A\ \ H_B)$ defined in (2.1), that is,

$$(sI-H_A)^{-1}H_B=N_H(s)D_H^{-1}(s) \tag{2.3}$$

Then the following lemma can be easily shown, which gives the relation between the coprime factorizations of the system $(A\ \ B)$ and its system Hessenberg upper form.

Lemma 2.2. Let $(H_A\ \ H_B)$ be the system upper Hessenberg form of system $(A\ \ B)$ associated with the orthogonal matrix *T*, and $N_H(s)$ and $D_H(s)$ be a pair of solution to the right coprime factorization of the system Hessenberg form $(H_A\ \ H_B)$. Then a pair of solution $N(s)$ and $D(s)$ to the right coprime factorization of system $(A\ \ B)$ is given by

$$N(s)=TN_H(s),\quad D(s)=D_H(s) \tag{2.4}$$

The above lemma indicates a right coprime factorization of the system $(A\ \ B)$ can be immediately obtained when a right coprime factorization of its system Hessenberg form is available. To derive a right coprime factorization of a system upper Hessenberg form, a simple method for solution to right coprime factorizations of linear systems using matrix elementary transformations is finally stated in this section. This method was first proposed and used by Duan in [8], and has been frequently used in some of his later works [9-21].

Due to the controllability of $(A\ \ B)$, there exist a pair of unimodular polynomial matrices $P(s)$ and $Q(s)$ of dimensions $n\times n$ and $(n+r)\times(n+r)$, respectively, such that

$$P(s)[B\quad A-sI]Q(s)=[I\quad 0] \tag{2.5}$$

Based on this relation, a solution to right coprime factorization of system $(A\ \ B)$ can be given following the method in [8].

Lemma 4.1: Let Assumption A1 hold, and $Q(s)$ be the unimodular matrix of dimension $(n+r)\times(n+r)$ satisfying (2.5), then a solution to the right coprime factorization for system (1.1) is given by

$$\begin{bmatrix} D(s) \\ N(s) \end{bmatrix}=\alpha Q(s)\begin{bmatrix} 0_n \\ I_r \end{bmatrix} \tag{2.6}$$

where α is an arbitrary nonzero scalar.

III. THE ITERATIVE FORMULA

For convenience, we will use in the following sections $M_{(i)}$ to denote the *i*-th column of the matrix *M*, and 0_k the zero vector of dimension *k*.

Suppose that the system upper Hessenberg form given by (2.1) for the controllable linear system (1.1) has been obtained. Let

$$n_0=0,\ n_{\mu+1}=r\ \text{ and }\ r_i=n_{i+1}-n_i,\ i=0,1,2,\cdots,\mu \tag{3.1}$$

then it is clear that

$$r_0+r_1+\cdots+r_\mu=r \tag{3.2}$$

Define

$$H_{i,1}^R=H_{i,1}\in R^{n_i\times r_0},\quad i=1,2,\cdots,\mu \tag{3.3}$$

and partition each H_i and H_{ij}, $j=2,3,\cdots,i,\ i=1,2,\cdots,\mu$, into two parts as follows

$$\begin{cases} H_i = \begin{bmatrix} H_i^L & H_i^R \end{bmatrix}, \ H_i^L \in R^{n_i \times n_i}, \ H_i^R \in R^{n_i \times r_i} \\ H_{i,j} = \begin{bmatrix} H_{i,j}^L & H_{i,j}^R \end{bmatrix}, H_{i,j}^L \in R^{n_i \times n_{j-1}}, \ H_i^R \in R^{n_i \times r_{j-1}} \\ j = 2,3,\cdots,i, \ i = 1,2,\cdots,\mu \end{cases} \quad (3.4)$$

Then, it is easy to see that the matrices $H_i^L \in R^{n_i \times n_i}$, $i=1,2,\cdots,\mu$, are non-singular upper-triangular matrices since H_i, $i=1,2,\cdots,\mu$, are all full-row rank upper-triangular matrices.

Further introduce

$$I_i^L = \begin{bmatrix} I_{n_{i-1}} \\ 0_{r_{i-1} \times n_{i-1}} \end{bmatrix}, \quad I_i^R = \begin{bmatrix} 0_{n_{i-1} \times r_{i-1}} \\ I_{r_{i-1}} \end{bmatrix}, \ i = 1,2,\cdots,\mu \quad (3.5)$$

then it is obvious that

$$[I_i^L \quad I_i^R] = I_{n_i}, \ i = 1,2,\cdots,\mu \quad (3.6)$$

Finally define

$$\begin{cases} \tilde{H}_i^R = \left(H_i^L\right)^{-1} H_i^R \\ H_{i,j}^L = \left(H_i^L\right)^{-1} H_{i,j}^L, \ H_{ij}^R = \left(H_i^L\right)^{-1} H_{ij}^R \\ j = 1,2,\cdots,i, \ i = 1,2,\cdots,\mu \end{cases} \quad (3.7)$$

and

$$\tilde{I}_i^R = \left(H_i^L\right)^{-1} I_i^R, \quad \tilde{I}_i^R = \left(H_i^L\right)^{-1} I_i^R, \quad i = 1,2,\cdots,\mu \quad (3.8)$$

Then the main result in this paper can be stated as follows.

Theorem 3.1: Let $(H_A \quad H_B)$ be a controllable system in system upper Hessenberg form. Then a pair of solution $N_H(s)$ and $D_H(s)$ to the right coprime factorization (2.3) of the system $(H_A \quad H_B)$ is given by

$$\begin{bmatrix} D_H(s) \\ N_H(s) \end{bmatrix} = \begin{bmatrix} F_{1\mu}(s) & F_{2\mu}(s) & F_{3\mu}(s) & \cdots & F_{\mu\mu}(s) & F_\mu \\ 0 & 0 & 0 & \cdots & 0 & I_{r_\mu} \\ \hline \vdots & \vdots & \vdots & & & \\ \hline F_{12}(s) & F_{22}(s) & F_2 & & & \\ 0 & 0 & I_{r_2} & & & \\ \hline F_{11}(s) & F_1 & & & & \\ 0 & I_{r_1} & & & & \\ \hline I_{r_0} & & & & & \end{bmatrix} \quad (3.9)$$

with

$$F_i = -\tilde{H}_i^R, \ i = 1,2,\cdots,\mu \quad (3.10)$$

and

$$\begin{cases} F_{ij}(s) = \sum_{k=0}^{j-i+1} F_{ij}^k s^k \\ j = i+1, i+2, \cdots, \mu, \ i = 1,2,\cdots,\mu \end{cases} \quad (3.11)$$

where the coefficients F_{ij}^k, $k=1,2,\cdots,j-i+1$, $j=i+1, i+2, \cdots$, μ, $i=1,2,\cdots,\mu$, are given iteratively by

$$\begin{cases} F_{ij}^0 = -\sum_{l=0}^{j-i} \tilde{H}_{j,i+l}^L F_{i,i+l-1}^0 - \tilde{H}_{ji}^L F_{i-1} - \tilde{H}_{ji}^R, \quad F_0 = 0 \\ F_{ij}^k = \tilde{I}_j^L F_{i,j-1}^{k-1} - \sum_{l=k}^{j-i} \tilde{H}_{j,i+l}^L F_{i,i+l-1}^k, \quad k = 1,2,\cdots, j-i+1 \end{cases} \quad (3.12)$$

$$j = i+1, \ i+2, \ \cdots, \mu, \ i = 1,2,\cdots,\mu$$

with initial values

$$\begin{cases} F_{ii}^0 = -\left(\tilde{H}_{ii}^L F_{i-1} + \tilde{H}_{ii}^R\right), \ F_0 = 0 \\ F_{ii}^1 = \left(\tilde{I}_i^L F_{i-1} + \tilde{I}_i^R\right), \ F_0 = 0 \\ i = 1,2,\cdots,\mu \end{cases} \quad (3.13)$$

The proof of Theorem 3.1 is provided in the Appendix.

It follows from Lemma 2.2 and Theorem 3.1 that a solution to the right coprime factorization of a given controllable linear system ($A \quad B$) can always be obtained numerically through the following steps:

1. Convert the given controllable linear system ($A \quad B$) into system upper Hessenberg form $(H_A \quad H_B)$ using orthogonal similarity transformations.
2. Find a pair of solutions $N_H(s)$ and $D_H(s)$ to the right coprime factorization of the system upper Hessenberg form $(H_A \quad H_B)$ using Theorem 3.1.
3. Obtain the pair of solutions $N(s)$ and $D(s)$ to the right coprime factorization of the original system (A B) using the relations in (2.4).

The proposed approach may be used to provide solutions to generalized Sylvester matrix equations, pole assignment and observer design in linear systems (refer to [8-21]).

IV. An Illustrative Example

In order to show the application of our iterative formula, we consider a simple system in the form of (1.1) with the following parameters:

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \quad (4.1)$$

There are respectively the transposes of the matrices A and C in the example system considered in Duan 1993 [19].

Let

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad (4.2)$$

Then, it is clear that this matrix T is orthogonal. Moreover, it can be easily seen that with this orthogonal matrix T system ($A \quad B$) can be transformed in to the following upper-triangular form

$$\begin{pmatrix} H_A & H_B \end{pmatrix} = \left(\left[\begin{array}{cc|c} 0 & -1 & 1 \\ 0 & 0 & 0 \\ \hline 1 & 0 & 1 \end{array}\right] \quad \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hline 0 & 0 \end{array}\right] \right) \quad (4.3)$$

Thus for this system we have

$$\mu = 2,\ \ n_1 = 1,\ \ n_2 = \text{rank}(B) = 2,\ n_3 = r = 2$$

and

$$r_0 = 1,\ \ r_1 = 1,\ \ r_2 = 0$$

Therefore, it follows from Theorem 3.1 that a solution to the right coprime factorisation of the system Hessenberg form possesses the following structure

$$\begin{bmatrix} D_H(s) \\ N_H(s) \end{bmatrix} = \begin{bmatrix} F_{12}(s) & F_{22}(s) \\ F_{11}(s) & F_1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{4.4}$$

Notice

$$H_1 = [1 \quad 0],\ \ H_2 = I_2 \tag{4.5a}$$

$$H_{11} = 1,\ \ H_{22} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ H_{22} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \tag{4.5b}$$

we have

$$H_1^L = 1,\ \ H_1^R = 0,\ H_2^L = I_2 \tag{4.6a}$$

$$H_{11}^R = 1,\ H_{21}^R = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ \ H_{22}^L = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ H_{22}^R = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \tag{4.6b}$$

Since both H_1^L and H_2^L are identity matrices, we have

$$\tilde{H}_i^R = H_i^R,\ \ \tilde{H}_{ij}^L = H_{ij}^L,\ \ \tilde{H}_{ij}^R = H_{ij}^R,\ i, j = 1,2 \tag{4.7}$$

Applying formula (3.12), yields

$$\begin{cases} F_{11}^0 = -H_{11}^R = -1 \\ F_{11}^1 = I_1^R = 1, \end{cases} \tag{4.8}$$

$$\begin{cases} F_{22}^0 = -(H_{22}^L F_1 + H_{22}^R) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ F_{22}^1 = I_2^L F_1 + I_2^R = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{cases} \tag{4.9}$$

and

$$\begin{cases} F_{12}^0 = -H_{22}^L F_{11}^0 - H_{21}^R = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \\ F_{12}^1 = I_2^L F_{11}^0 - H_{22}^L F_{11}^1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \\ F_{12}^2 = I_2^L F_{11}^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{cases} \tag{4.10}$$

which respectively give

$$F_{11}(s) = s - 1 \tag{4.11}$$

$$F_{22}(s) = \begin{bmatrix} 1 \\ s \end{bmatrix} \tag{4.12}$$

and

$$F_{12}(s) = \begin{bmatrix} s^2 - s - 1 \\ 0 \end{bmatrix} \tag{4.13}$$

Substituting (4.11-4.13) into (4.4) gives

$$N_H(s) = \begin{bmatrix} s-1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix},\quad D_H(s) = \begin{bmatrix} s^2 - s - 1 & 1 \\ 0 & s \end{bmatrix} \tag{4.14}$$

Therefore, a pair of solutions to the right coprime factorization of the original system (A B) are

$$\begin{aligned} N(s) = TN_H(s) &= \begin{bmatrix} s-1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \\ D(s) = D_H(s) &= \begin{bmatrix} s^2 - s - 1 & 1 \\ 0 & s \end{bmatrix} \end{aligned} \tag{4.15}$$

V. Conclusion

This note presents a simple, efficient numerical approach for solution to the right coprime factorization of linear systems. It is shown that when the given system is converted into system upper Hessenberg form using orthogonal similarity transformations, the solution to the right coprime factorization of the system can be easily calculated by an iterative formula which gives directly the coefficient matrices of the solutions to the right coprime factorization of the system Hessenberg form. This iterative formula possesses good numerical property because it involves only manipulations of matrix multiplications and summations and inverses of a few triangular matrices. Extensions of the idea in this note to the case of multi-input singular linear systems will be considered in a separate paper.

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APPENDIX. PROOF OF THEOREM 3.1 (OUTLINE)

The proof of Theorem 3.1 is composed of three steps. The first step involves the proof of the following result.

Lemma A1: Let $(H_A \quad H_B)$ be a controllable system in system upper Hessenberg form. Then a pair of solution $N_H(s)$ and $D_H(s)$ to the right coprime factorization (2.3) of the system $(H_A \quad H_B)$ is given by

$$\begin{bmatrix} D_H(s) \\ N_H(s) \end{bmatrix} = Q_0 Q_\mu(s) Q_{\mu-1}(s) \cdots Q_1(s) \begin{bmatrix} 0_n \\ I_r \end{bmatrix} \tag{A1}$$

where

$$Q_0 = \begin{bmatrix} I^L_{\mu+1} & & & & & & I^R_{\mu+1} \\ & I^L_\mu & & & & I^R_\mu & \\ & & \ddots & & \cdot^{\cdot^{\cdot}} & & \\ & & & I^L_2 \quad I^R_2 & & & \\ & & & I_{r_0} & & & \end{bmatrix} \tag{A2}$$

and the matrix $Q_i(s)$ is given at the end of the paper.

The second step proves the following result based on Lemma A1.

Lemma A2: Let $(H_A \quad H_B)$ be a controllable system in system upper Hessenberg form. Then a pair of solution $N_H(s)$ and $D_H(s)$ to the right coprime factorization (2.3) of the system $(H_A \quad H_B)$ is given by (3.9) with $F_i,\ i=1,2,\cdots,\mu$, given by (3.10) and the polynomial matrices $F_{ij}(s) \in R^{n_i \times r_{j-1}},\ j=i+1,i+2,\cdots,\mu,\ i=1,2,\cdots,\mu$, given iteratively by

$$\begin{cases} F_{ii}(s) = (s\tilde{I}_i^L - \tilde{H}_{ii}^L)F_{i-1} + (s\tilde{I}_i^R - \tilde{H}_{ii}^R), \quad F_0 = 0 \\ F_{ij}(s) = (s\tilde{I}_j^L - \tilde{H}_{jj}^L)F_{i,j-1}(s) \\ \qquad - \sum_{k=i}^{j-2} \tilde{H}^L_{j,k+1} F_{ik}(s) - \tilde{H}^L_{ji} F_{i-1} - \tilde{H}^R_{ji}, \quad F_0 = 0 \\ \qquad\qquad j = i+1, i+2, \cdots, \mu,\ i = 1,2,\cdots,\mu \end{cases} \tag{A3}$$

Based on Lemma A2, we can now in the final step derive the iterative formula (3.12) in Theorem (3.1).

According to Lemma A2, a pair of solution $N_H(s)$ and $D_H(s)$ to the right coprime factorization (2.3) of the system $(H_A \quad H_B)$ is given by (3.9) with the polynomial matrices $F_{ij}(s)\in R^{n_i\times r_{j-1}}$, $j=i+1,i+2,\cdots,\mu,\ i=1,2,\cdots,\mu$, given iteratively by the following formula

$$\begin{aligned} F_{ij}(s) &= s\tilde{I}_j^L F_{i,j-1}(s) - \sum_{l=i}^{j-1}\tilde{H}_{j,l+1}^L F_{il}(s) \\ &\quad - \tilde{H}_{ji}^L F_{i-1} - \tilde{H}_{ji}^R,\ F_0=0 \\ & j=i+1,i+2,\cdots,\mu,\ i=1,2,\cdots,\mu \end{aligned} \tag{A4}$$

and the following initial values

$$\begin{aligned} F_{ii}(s) &= (s\tilde{I}_i^L - \tilde{H}_{ii}^L)F_{i-1} + (s\tilde{I}_i^R - \tilde{H}_{ii}^R) \\ &= \left(\tilde{I}_i^L F_{i-1} + \tilde{I}_i^R\right)s - \left(\tilde{H}_{ii}^L F_{i-1} + \tilde{H}_{ii}^R\right),\quad F_0=0 \\ & i=1,2,\cdots,\mu \end{aligned} \tag{A5}$$

It can be easily seen from (A3) or (A4) and (A5) that the matrix polynomial $F_{ij}(s)$ is of order $j-i+1$ in s, therefore, it can be written in the form of (3.11). Using (3.11), we can obtain

$$\begin{aligned} \sum_{l=i}^{j-1}\tilde{H}_{j,l+1}^L F_{il}(s) &= \sum_{l=i}^{j-1}\tilde{H}_{j,l+1}^L \sum_{k=0}^{l-i+1} F_{il}^k s^k \\ &= \sum_{l=i}^{j-1}\sum_{k=0}^{l-i+1}\tilde{H}_{j,l+1}^L F_{il}^k s^k \\ &= \sum_{q=0}^{j-i-1}\sum_{k=0}^{q+1}\tilde{H}_{j,i+q+1}^L F_{i,i+q}^k s^k \\ &= \sum_{k=0}^{j-i}\sum_{q=k-1}^{j-i-1}\tilde{H}_{j,i+q+1}^L F_{i,i+q}^k s^k \\ &= \sum_{k=0}^{j-i}\left(\sum_{l=k}^{j-i}\tilde{H}_{j,i+l}^L F_{i,i+l-1}^k\right)s^k \\ & j=i+1,i+2,\cdots,\mu,\ i=1,2,\cdots,\mu \end{aligned} \tag{A6}$$

and

$$\begin{aligned} s\tilde{I}_j^L F_{i,j-1}(s) &= s\tilde{I}_j^L \sum_{k=0}^{j-i} F_{i,j-1}^k s^k \\ &= \sum_{k=0}^{j-i}\tilde{I}_j^L F_{i,j-1}^k s^{k+1} \\ &= \sum_{k=1}^{j-i+1}\tilde{I}_j^L F_{i,j-1}^{k-1} s^k \end{aligned} \tag{A7}$$

Substituting (A6) and (A7) into (A4), yields

$$\begin{aligned} F_{ij}(s) &= \sum_{k=1}^{j-i+1}\tilde{I}_j^L F_{i,j-1}^{k-1}s^k - \sum_{k=0}^{j-i}\left(\sum_{l=k}^{j-i}\tilde{H}_{j,i+l}^L F_{i,i+l-1}^k\right)s^k \\ &\quad - \tilde{H}_{ji}^L F_{i-1} - \tilde{H}_{ji}^R \\ &= \tilde{I}_j^L F_{i,j-1}^{j-i}s^{j-i+1} + \sum_{k=1}^{j-i}\tilde{I}_j^L F_{i,j-1}^{k-1}s^k - \sum_{k=1}^{j-i}\left(\sum_{l=k}^{j-i}\tilde{H}_{j,i+l}^L F_{i,i+l-1}^k\right)s^k \\ &\quad - \sum_{l=0}^{j-i}\tilde{H}_{j,i+l}^L F_{i,i+l-1}^0 - \tilde{H}_{ji}^L F_{i-1} - \tilde{H}_{ji}^R \\ &= \tilde{I}_j^L F_{i,j-1}^{j-i}s^{j-i+1} + \sum_{k=1}^{j-i}\left(\tilde{I}_j^L F_{i,j-1}^{k-1} - \sum_{l=k}^{j-i}\tilde{H}_{j,i+l}^L F_{i,i+l-1}^k\right)s^k \\ &\quad - \sum_{l=0}^{j-i}\tilde{H}_{j,i+l}^L F_{i,i+l-1}^0 - \tilde{H}_{ji}^L F_{i-1} - \tilde{H}_{ji}^R,\quad F_0=0 \\ & j=i+1,i+2,\cdots,\mu,\ i=1,2,\cdots,\mu \end{aligned} \tag{A8}$$

From (A8) the iterative formula (3.12) can be immediately obtained. It follows from (A5) that the initial values are obviously given as in (3.13). Thus the whole proof of Theorem 3.1 is complete.

The $Q_i(s)$ matrix mentioned in Lemma A1:

$$Q_i(s)=\begin{bmatrix} I_{n_\mu} &&&&&&&&&&&&&& \\ & \ddots &&&&&&&&&&&&& \\ && I_{n_i} & s\tilde{I}_i^L-\tilde{H}_{i,i}^L & -\tilde{H}_{i,i-1}^L & \cdots & -\tilde{H}_{i,2}^L & -\tilde{H}_{i,1}^R & \cdots & -\tilde{H}_{i,i-1}^R & s\tilde{I}_i^R-\tilde{H}_{i,i}^R & -\tilde{H}_i^R & 0 & \cdots & 0 \\ &&& I_{n_{i-1}} &&&&&&&&&&& \\ &&&& I_{n_{i-2}} &&&&&&&&&& \\ &&&&& \ddots &&&&&&&&& \\ &&&&&& I_{n_1} &&&&&&&& \\ &&&&&&& I_{r_0} &&&&&&& \\ &&&&&&&& \ddots &&&&&& \\ &&&&&&&&& I_{r_{i-2}} &&&&& \\ &&&&&&&&&& I_{r_{i-1}} &&&& \\ &&&&&&&&&&& I_{r_i} &&& \\ &&&&&&&&&&&& I_{r_i} && \\ &&&&&&&&&&&&& \ddots & \\ &&&&&&&&&&&&&& I_{r_\mu} \end{bmatrix}$$