Uniform Invariance Principle and Synchronization. Robustness with Respect to Parameter Variation.

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Abstract

The object of this work is to obtain uniform estimates, with respect to parameters, of the attractor and of the basin of attraction of a dynamical system and to apply these results to analyze the roughness of the synchronization of two subsystems. These estimates are obtained through an uniform version of the Invariance Principle of La Salle which is stated and proved in this work.

Keywords: Invariance Principle, synchronization.

1 Introduction

The Invariance Principle has been one of the most important tools to study the asymptotic behavior of solutions of differential equations. It was first stated and proved for autonomous differential equations defined on finite dimensional spaces[13] and [14] by J. P. LaSalle and it was successfully extended to differential equations defined on infinite dimensional spaces, see Hale[12], Slemrod[23], including to Functional Differential Equations, see Hale-Lunel[9]. It was also extended to non-autonomous differential equations LaSalle[15] for the periodic case, Miller[17] for the almost periodic case and Sell[22] for more general ordinary differential equations, and also to non-autonomous Retarded Equations, see Rodrigues[19]. LaSalle[16] also obtained an extension for difference equations.

Although, most applications of the Invariance Principle are concerned with convergence to equilibrium, in this paper it is shown that it can also be used to study synchronization between solutions of coupled differential equations.

Synchronization is an important concept that has been extensively used by researchers of applied sciences as Electrical and Mechanical Engineering, Biology, Physics, etc. It has successfully been used on Communication Systems for codification of information, see Cuomo-Hoppenheim[7], Yang-Chua[25] and Peccora et all[18].

Mathematical Methods to study synchronization between chaotic systems were presented in Fujisaka-Yamada [8], in Afraimovich et all[2] and in Wu-Chua [24]. Abstract results and the robustness with respect to parameters variation and uniform dissipativeness were obtained in Rodrigues[20] and Afraimovich-Rodrigues[1].

For infinite dimensional systems some results are presented in Rodrigues[20] , Carvalho-Dłotko-Rodrigues[4], Hale[10][11] and Afraimovich-Chow-Hale[3].

The object of this paper is to present a more general version of the Invariance Principle in which the derivative of the Liapunov function is not required to be always negative semidefinite and the parameters are allowed to vary on a certain range. In many complex engineering systems and systems whose solutions present a complicated or chaotic behavior, it may not be easy to find a Liapunov function such that its derivative along the solutions is not negative. Therefore, the results presented here will be helpful in such cases. It is important to point out that, in this paper, the expression "Liapunov Function" should be understood in a wider sense in which its derivative along the solutions may also be positive.

The Uniform Invariance Principle proposed in this paper is useful to obtain concrete upper bounds for attractors, for the attraction basin and also to study synchronization. Estimates in some examples, such as Lorenz equation, power systems, etc. are obtained.

2 The Uniform Invariance Principle

Before presenting the Uniform Invariance Principle, this section starts by reviewing the usual Invariance Principle. For that, consider the following autonomous differential equation:

$$\dot{x} = f(x) \tag{2.1}$$

Theorem 2.1 Let $V : \mathbb{R}^n \to \mathbb{R}$, $f : \mathbb{R}^n \to \mathbb{R}^n$ be C^1 functions. Let $L > 0$ be a constant such that $\Omega_L = \{x \in \mathbb{R}^n : V(x) < L\}$ is bounded. Suppose that $\dot{V}(x) \leq 0$ for every $x \in \Omega_L$ and define $E := \{x \in \Omega_L : \dot{V}(x) = 0\}$. Let B be the largest invariant set contained in E. Then every solution of (2.1) starting in Ω_L converges to B as $t \to \infty$.

A global version of this theorem can be stated as follows:

Theorem 2.2 Let $V : \mathbb{R}^n \to \mathbb{R}$, $f : \mathbb{R}^n \to \mathbb{R}^n$ be C^1 functions. Suppose that $\dot{V}(x) \le 0$ for every $x \in \mathbb{R}^n$ and define $E := \{x \in \mathbb{R}^n : \dot{V}(x) = 0\}$. Let B be the largest invariant set contained in E. Then every solution of (2.1) which is bounded for $t \ge 0$ converges to B as $t \to \infty$.

In this work, some results that are more general than the previous ones are given. For that, let $\lambda \in \Lambda \subset \mathbb{R}^m$, $x \in \mathbb{R}^n$ and consider the following autonomous differential equation:

$$\dot{x} = f(x, \lambda) \tag{2.2}$$

Theorem 2.3 (Uniform Invariance Principle) Suppose $f : \mathbb{R}^n \times \Lambda \to \mathbb{R}^n$ and $V : \mathbb{R}^n \times \Lambda \to \mathbb{R}$ are C^1 functions, $a, b, c : \mathbb{R}^n \to \mathbb{R}$ are continuous functions. Assume that for any $(x, \lambda) \in \mathbb{R}^n \times \Lambda$, one has:

$$a(x) \le V(x, \lambda) \le b(x), \qquad -\dot{V}(x, \lambda) \ge c(x)$$

For $L > 0$ let $\mathcal{A}_L := \{x \in \mathbb{R}^n : a(x) < L\}$. Assume that $\mathcal{A}_L$ is nonempty and bounded.
Consider the sets

$$\mathcal{B}_L := \{x \in \mathbb{R}^n : b(x) < L\},$$

$$C := \{x \in \mathbb{R}^n : c(x) < 0\} \text{ and}$$

$$E_L := \{x \in \mathcal{A}_L : c(x) = 0\}.$$

Suppose now that $\sup_{x \in C} b(x) \le l < L$ and define the sets

$$A_l := \{x \in \mathbb{R}^n : a(x) \le l\} \text{ and}$$

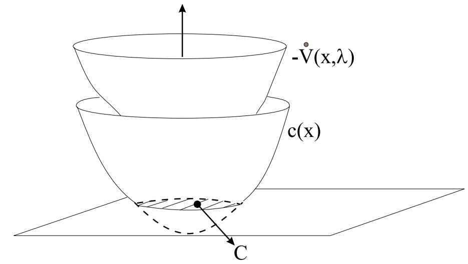
$$B_l := \{x \in \mathbb{R}^n : b(x) \le l\}.$$

If λ is a fixed parameter in Λ and all the previous conditions are satisfied then for $x_0 \in \mathcal{B}_L$ the solution $\varphi(t, x_0, \lambda)$ is defined in $[0, \infty)$ and the following holds:

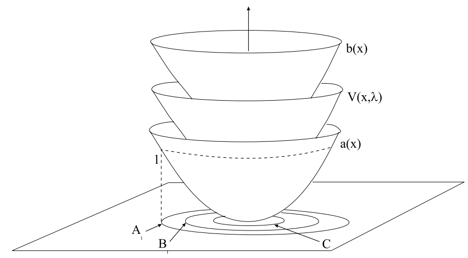
I) if $x_0 \in B_l$ then $\varphi(t, x_0, \lambda) \in A_l$, for $t \ge 0$ and $\varphi(t, x_0, \lambda)$ tends to the largest invariant set of (2.2) contained in A_l, as $t \to \infty$.

II) if $x_0 \in \mathcal{B}_L - B_l$ then $\varphi(t, x_0, \lambda)$ tends to the largest invariant set of (2.2) contained in $A_l \cup E_L$.

The proof of this and the following theorems can be found in [21]. Note that the uniformity is guaranteed by the existence of the functions a, b and c which are independent of the parameters. Figure 1 shows the relation between these functions and the estimates which are obtained with the theorem. Note that the set C contains the set where $\dot{V}$ is positive independently of the parameter $\lambda \in \Lambda$. As consequence, $l = \sup_{x \in C} b(x) \ge \sup_{(x,\lambda) \in x, \lambda : \dot{V} > 0} V(x, \lambda)$. Then, the level curve l of the function a is used to obtain an estimate of the attractor.



(a) Derivative of Liapunov Function



(b) Liapunov Function

Figure 1: Functions a,b and c of the Theorem 2.3

Figure 2 illustrates the application of the Uniform Invariance Principle. Note that $B_l \subset A_l$ and $\mathcal{B}_L \subset \mathcal{A}_L$. The invariance notion in this case is a little bit different. In this figure, x_1 and x_3 belongs to $\mathcal{B}_L$. The set $\mathcal{B}_L$ is not positively invariant with respect to (2.2), however one can guarantee that every solution starting into $\mathcal{B}_L$ does not leave the set $\mathcal{A}_L$. This is the case of the solutions starting at x_1 and x_3 in Figure 2. Nothing can be said about solutions starting in $\mathcal{A}_L - \mathcal{B}_L$. For example, the solution starting at x_2 abandon the set $\mathcal{A}_L$.
Every solution starting into $\mathcal{B}_L$ tends to the largest invariant set contained into $A_l \cup E_l$. If a solution enters in B_l, then one can guarantee that this solution will never abandon the set $A_l \supset B_l$. This is the case of the solution starting at x_3.
It is important to mention that Figure 2 does not show the general case. In this figure $A_l \subset \mathcal{B}_L$, however this is not a necessary condition.

Theorem 2.4 (The Global Uniform Invariance Principle.) Suppose $f : \mathbb{R}^n \times \Lambda \to \mathbb{R}^n$, $V : \mathbb{R}^n \times \Lambda \to \mathbb{R}$ are C^1 functions and that $a, b, c : \mathbb{R}^n \to \mathbb{R}$ are continuous functions. Suppose that for all $(x, \lambda) \in \mathbb{R}^n \times \Lambda$, one has:

$$a(x) \le V(x, \lambda) \le b(x)\ , \quad -\dot{V}(x, \lambda) \ge c(x).$$

Consider the sets:

$$C := \{x \in \mathbb{R}^n : c(x) < 0\}, \quad E := \{x \in \mathbb{R}^n : c(x) = 0\}.$$

Suppose that $\sup_{x \in C} b(x) \le l < \infty$ and consider the sets

$$A_l := \{x \in \mathbb{R}^n : a(x) \le l\} \text{ and}$$

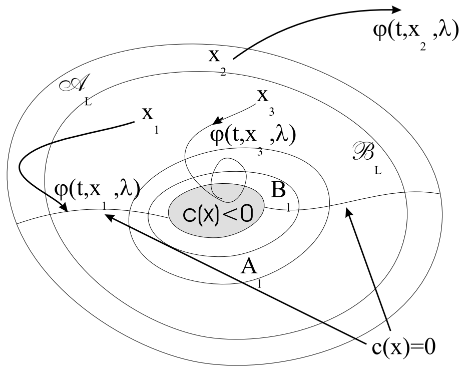


Figure 2: Geometric Interpretation of the Uniform Invariance Principle

$$B_l := \{x \in \mathbb{R}^n : b(x) \leq l\}.$$

Assume that A_l is nonempty and bounded.
If λ is a fixed parameter in Λ and all the previous conditions are satisfied then the following holds:

I) if $x_0 \in B_l$ then $\varphi(t, x_0, \lambda)$ is defined and belongs to A_l, for every $t \geq 0$ and tends to the largest invariant set of (2.2) contained in A_l, as $t \to \infty$.

II) If x_0 is such that the solution $\varphi(t, x_0, \lambda)$ is bounded for $t \geq 0$, then $\varphi(t, x_0, \lambda)$ tends to the largest invariant set of (2.2) contained in $A_l \cup E$, as $t \to \infty$.

Remark 2.1 If $a(x) \to \infty$, as $\|x\| \to \infty$, then for every $r > 0$ the set $A_r := \{x \in \mathbb{R}^n : a(x) \leq r\}$ is bounded. If such condition is satisfied, then every solution is bounded for $t \geq 0$ and the conclusion of the previous theorem holds true for every solution.
If $c(x) > 0$ for every $x \in \mathbb{R}^n - \bar{C}$, or if for every $x_0 \in E - \bar{C}$, $\varphi(t, x_0, \lambda) \notin E$, for every $t > 0$, sufficiently small and the previous conditions of the theorem are satisfied, then it is possible to show that every solution tends to the largest invariant set contained in A_l, as $t \to \infty$. In such case one has that the set A_l is an estimate of the attractor and $\mathbb{R}^n$ is the basin of attraction.

When using Theorem 2.3 or Theorem 2.4 in some applications, some technical difficulties may arise. The function $c(x)$ may not be smooth, the set C may not be convex and so $\sup_{x \in C} b(x)$ may not be attained on the boundary of the set C. Therefore the Lagrange multipliers technique cannot be used to compute $\sup_{x \in C} b(x)$ even if b is a convex function. When some symmetries are present in the problem, the next lemma provides an alternative approach to avoid these difficulties.

Lemma 2.5 Let $h, b, f_1, f_2, ..., f_k : \mathbb{R}^n \to \mathbb{R}$, continuous functions and assume that $h(x) \geq \inf\{f_1(x), f_2(x), ..., f_k(x)\}, \quad \forall x \in \mathbb{R}^n$.
Let $F_i := \{x \in \mathbb{R}^n : f_i(x) < 0\}$ and $H := \{x \in \mathbb{R}^n : h(x) < 0\}$. Then the following hold:

- $H \subset \bigcup_{i=1}^k F_i$ and $\sup_{x \in H} b(x) \leq \sup_{x \in \bigcup_{i=1}^k F_i} b(x)$.
- Suppose F_i is bounded and that there exists a sequence of homeomorphisms $S_i : \mathbb{R}^n \to \mathbb{R}^n$ $(i = 1, \ldots, k)$, such that, $F_j = S_{j-1}(F_{j-1})$, $\forall\ j = 2, \ldots, k$ and $F_1 = S_k F_k$. If $b(S_i(x)) = b(x), \forall\ x \in \mathbb{R}^n, \forall\ i = 1, \ldots, k$ then $\sup_{x \in \bigcup_{i=1}^k F_i} b(x) = \sup_{x \in F_j} b(x) \geq \sup_{x \in H} b(x)$, $\forall j \in \{1, 2, \ldots, k\}$.

The following lemma has an obvious proof, but it is very useful to reduce the dimension in a problem of maximization. In fact in the applications of this paper the reduction will be from dimension $2n$ to dimension n.

Lemma 2.6 Let $A \subset \mathbb{R}^n$ be a compact set and $b : A \to \mathbb{R}$ be a continuous function. Let $D \subset \mathbb{R}^n$ a closed set such that $A \cap D \neq \emptyset$ and for every $x \in A$ there exists $\bar{x} \in A \cap D$ such that $b(\bar{x}) \geq b(x)$. Then $\sup_{x \in A} b(x) = \sup_{x \in A \cap D} b(x)$.

Example 2.1 Uniform estimate of the attractor of Lorenz System with parameter variation.
Consider the Lorenz system:

$$\begin{cases} \dot{u} &= -\sigma u + \sigma v \\ \dot{v} &= -v - uz + ru \\ \dot{w} &= -bw + uv \end{cases}$$

With a change of variables in the previous system, as $x := u,\ y := v,\ z := w - \frac{5}{4}r$, the following system is obtained:

$$\begin{cases} \dot{x} &= -\sigma x + \sigma y \\ \dot{y} &= -y - x(z + \frac{5}{4}r) + rx \\ \dot{z} &= -b(z + \frac{5}{4}r) + xy \end{cases} \tag{2.3}$$

The nominal values of these parameters are $\sigma_N = 10, r_N = 28$ and $b_N = \frac{8}{3}$. An uncertainty of $\pm 5\%$ is admitted to exist in the determination of these parameters.
Let $\sigma_m := 9.5,\ \sigma_M := 10.5,\ r_m := 28 - \frac{28}{20},\ r_M := 28 + \frac{28}{20},\ b_m := \frac{8}{3} - \frac{8}{60},\ b_M := \frac{8}{3} + \frac{8}{60}$.
Consider the set:

$$\Lambda := \{\lambda := (\sigma, r, b) \in \mathbb{R}^3 : \sigma_m \leq \sigma \leq \sigma_M,\ r_m \leq r \leq r_M,\ b_m \leq b \leq b_M\}.$$

Let $V(x, y, z) = rx^2 + 4\sigma y^2 + 4\sigma z^2$ be a Liapunov function for (2.3). Then functions a and b can be chosen as $a(x, y, z) := r_m x^2 + 4\sigma_m y^2 + 4\sigma_m z^2$ and $b(x, y, z) := r_M x^2 + 4\sigma_M y^2 + 4\sigma_M z^2$
If we estimate the derivative of V along the solutions of (2.3) we obtain:

$$-\dot{V}(r, \sigma, b, x, y, z) = 2\sigma(rx^2 + 4y^2) + 8\sigma bz^2 + 10\sigma rbz \geq$$

$$2\sigma_m(r_m x^2+4y^2)+8\sigma_m b_m(|z|-\frac{5\sigma_M r_M b_M}{8\sigma_m b_m})^2-\frac{(5\sigma_M r_M b_M)^2}{8\sigma_m b_m}:=$$

$$c(x,y,z):=\alpha x^2+\beta y^2+\gamma(|z|-\rho)^2-\mu$$

The above expression defines naturally the numbers $\alpha,\ \beta,\gamma$.
Now we will use Lemma 2.5 with $h=c$, $f_1(x,y,z):=\alpha x^2+\beta y^2+\gamma(z-\rho)^2-\mu$ and $f_2(x,y,z):=\alpha x^2+\beta y^2+\gamma(z+\rho)^2-\mu$.
If we let $C:=\{(x,y,z)\in\mathbb{R}^3 : c(x,y,z)<0\}$, $F_1:=\{(x,y,z)\in\mathbb{R}^3 : f_1(x,y,z)<0\}$ from Lemma 2.5 it follows that $\sup_C b\leq\sup_{F_1} b$.
Using the Lagrange function:

$$\mathcal{L}(x,y,z,\mu)=r_M x^2+4\sigma_M y^2+4\sigma_M z^2+\mu[2\sigma_m r_m x^2$$
$$+8\sigma_m y^2+8\sigma_m b_m(z-\frac{5\sigma_M b_M r_M}{8\sigma_m b_m})^2-\frac{(5\sigma_M b_M r_M)^2}{8\sigma_m b_m}]$$

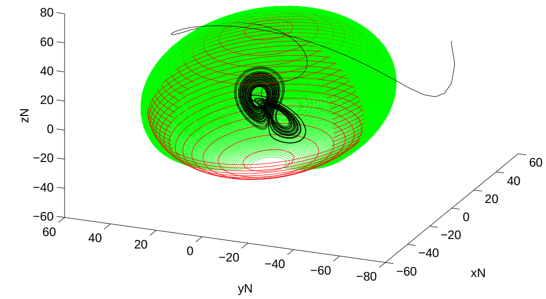
one obtains that the maximum is attained at $x=0,\ y^2=\frac{25\sigma_M^2 b_M^2 r_M^2(b_m-2)}{64\sigma_m(1-b_m)^2},\ z=\frac{5\sigma_M b_M r_M}{8\sigma_m(b_m-1)}$ and

$$\sup_{F_1} b=\frac{25\sigma_M^3 b_M^2 r_M^2}{16\sigma_m^2(b_m-1)}<88575,75<l:=88576$$

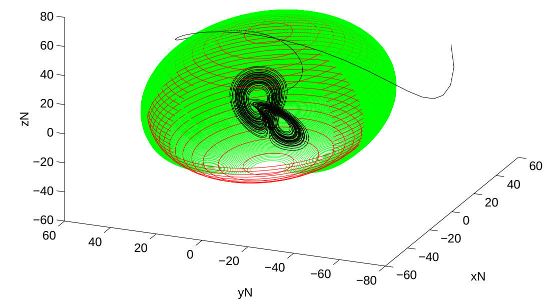
Therefore Lorenz attractor is contained in the ellipsoid:

$$\{(x,y,z)\in\mathbb{R}^3 :\ r_m x^2+4\sigma_m y^2+4\sigma_m z^2<l=88576\}$$

The above estimation as well an idea of Lorenz attractor is shown in Figure 3, where the external ellipsoid corresponds to the set A_l and the internal ellipsoid corresponds to the set B_l.



(a) $(x(t),y(t),z(t)),\ (\sigma,r,b)=(9.5,26.6,2.53)$



(b) $(x(t),y(t),z(t)),\ (\sigma,r,b)=(10.5,29.4,2.8)$

Figure 3: The uniform estimate of Lorenz Attractor. Initial condition: $(x(0),y(0),z(0))=(20,-70,40)$

3 Invariance Principle Applied to Synchronization Analysis

This paper is concerned with the synchronization of two coupled systems of the following form:

$$\begin{cases}\dot{x}=f(x,y,\lambda_1)\\ \dot{y}=g(x,y,\lambda_2)\end{cases} \tag{3.4}$$

where $(x,y)\in\mathbb{R}^n\times\mathbb{R}^n$.
One advantage of the technique proposed in this paper is that it gives an estimate of the minimum parameter value which is necessary to guarantee the synchronization.
Let A a nonempty open set in $\mathbb{R}^n\times\mathbb{R}^n$, $M\subset A$ a C^1-manifold and let $dist(\cdot,\cdot)$ be a distance in $\mathbb{R}^n\times\mathbb{R}^n$.

Definition 3.1 System (3.4) synchronizes with respect to A and M if, for any $\varepsilon>0$ there exists $\delta>0$ such that, if $\|\lambda_1-\lambda_2\|<\delta$, then

$$\limsup_{t\to\infty} dist((x(t,x_o,y_o),y(t,x_o,y_o)),M)\leq\varepsilon$$

for all initial conditions $(x_o,y_o)\in A$.

If $A=\mathbb{R}^n\times\mathbb{R}^n$, we say that the system (3.4) synchronizes globally.
In many applications, the synchronization is global and the manifold M is the diagonal $\{(x,y)\in\mathbb{R}^n\times\mathbb{R}^n : x=y\}$. In that case the above definition may be stated as follows:

Definition 3.2 System (3.4) synchronizes globally, with respect to the diagonal, if for any $\varepsilon>0$ there exists $\delta>0$ such that, if $\|\lambda_1-\lambda_2\|<\delta$, then

$$\limsup_{t\to\infty}\|y(t,x_o,y_o,\lambda_1,\lambda_2)-x(t,x_o,y_o,\lambda_1,\lambda_2)\|\leq\varepsilon$$

for all initial conditions $(x_o,y_o)\in\mathbb{R}^n$.

See Hale[10], for related definitions.

Example 3.1 Uniform estimate of the attractor and synchronization of coupled Lorenz System with parameter variation.

Consider two Lorenz system coupled through a linear term:

$$\begin{cases} \dot{u_1} &=& -\sigma_1 u_1 + \sigma_1 v_1 - k(u_1 - u_2) \\ \dot{v_1} &=& -v_1 - u_1 w_1 + r_1 u_1 \\ \dot{w_1} &=& -b_1 w_1 + u_1 v_1 \\ \dot{u_2} &=& -\sigma_2 u_2 + \sigma_2 v_2 - k(u_2 - u_1) \\ \dot{v_2} &=& -v_2 - u_2 w_2 + r_2 u_2 \\ \dot{w_2} &=& -b_2 w_2 + u_2 v_2 \end{cases} \tag{3.5}$$

The nominal values of these parameters are $\sigma_N = 10, r_N = 28$ and $b_N = \dfrac{8}{3}$. Allowing an uncertainty of $\pm 5\%$ on the determination of these parameters, we define $\sigma_m := 9.5,\ \sigma_M := 10.5,\ r_m := 28 - \dfrac{28}{20},\ r_M := 28 + \dfrac{28}{20},\ b_m := \dfrac{8}{3} - \dfrac{8}{60},\ b_M := \dfrac{8}{3} + \dfrac{8}{60}$. Consider the following set: $\Lambda := \{\lambda \in \mathbb{R}^6 : \sigma_m \le \sigma_1, \sigma_2 \le \sigma_M, r_m \le r_1, r_2 \le r_M, b_m \le b_1, b_2 \le b_M\}$, where $\lambda := (\sigma_1, r_1, b_1, \sigma_2, r_2, b_2)$.

Our purpose is the study of synchronization of this system. With a change of variables in the previous system, as $x_i := u_i,\ y_i := v_i,\ z_i := w_i - \dfrac{5}{4} r_i$, the following system is obtained:

$$\begin{cases} \dot{x}_1 &=& -\sigma_1 x_1 + \sigma_1 y_1 - k(x_1 - x_2) \\ \dot{y}_1 &=& -y_1 - x_1(z_1 + \frac{5}{4} r_1) + r_1 x_1 \\ \dot{z}_1 &=& -b_1(z_1 + \frac{5}{4} r_1) + x_1 y_1 \\ \dot{x}_2 &=& -\sigma_2 x_2 + \sigma_2 y_2 - k(x_2 - x_1) \\ \dot{y}_2 &=& -y_2 - x_2(z_2 + \frac{5}{4} r_2) + r_2 x_2 \\ \dot{z}_2 &=& -b_2(z_2 + \frac{5}{4} r_2) + x_2 y_2 \end{cases}$$

Let $V(x_1, y_1, z_1, x_2, y_2, z_2, \lambda) = x_1^2 + x_2^2 + 4\frac{\sigma_1}{r_1} y_1^2 + 4\frac{\sigma_2}{r_2} y_2^2 + 4\frac{\sigma_1}{r_1} z_1^2 + 4\frac{\sigma_2}{r_2} z_2^2$ be a Liapunov function for the previous system.

Our next purpose is to show that the conditions of Remark 2.1, and Theorem 2.4 are satisfied.

Functions a and b can be chosen as

$$a(x_1, y_1, z_1, x_2, y_2, z_2) := x_1^2 + x_2^2 + 4\frac{\sigma_m}{r_M}(y_1^2 + y_2^2) + 4\frac{\sigma_m}{r_M}(z_1^2 + z_2^2)$$

$$b(x_1, y_1, z_1, x_2, y_2, z_2) := x_1^2 + x_2^2 + 4\frac{\sigma_M}{r_m}(y_1^2 + y_2^2) + 4\frac{\sigma_M}{r_m}(z_1^2 + z_2^2)$$

The derivative of V is given by:

$$\begin{aligned} &-\dot{V}(x_1, y_1, z_1, x_2, y_2, z_2, \lambda, k) = 2\sigma_1(x_1^2 + \tfrac{4}{r_1} y_1^2 + 4\tfrac{b_1}{r_1} z_1^2 + 5b_1 z_1) \\ &+2\sigma_2(x_2^2 + \tfrac{4}{r_2} y_2^2 + 4\tfrac{b_2}{r_2} z_2^2 + 5b_2 z_2) + 2k(x_1 - x_2)^2 \ge \\ &2\sigma_m(x_1^2 + x_2^2) + \tfrac{8\sigma_m}{r_M}(y_1^2 + y_2^2) + 8\tfrac{\sigma_m b_m}{r_M}[(|z_1|^2 - \tfrac{5\sigma_M b_M r_M}{8\sigma_m b_m})^2 + \\ &(|z_2|^2 - \tfrac{5\sigma_M b_M r_M}{8\sigma_m b_m})^2] - \tfrac{(5\sigma_M b_M)^2}{4\sigma_m b_m} r_M := \\ &\alpha(x_1^2 + x_2^2) + \beta(y_1^2 + y_2^2) + \gamma[(|z_1| - \rho)^2 + (|z_2| - \rho)^2] - \mu \\ &\qquad := c(x_1, y_1, z_1, x_2, y_2, z_2), \end{aligned}$$

for every $\lambda \in \Lambda$ and $k > 0$. The previous identities also define the parameters $\alpha,\ \beta,\ \gamma,\ \rho,\ \mu$.

Note that the functions $a,\ b$ previously obtained are regular functions. However the function c is not regular and the set where $c < 0$ is not convex, which brings some technical difficulties in the application of the Lagrange multipliers technique. In order to overcome this difficulty, the Lemma 2.5 can be used to transform the problem in a convex problem andthe Lemma 2.6 can be used to reduce the problem from $\mathbb{R}^6$ to $\mathbb{R}^3$

Finally the technique of Lagrange multipliers can be used to compute the sup of b and one obtains that the maximum is attained at $x = y = 0$, and $z = \frac{10\sigma_m r_M b_M r_m}{4\sigma_m b_m(2r_m+3)}$. Substituting these values in the expression of b, the number l is obtained:

$$l = sup_{(x,y,z)\in F} b(x) = \frac{50\sigma_M^3 r_M^2 b_M^2 r_m}{\sigma_m^2 b_m^2 (2r_m + 3)^2} < 5703.3$$

The sets B_l and A_l are the ellipsoids: $B_l := \{(x_1, y_1, z_1, x_2, y_2, z_2) \in \mathbb{R}^6 : b(x_1, y_1, z_1, x_2, y_2, z_2) \le l\}$ and $A_l := \{(x_1, y_1, z_1, x_2, y_2, z_2) \in \mathbb{R}^6 : a(x_1, y_1, z_1, x_2, y_2, z_2) \le l\}$

The set in which $c(x_1, y_1, z_1, x_2, y_2, z_2) = 0$ is contained in A_l and so every solution converges to the largest invariant set contained in A_l. The set A_l is an estimate of the attractor which is independent of the parameters $\lambda \in \Lambda$ and $k > 0$. Therefore every solution of (3.5) enters in A_l in finite time and stays there in the future.

In order to study the synchronization, either Theorem 2.3 of [20] or Theorem 3.1 of [1] can be used.

Following the ideas of Theorem 2.3 of [20] and the fact that into the attractor estimation, one has: $y_1^2 \le 4424.3$ and $z_1^2 \le 4424.3 \Rightarrow -67 \le z_1 \le 67$.

One obtains that the system (3.5) synchronizes for $k > 377$.

Figure 4a and 4b show respectively the projection of the orbits of both systems on the plane x-z and the norm of the difference between the system solutions when the system 2 has an error of $+5\%$ in the parameters.

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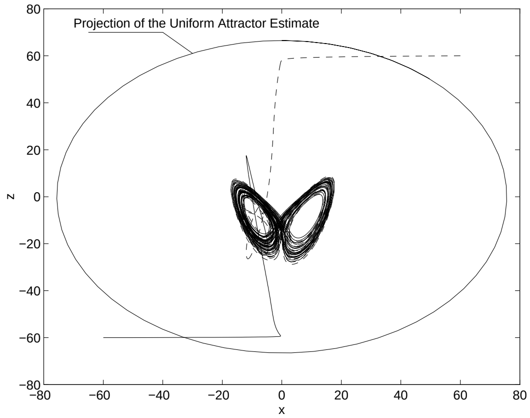
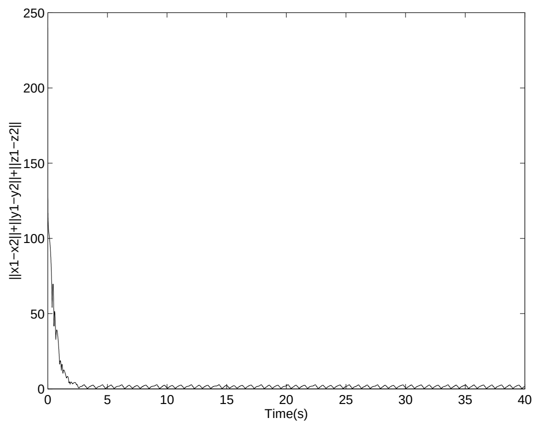


(a) $(x_1(0), y_1(0), z_1(0)) = (-60, 0, -60)$, $(x_2(0), y_2(0), z_2(0)) = (60, 0, 60)$,for $\sigma_1 = 10, \sigma_2 = 10.5, r_1 = 28, r_2 = 29.4, b_1 = 8/3, b_2 = 2.8$ and $k = 400$



(b) $(x_1(0), y_1(0), z_1(0)) = (-60, 0, -60)$, $(x_2(0), y_2(0), z_2(0)) = (60, 0, 60)$,for $\sigma_1 = 10, \sigma_2 = 10.5, r_1 = 28, r_2 = 29.4, b_1 = 8/3, b_2 = 2.8$ and $k = 400$

Figure 4: Lorenz Attractor

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