Global Estimation of n Unknown Frequencies [1]

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Abstract

Given a measurable signal consisting of the sum of n sinusoids with unknown amplitudes, frequencies and phases, a dynamic algorithm is designed which is able to recover, asymptotically, the unknown values of the frequencies, for any initial condition and any value of the frequencies.

Keywords: Frequency estimation, adaptive observer.

1 Introduction

Consider the sinusoidal signal

$$y = \sum_{i=1}^{n} A_i \sin(\omega_i t + \phi_i) + A_0 \qquad (1)$$

in which the amplitudes $A_i \neq 0$, the phases ϕ_i and the frequencies $\omega_i > 0$, $\omega_i \neq \omega_j$ for $i \neq j$, are constant and unknown, while A_0 is an unknown constant bias. Assuming that the signal y is available, we address the problem of designing a dynamic algorithm which asymptotically recovers on-line the unknown values ω_i, for any initial condition and for any arbitrary unknown constants A_i, ω_i, ϕ_i.

A globally convergent frequency estimator was recently proposed in [1] to estimate on-line the frequency ω of a given sinusoidal signal

$$y_1 = A \sin \omega t \ . \qquad (2)$$

The estimator in [1] is a modification of the continuous-time version given in [2] of the discrete-time Regalia's algorithm [3]: it is a third order algorithm

$$\begin{aligned} &\ddot{x} + 2\zeta\hat{\theta}\dot{x} + \hat{\theta}^2 x = \hat{\theta}^2 y_1 \\ &\dot{\hat{\theta}} = \gamma(2\zeta\dot{x} - \hat{\theta}y_1)x\hat{\theta} \\ &\gamma = \frac{\epsilon}{\left\{1 + N\left[x^2 + \left(\frac{\dot{x}}{\hat{\theta}}\right)^2\right]\right\}(1 + \mu|\hat{\theta}|^\alpha)} \end{aligned} \qquad (3)$$

with $\alpha \geq 1$ and ϵ, N, μ and ζ positive reals. It is shown in [1] that, given an upper bound for the amplitude A in (2) with $n = 1$, there exist constants ϵ_* and N such that for any $\epsilon \leq \epsilon_*$, $x(t)$, $\dot{x}(t)$ and $\hat{\theta}(t)$ are bounded for any initial condition and, furthermore, $\lim_{t\to\infty} \hat{\theta}(t) = \omega$.

In this paper we show how to solve the same problem via the adaptive observers developed in [4] (see also [5], Ch. 5), without imposing any restriction on the amplitude A. We then show how the same approach naturally extends to the estimation of two frequencies even in the presence of an unknown bias A_0. Finally, the extension to the general case of n unknown frequencies is presented.

2 Global estimation of one frequency

The signal $y = A\sin(\omega t + \phi)$ satisfies the linear model

$$\begin{aligned} \dot{w}_1 &= w_2 \\ \dot{w}_2 &= -\omega^2 w_1 \triangleq -\theta w_1 \\ y &= w_1 \end{aligned} \qquad (4)$$

in which $\theta = \omega^2$ is an unknown positive parameter and the initial conditions $w_1(0) = A\sin\phi$, $w_2(0) = A\omega\cos\phi$ are unknown. System (4) is observable for any θ and belongs to the class of systems for which adaptive observers can be designed, according to [4]. In fact, the filtered transformation ($\lambda > 0$ is a design parameter)

$$\dot{\xi} = -\lambda\xi - y, \quad \xi(0) = 0 \qquad (5)$$

$$\begin{aligned} y &= w_1 \\ \eta &= w_2 - \lambda w_1 - \theta\xi \end{aligned} \qquad (6)$$

transforms (4) into an adaptive observer form

$$\begin{aligned} \dot{y} &= \lambda y + \eta + \theta\xi \\ \dot{\eta} &= -\lambda\eta - \lambda^2 y \end{aligned} \qquad (7)$$

for which an adaptive observer is given by ($k > \frac{1}{4\lambda}$ and $\gamma > 0$ are design parameters, $(\hat{y}, \hat{\eta}, \hat{\theta})$ are estimates for (y, η, θ))

$$\dot{\hat{y}} = \hat{\eta} + \lambda y + \hat{\theta}\xi + k(y - \hat{y})$$

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$$\dot{\hat{\eta}} = -\lambda\hat{\eta} - \lambda^2 y$$
$$\dot{\hat{\theta}} = \gamma\xi(y - \hat{y}) \quad (8)$$

Defining the estimation errors $\tilde{y} = y - \hat{y}$, $\tilde{\eta} = \eta - \hat{\eta}$, $\tilde{\theta} = \theta - \hat{\theta}$, from (7) and (8) we obtain the error dynamics

$$\begin{aligned}\dot{\tilde{y}} &= -k\tilde{y} + \tilde{\eta} + \tilde{\theta}\xi \\ \dot{\tilde{\eta}} &= -\lambda\tilde{\eta} \\ \dot{\tilde{\theta}} &= -\gamma\xi\tilde{y}\ .\end{aligned} \quad (9)$$

The analysis of (9) is easily carried out using standard arguments in adaptive control. Since $\xi(t)$ is bounded and smooth from (5), the solutions of (9) exist for any initial condition and are bounded, given that the time derivative of

$$V = \frac{1}{2}(\tilde{y}^2 + \tilde{\eta}^2 + \frac{1}{\gamma}\tilde{\theta}^2) \quad (10)$$

is

$$\dot{V} = -k\tilde{y}^2 + \tilde{y}\tilde{\eta} - \lambda\tilde{\eta}^2 \quad (11)$$

which is negative semidefinite (recall that $k > \frac{1}{4}$). According to [6],

$$\begin{aligned}\lim_{T\to\infty}\int_t^{t+T} \xi^2(\tau)d\tau &= \int_{-\infty}^{+\infty} \left|\frac{1}{j\nu + \lambda}\right|^2 S_y(d\nu) \\ &= \left|\frac{1}{j\omega + \lambda}\right|^2 A^2\end{aligned} \quad (12)$$

where $S_y(d\nu)$ is the spectral measure of the signal y; it follows that for every $t \geq 0$, there exist positive reals T and K_p such that

$$\int_t^{t+T} \xi^2(\tau)d\tau \geq K_p > 0, \quad (13)$$

that is the signal $\xi(t)$ is persistently exciting. This implies that the origin of (9) is globally exponentially stable, as it can be directly proved using the Lyapunov function

$$W = V + \frac{1}{2}\gamma_1(\xi\tilde{\theta} - \gamma_2\xi\tilde{y})^2 \quad (14)$$

with γ_1 and γ_2 suitable positive constants. In particular, $\tilde{\theta}(t)$ tends exponentially to zero for every initial condition of the fourth order algorithm (5), (8): hence $\hat{\theta}$ is a global exponential estimator of $\theta = \omega^2$. When compared with the third order estimator (3), the proposed algorithm has the advantage that no upper bound for the amplitude A is required. It has the additional advantage of being designed on the basis of a constructive procedure which may be extended to the two frequencies case. If the given signal contains an unknown bias A_0, that is $y = A_0 + A\sin(\omega t + \phi)$, it can be modelled as

$$\begin{aligned}\dot{w}_1 &= w_2 \\ \dot{w}_2 &= -\omega^2 w_1 + A_0\omega^2 = \theta_0 - \theta_1 w_1 \\ y &= w_1\end{aligned}$$

in which $\theta = [\theta_0, \theta_1]^T = [A_0\omega^2, \omega^2]^T$ is the vector of unknown parameters. The adaptive observer is then modified as follows

$$\begin{aligned}\dot{\xi} &= -\lambda\xi - y, \quad \xi(0) = 0 \\ \dot{\hat{y}} &= \hat{\eta} + \lambda y + \frac{1}{\lambda}y\hat{\theta}_0 + \hat{\theta}_1\xi + k(y - \hat{y}) \\ \dot{\hat{\eta}} &= -\lambda\eta - \lambda^2 y \\ \dot{\hat{\theta}}_0 &= \gamma_0(y - \hat{y}) \\ \dot{\hat{\theta}}_1 &= \gamma_1\xi(y - \hat{y})\end{aligned}$$

where $(\lambda, k > \frac{1}{4\lambda}, \gamma_0, \gamma_1)$ are positive design parameters. By using the same arguments we can conclude that θ_0 and θ_1 tend exponentially to $A_0\omega^2$ and ω^2, respectively, for any initial condition.

3 Global estimation of two frequencies

The signal y given in (1) when $n = 2$ (let $A_0 = 0$ for simplicity) satisfies the linear model of order 4

$$\begin{aligned}\dot{w}_1 &= w_2 \\ \dot{w}_2 &= -\omega_1^2 w_1 \\ \dot{w}_3 &= w_4 \\ \dot{w}_4 &= -\omega_2^2 w_3 \\ y &= w_1 + w_3\end{aligned} \quad (15)$$

with unknown initial conditions. Its characteristic polynomial is

$$s^4 + (\omega_1^2 + \omega_2^2)s^2 + \omega_1^2\omega_2^2 = s^4 + \theta_1 s^2 + \theta_2 \quad (16)$$

where $\theta_1 = \omega_1^2 + \omega_2^2$, $\theta_2 = \omega_1^2\omega_2^2$ is a reparametrization of the two unknowns (ω_1^2, ω_2^2). Since (15) is observable (recall that $\omega_1 \neq \omega_2$), it is transformed by the linear change coordinates

$$x = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \omega_2^2 & 0 & \omega_1^2 & 0 \\ 0 & \omega_2^2 & 0 & \omega_1^2 \end{bmatrix} w \quad (17)$$

into an observer canonical form ($x \in R^4$)

$$\begin{aligned}\dot{x} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\theta_1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\theta_2 & 0 & 0 & 0 \end{bmatrix} x \\ &= A_c x + \theta_1 \begin{bmatrix} 0 \\ -y \\ 0 \\ 0 \end{bmatrix} + \theta_2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ -y \end{bmatrix} \\ y &= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x = C_c x\end{aligned} \quad (18)$$

with θ_1, θ_2 unknown parameters. According to [4, 5], given an arbitrary Hurwitz vector $d = [1, d_2, d_3, d_4]^T$, i.e. a vector such that all of the zeros of the polynomial $s^3 + d_2 s^2 + d_3 s + d_4$ have real part less than zero, the filtered transformation

$$\begin{aligned}
\dot{\xi}_1 &= \Gamma\xi_1 - \begin{bmatrix} y \\ 0 \\ 0 \end{bmatrix}, \quad \xi_1(0) = 0 \\
\mu_1 &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \xi_1 \\
\dot{\xi}_2 &= \Gamma\xi_2 - \begin{bmatrix} 0 \\ 0 \\ y \end{bmatrix}, \quad \xi_2(0) = 0 \\
\mu_2 &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \xi_2 \qquad (19) \\
z_1 &= x_1 = y \\
z_j &= x_j - \xi_{1,j-1}\theta_1 - \xi_{2,j-1}\theta_2, \quad 2 \le j \le 4 \qquad (20)
\end{aligned}$$

with

$$\Gamma = \begin{bmatrix} -d_2 & 1 & 0 \\ -d_3 & 0 & 1 \\ -d_4 & 0 & 0 \end{bmatrix}$$

a Hurwitz matrix, transforms (18) into an adaptive observer form ($z \in R^4$)

$$\begin{aligned}
\dot{z} &= A_c z + d(\mu_1\theta_1 + \mu_2\theta_2) \\
y &= C_c z \; . \qquad (21)
\end{aligned}$$

Defining $\eta_j = z_{j+1} - d_{j+1} z_1$, $j = 1, 2, 3$, (21) is equivalently expressed as ($\eta = [\eta_1, \eta_2, \eta_3]^T$)

$$\begin{aligned}
\dot{y} &= \eta_1 + d_2 y + \mu_1\theta_1 + \mu_2\theta_2 \\
\dot{\eta} &= \Gamma\eta + \beta y \qquad (22)
\end{aligned}$$

with $\beta = [d_3 - d_2^2, d_4 - d_3 d_2, -d_4 d_2]^T$. The signals $\mu = [\mu_1, \mu_2]^T$ given in (19) can be equivalently generated by the minimal realization in controller canonical form

$$\begin{aligned}
\dot{\chi} &= \Gamma^T\chi + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} y, \quad \chi(0) = 0 \\
\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \chi \qquad (23)
\end{aligned}$$

with transfer matrix:

$$\begin{aligned}
\begin{bmatrix} \mu_1(s) \\ \mu_2(s) \end{bmatrix} &= \frac{1}{s^3 + d_2 s^2 + d_3 s + d_4} \begin{bmatrix} -s^2 \\ -1 \end{bmatrix} y(s) \\
&\triangleq H_\mu(s) y(s) \; .
\end{aligned}$$

The adaptive observer for (22) is given by ($k > \frac{1}{4\|\Gamma\|}$ is a design parameter)

$$\begin{aligned}
\dot{\hat{y}} &= \hat{\eta}_1 + d_2 y + \mu_1\hat{\theta}_1 + \mu_2\hat{\theta}_2 + k(y - \hat{y}) \\
\dot{\hat{\eta}} &= \Gamma\hat{\eta} + \beta y \\
\dot{\hat{\theta}}_1 &= \mu_1(y - \hat{y}) \\
\dot{\hat{\theta}}_2 &= \mu_2(y - \hat{y}) \qquad (24)
\end{aligned}$$

which, along with (23), gives the proposed 9-th order estimator with $(\hat{\theta}_1, \hat{\theta}_2)$ estimates for (θ_1, θ_2). Defining $\tilde{y} = y - \hat{y}$, $\tilde{\eta} = \eta - \hat{\eta}$, $\tilde{\theta} = [\theta_1 - \hat{\theta}_1, \theta_2 - \hat{\theta}_2]^T$, the error system is

$$\begin{aligned}
\dot{\tilde{\eta}} &= \Gamma\tilde{\eta} \\
\dot{\tilde{y}} &= -k\tilde{y} + \tilde{\eta}_1 + \mu^T\tilde{\theta} \\
\dot{\tilde{\theta}} &= -\mu\tilde{y} \; . \qquad (25)
\end{aligned}$$

According to [4, 5, 6], if the persistency of excitation condition

$$\int_t^{t+T} \mu(\tau)\mu^T(\tau) d\tau \ge k_p I > 0 \qquad (26)$$

is satisfied for some $T > 0$, $k_p > 0$ and for every $t \ge 0$, then $\tilde{\theta}_i(t)$, $i = 1, 2$, tend exponentially to zero for any initial condition as t goes to infinity, while in any case the state variables in (25) are bounded and $\tilde{y}(t)$ tends to zero as t goes to infinity, for any initial condition. Following [6] (Theorem 2.7.2) (see also [7]), since the signal $y = A_1 \sin(\omega_1 t + \phi_1) + A_2 \sin(\omega_2 t + \phi_2)$ is sufficiently rich of order 2, the transfer matrix $H_\mu(s)$ is stable and $H_\mu(j\omega_i)$ are linearly independent for $i = 1, 2$, then $\mu(t)$ is persistently exciting and (26) is satisfied.

If the signal y in (1) contains an unknown bias A_0, then it satisfies the linear observable model

$$\begin{aligned}
\dot{x} &= A_c x + \theta_1 \begin{bmatrix} 0 \\ -y \\ 0 \\ 0 \end{bmatrix} + \theta_2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ -y \end{bmatrix} \\
&\quad + A_0\theta_1 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + A_0\theta_2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\
y &= C_c x \qquad (27)
\end{aligned}$$

which is an obvious modification of (18). Then there exists a filtered transformation mapping (27) into

$$\begin{aligned}
\dot{z} &= A_c z + d(\mu_1\theta_1 + \mu_2\theta_2 + \theta_0) \\
y &= C_c z \qquad (28)
\end{aligned}$$

with (μ_1, μ_2) given by (23) and

$$\theta_0 = A_0 C_c \Gamma^{-1} \left(\theta_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \theta_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) .$$

Hence (28) can be written as

$$\begin{aligned}
\dot{y} &= \eta_1 + d_2 y + \mu_1\theta_1 + \mu_2\theta_2 + \theta_0 \\
\dot{\eta} &= \Gamma\eta + \beta y
\end{aligned}$$

so that the adaptive observer is

$$\dot{\hat{y}} = \hat{\eta}_1 + d_2 y + \mu_1\hat{\theta}_1 + \mu_2\hat{\theta}_2 + \hat{\theta}_0 + k(y - \hat{y})$$

$$
\begin{aligned}
\dot{\hat{\eta}} &= \Gamma\hat{\eta} + \beta y \\
\dot{\hat{\theta}}_0 &= y - \hat{y} \\
\dot{\hat{\theta}}_1 &= \mu_1(y - \hat{y}) \\
\dot{\hat{\theta}}_2 &= \mu_2(y - \hat{y}) \ .
\end{aligned}
$$

4 Global estimation of n frequencies

A signal

$$
y = \sum_{i=1}^{n} A_i \sin(\omega_i t + \phi_i) \tag{29}
$$

with unknown amplitudes $A_i \neq 0$, phases ϕ_i and pulsations $\omega_i > 0$, $1 \leq i \leq n$, $\omega_i \neq \omega_j$ for $i \neq j$, is generated by the linear observable model of order $2n$

$$
\begin{aligned}
\dot{w}_{i1} &= w_{i2} \\
\dot{w}_{i2} &= -\omega_i^2 w_{i1}, \quad 1 \leq i \leq n \\
y &= \sum_{i=1}^{n} w_{i1}
\end{aligned} \tag{30}
$$

with unknown initial conditions. Its characteristic polynomial is

$$
\begin{aligned}
\prod_{i=1}^{n}(s^2 + \omega_i^2) &= s^{2n} + \sum_{i=1}^{n} \omega_i^2 s^{2(n-1)} \\
&+ \sum_{i=1}^{n-1} \sum_{j=1+1}^{n} \omega_i^2 \omega_j^2 s^{2(n-2)} + \ldots + \prod_{i=1}^{n} \omega_i^2 \\
&\triangleq s^{2n} + \theta_1 s^{2(n-1)} + \ldots + \theta_{n-1} s^2 + \theta_n
\end{aligned} \tag{31}
$$

with $(\theta_1, \ldots, \theta_n)$ a reparametrization of the n unknowns $(\omega_1^2, \ldots, \omega_n^2)$. Actually, $(\theta_1, \ldots, \theta_n)$ are the coefficients of a characteristic polynomial of order n whose real positive zeros are $(\omega_1^2, \ldots, \omega_n^2)$. Since (30) is observable (recall that $\omega_i \neq \omega_j$, $i \neq j$), it is transformable by a linear change of coordinates into an observer canonical form ($x \in R^{2n}$)

$$
\begin{aligned}
\dot{x} &= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ -\theta_1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\theta_n & 0 & 0 & \cdots & 0 \end{bmatrix} x \\
&\triangleq A_c x - \sum_{i=1}^{n} \theta_i e_i y \\
y &= \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} x \triangleq C_c x
\end{aligned} \tag{32}
$$

with $e_i = [0, \ldots, 0, 1, 0, \ldots, 0]^T$ a $(2n \times 1)$ vector whose only nonzero entry is the $2i$-th one. According to [4, 5], given an arbitrary Hurwitz vector $d = [1, d_2, \ldots, d_{2n}]^T$, i.e. a vector such that all the $2n-1$ zeros of the polynomial $s^{2n-1} + d_2 s^{2n-2} + \ldots + d_{2n-1}s + d_{2n}$ have negative real part, the filtered transformation

$$
\begin{aligned}
\dot{\xi}_i &= \Gamma\xi_i + e_i y, \quad 1 \leq i \leq n, \ \xi \in R^{2n-1} \\
\mu_i &= \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \xi_i
\end{aligned} \tag{33}
$$

$$
\begin{aligned}
z_1 &= x_1 = y \\
z_j &= x_j - \sum_{i=1}^{n} \xi_{i,j-1}\theta_i, \quad 2 \leq j \leq 2n
\end{aligned} \tag{34}
$$

with

$$
\Gamma = \begin{bmatrix} -d_2 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -d_{2n-2} & 0 & \cdots & 1 \\ -d_{2n-1} & 0 & \cdots & 0 \end{bmatrix}
$$

a $(2n-1) \times (2n-1)$ Hurwitz matrix, transforms (30) into an adaptive observer form ($z \in R^{2n}$, $\mu = [\mu_1, \ldots, \mu_n]^T$, $\theta = [\theta_1, \ldots, \theta_n]^T$)

$$
\begin{aligned}
\dot{z} &= A_c z + d\mu^T\theta \\
y &= C_c z \ .
\end{aligned} \tag{35}
$$

Defining

$$
\eta_j = z_{j+1} - d_{j+1} z_1, \quad 1 \leq j \leq 2n-1
$$

system (35) is equivalently expressed as ($\eta = [\eta_1, \ldots, \eta_{2n-1}]^T$)

$$
\begin{aligned}
\dot{y} &= \eta_1 + d_2 y + \mu^T\theta \\
\dot{\eta} &= \Gamma\eta + \beta y
\end{aligned} \tag{36}
$$

with $\beta = [d_3 - d_2^2, d_4 - d_3 d_2, \ldots, d_{2n} - d_{2n-1}d_2, -d_{2n}d_2]^T$. The signals μ given as outputs of the n filters (33) can be equivalently generated by the minimal realization

$$
\begin{aligned}
\dot{\chi} &= \Gamma^T\chi + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} y, \quad \chi \in R^{2n-1} \\
\mu_i &= -\chi_{2i-1}, \quad 1 \leq i \leq n
\end{aligned} \tag{37}
$$

with transfer matrix

$$
\begin{aligned}
\mu(s) &= \frac{-1}{s^{2n-1} + \cdots + d_{2n-1}s + d_{2n}} \begin{bmatrix} s^{2n-2} \\ \vdots \\ s^2 \\ 1 \end{bmatrix} y(s) \\
&\triangleq H_\mu(s) y(s) \ .
\end{aligned} \tag{38}
$$

The adaptive observer for (36) is given by ($k > \frac{1}{4\|\Gamma\|}$ a design parameter)

$$
\begin{aligned}
\dot{\hat{y}} &= \hat{\eta}_1 + d_2 y + \mu^T\hat{\theta} \\
\dot{\hat{\eta}} &= \Gamma\hat{\eta} + \beta y \\
\dot{\hat{\theta}} &= \mu(y - \hat{y})
\end{aligned} \tag{39}
$$

which, along with (37), gives the proposed estimator of order $2(2n-1)+n+1=5n-1$, with $\hat{\theta}$ being the estimate for θ. Defining $\tilde{y}=y-\hat{y}$, $\tilde{\eta}=\eta-\hat{\eta}$, $\tilde{\theta}=\theta-\hat{\theta}$, the error system is

$$\begin{aligned} \dot{\tilde{\eta}} &= \Gamma\tilde{\eta} \\ \dot{\tilde{y}} &= -k\tilde{y}+\tilde{\eta}_1+\mu^T\tilde{\theta} \\ \dot{\tilde{\theta}} &= -\mu\tilde{y}\ . \end{aligned} \tag{40}$$

According to [4, 5, 6] if the persistency of excitation condition

$$\int_t^{t+T}\mu(\tau)\mu^T(\tau)d\tau \geq k_p I > 0 \tag{41}$$

is satisfied for some $T>0$, $k_p>0$ and every $t\geq 0$, then $\tilde{\theta}(t)$ tends exponentially to zero for any initial condition, as t goes to infinity. In any case the states of (40) are bounded and $\tilde{y}(t)$ tends to zero as t goes to infinity. Following [6] (Theorem 2.7.2) (see also [7]), since the signal $y(t)$ in (29) is sufficiently rich of order n, $H_\mu(s)$ is a stable transfer matrix and $H_\mu(j\omega_i)$ are linearly independent for $i=1,\ldots,n$, then $\mu(t)$ is persistently exciting and (41) is satisfied.

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