

Optimal Control Law for Fault Tolerant Control Systems

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Abstract

In this paper, an optimal control law is designed for Fault Tolerant Control Systems with Markovian Parameters (FTCSMP). The matrix maximum principle is used to minimize an equivalent deterministic cost function. Three scenarios are considered. The first assumes that both the failure process and the Fault Detection and Isolation (FDI) process are accessible for the controller. In the second scenario, the controller is reconfigured based on the decisions of the FDI process and does not need an accessible failure process. The case where the FDI process itself is not able to give any decision due to physical malfunction or excessive computational time is the third scenario. In these scenarios, optimal control laws are developed to reduce the risk of losing system stability. A computational algorithm is constructed to calculate the optimal control law.

1 INTRODUCTION

Fault tolerant control systems with Markovian parameters (FTCSMP) have recently been developed. The mathematical model of FTCSMP defines two random processes: one to represent the random failure that may take place in the system, and the other is to represent the decisions of the Fault Detection and Isolation (FDI) process. The uniqueness of this modeling is that it does not assume perfect performance for the FDI process, but it allows the study of the effect of detection delays and errors in FDI. The existing research is more focused on the stochastic stability of FTCSMP. Necessary and sufficient conditions for the exponential stability in the mean square were derived for FTCSMP with actuator failures [8]. The work was extended for two independent failure processes in [4], one for actuator failures and the other for plant components failures. Substantial results were obtained in [5] where stochastic stability for FTCSMP is studied in noisy environment. Three types of noise were considered: state-dependent, control-dependent and purely additive white Gaussian noise. The effect of modeling uncertainties on the stochastic stability of FTCSMP was addressed in [6]. To the best of our knowledge, the problem concerned with the existence and the design of optimal control laws for FTCSMP has not been considered yet.

The inadequacy of the existing results in synthesizing an

optimal controller for FTCSMP is the primary motivation for the current study. Specifically, in this paper, an optimal control law will be designed for FTCSMP. To achieve this objective, the initial probability distributions for the failure process and the FDI process are used. Depending upon the complexity of the system and the assumed distribution functions, the initial probabilities are determined either analytically using stochastic modeling or numerically using Monte Carlo simulations. In some extreme cases, the FDI process is not able to make any decision. This situation may occur due to physical damage to the data acquisition hardware of the FDI algorithm or due to the excessive time needed for the statistical tests to converge to a decision. An optimal control law is designed for both the finite and the infinite time-horizon. Necessary conditions which guarantee the existence of such controllers are derived and verified.

The paper is organized as follows: The optimization problem is stated in Section 2. Optimal control laws for the three scenarios are designed in Section 3. Section 4 extends the optimization problem to the infinite time horizon. Section 5 outlines a computational algorithm. A numerical example to illustrate the theory is given in Section 6. Finally, a concluding summary is presented in Section 7.

2 PROBLEM STATEMENT

Consider a FTCSMP shown in Figure 1, which is modeled as [4, 8]

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B(\eta(t)) u(x(t), \Psi(t), t) \\ u(x(t), \Psi(t), t) &= -K(\Psi(t))x(t) \end{aligned} \quad (1)$$

where $\eta(t)$ is the actuator failure process and $\Psi(t)$ is the FDI process. Both processes are assumed to have Markovian transition characteristics with finite state spaces $S = \{1, 2, \dots, s\}$, $R = \{1, 2, \dots, r\}$, respectively. $x(t) \in \mathbb{R}^n$ is the system state, $u(x(t), \Psi(t), t) \in \mathbb{R}^m$ is the input, and $K(\Psi(t))$ is a piecewise constant gain matrix.

The transition probability of the actuator failure process, $\eta(t)$, is

$$p_{kj}^k(\Delta t) = \alpha_{kj} \Delta t + o(\Delta t) \quad (k \neq j) \quad (2)$$

and the conditional transition probability of the FDI process, $\Psi(t)$, is

$$p_{ij}^k(\Delta t) = q_{ij}^k \Delta t + o(\Delta t) \quad (i \neq j) \quad (3)$$

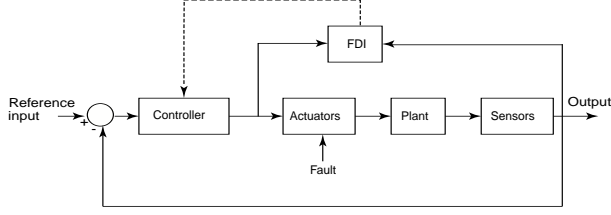


Figure 1: General Schematic Diagram for FTCS.

The conditional transition rate, q_{ij}^k , is the probability that the FDI process will decide that the next state is j leaving the state i given that the actuator failure process is in the state k . It is important to recall that the stochastic stability of FTCSMP depends upon the performance of the FDI process through q_{ij}^k [4, 5, 6, 8].

In the sequel, $B(\eta(t)) = B_k$ when $\eta(t) = k \in S$ and $u(x(t), \Psi(t), t) = u_i$ when $\Psi(t) = i \in R$. To simplify notations an indicator function $\Theta_{ki} \in \mathbb{R}^{S \times R}$ will be used to describe that the failure process, $\eta(t)$, is currently in state k , and the FDI process, $\Psi(t)$, is in state i .

The objective is to synthesize a fault tolerant control law for the FTCSMP (1) that minimizes the following quadratic performance index

$$J = \mathcal{E} \left\{ \int_{t_o}^{t_f} \{x^T(t)Q(\eta(t), \Psi(t))x(t) + u^T(t)R(\eta(t), \Psi(t))u(t)\} dt \right\} \quad (4)$$

where \mathcal{E} is the mathematical expectation. $Q(\eta(t), \Psi(t)) \geq 0$ and $R(\eta(t), \Psi(t)) > 0$. Denote Q_{ki} and R_{ki} when $\eta = k \in S$, $\Psi = i \in R$. The optimal control problem is

$$\begin{aligned} \min_u \mathcal{E} \left\{ \int_{t_o}^{t_f} \{x^T(t)Q(\eta(t), \Psi(t))x(t) + u^T(t)R(\eta(t), \Psi(t))u(t)\} dt \right\} \\ \text{Subject to} \\ \dot{x}(t) = Ax(t) - B(\eta(t))K(\Psi(t))x(t) \end{aligned} \quad (5)$$

Define covariance-like matrices as

$$X_{ki}(t) = \mathcal{E}\{x(t)x(t)^T | \eta(t) = k, \Psi(t) = i\} = \mathcal{E}\{x(t)x(t)^T | \Theta_{ki}\} \quad (6)$$

Then, the equivalent deterministic cost function is given in the following lemma.

Lemma 1: *Let the current state of the FDI process be $i \in R$, and the actuator failure process is in state $k \in S$. If $\eta(t_o), \Psi(t_o)$ are independent of $x(t_o)$, then for the set of matrices $(Q_{ki} + K_i^T R_{ki} K_i)$, the cost function is*

$$J_{d1} = \int_{t_o}^{t_f} \text{tr}\{X_{ki}(t)(Q_{ki} + K_i^T R_{ki} K_i)\} dt$$

The time-derivatives of $X_{ki}(t)$ can be calculated as

$$\begin{aligned} \dot{X}_{ki} &= (A - B_k K_i)X_{ki} + X_{ki}(A - B_k K_i)^T \\ &+ \sum_{\substack{j \in S \\ j \neq k}} \alpha_{jk} [X_{ji} - X_{ki}] + \sum_{\substack{j \in R \\ j \neq i}} q_{ji}^k [X_{kj} - X_{ki}] \end{aligned} \quad (7)$$

Define

$$\tilde{A}_{ki} = A - B_k K_i - 0.5I \sum_{\substack{j \in S \\ j \neq k}} \alpha_{jk} - 0.5I \sum_{\substack{j \in R \\ j \neq i}} q_{ji}^k \quad (8)$$

Then

$$\dot{X}_{ki} = \tilde{A}_{ki} X_{ki} + X_{ki} \tilde{A}_{ki}^T + \sum_{\substack{j \in S \\ j \neq k}} \alpha_{jk} X_{ji} + \sum_{\substack{j \in R \\ j \neq i}} q_{ji}^k X_{kj} \quad (9)$$

X_{ki} are solved forward in time with the initial conditions

$$X_{ki}(t_o) = \mathcal{E}\{x(t_o)x(t_o)^T | \Theta_{ki}\} \quad (10)$$

Therefore, the equivalent deterministic optimization problem to be solved is

$$\begin{aligned} \min_{K_i} \int_{t_o}^{t_f} \text{tr}\{X_{ki}(t)(Q_{ki} + K_i^T R_{ki} K_i)\} dt \\ \text{subject to} \\ \dot{X}_{ki} = \tilde{A}_{ki} X_{ki} + X_{ki} \tilde{A}_{ki}^T + \sum_{\substack{j \in S \\ j \neq k}} \alpha_{jk} X_{ji} + \sum_{\substack{j \in R \\ j \neq i}} q_{ji}^k X_{kj} \end{aligned} \quad (11)$$

3 OPTIMAL CONTROL ON A FINITE TIME-HORIZON

In this section, necessary conditions for the existence of optimal control law that minimizes the cost function (4) for the FTCSMP (1) are developed. The optimal control law is derived using the equivalent deterministic cost function.

3.1 Both Failure and FDI Processes Accessible

As mentioned earlier the FTCSMP defines two separate processes: $\eta(t)$ to represent the random failures in the system and $\Psi(t)$ to represent the decisions of the FDI process. Therefore, any attempt to design an optimal control law for the FTCSMP leads to a controller function of both $\eta(t)$ and $\Psi(t)$ which is the first scenario considered in this work. In the following, an optimal controller will be synthesized applying the technique of matrix maximum principle [1].

Theorem 1 *The optimal gain matrices are obtained from*

$$R_{ki} K_{ki} X_{ki}(t) - B_k^T P_{ki}(t) X_{ki}(t) = 0$$

where $P_{ki}(t)$ are the solutions of the matrix Riccati-like differential equations:

$$\begin{aligned} \dot{P}_{ki}(t) + \tilde{A}_{ki}^T P_{ki}(t) + P_{ki}(t) \tilde{A}_{ki} + \sum_{\substack{j \in S \\ j \neq k}} \alpha_{kj} P_{ji}(t) + \\ \sum_{\substack{j \in R \\ j \neq i}} q_{ij}^k P_{kj}(t) + K_{ki}^T R_{ki} K_{ki} + Q_{ki} = 0, \quad \forall P_{ki}(t_f) = 0 \end{aligned}$$

and $X_{ki}(t)$ are the solutions of the following covariance-like differential equations

$$\begin{aligned} \dot{X}_{ki}(t) &= \tilde{A}_{ki} X_{ki}(t) + X_{ki}(t) \tilde{A}_{ki}^T + \sum_{j \in S, j \neq k} \alpha_{jk} X_{ji}(t) + \sum_{j \in R, j \neq i} q_{ji}^k X_{kj}(t) \\ \forall X_{ki}(t_0) &= \mathcal{E}\{x(t_0)x(t_0)^T | \Theta_{ki}\}, \tilde{A}_{ki} \text{ is defined in (8)} \end{aligned}$$

Proof: Recall that the equivalent deterministic optimization problem is given in (11). Applying the techniques of matrix maximum principle [1], the associated Hamiltonian is

$$\begin{aligned} H &= \text{tr}\{X_{ki}(t)(Q_{ki} + K_{ki}^T R_{ki} K_{ki})\} + \text{tr}\{(\tilde{A}_{ki} X_{ki}(t) \\ &+ X_{ki}(t) \tilde{A}_{ki}^T + \sum_{j \in S, j \neq k} \alpha_{jk} X_{ji} + \sum_{j \in R, j \neq i} q_{ji}^k X_{kj}) P_{ki}^T\} \end{aligned} \quad (12)$$

The state optimality condition is

$$\begin{aligned} \dot{X}_{ki}^* &= \frac{\partial H}{\partial P_{ki}} \\ &= \tilde{A}_{ki} X_{ki}^*(t) + X_{ki}^*(t) \tilde{A}_{ki}^T + \sum_{j \in S, j \neq k} \alpha_{jk} X_{ji}^* + \sum_{j \in R, j \neq i} q_{ji}^k X_{kj}^* \end{aligned} \quad (13)$$

The co-state optimality condition is

$$\begin{aligned} \dot{P}_{ki}^* &= \frac{-\partial H}{\partial X_{ki}} = -\{K_{ki}^{*T} R_{ki} K_{ki}^* + Q_{ki}\} - \\ &\{\tilde{A}_{ki}^T P_{ki}^*(t) + P_{ki}^*(t) \tilde{A}_{ki} + \sum_{j \in S, j \neq k} \alpha_{jk} P_{ji}^* + \sum_{j \in R, j \neq i} q_{ji}^k P_{kj}^*\} \end{aligned} \quad (14)$$

and
$$\begin{aligned} 0 &= \frac{\partial H}{\partial K_{ki}} \\ &= -R_{ki} K_{ki}^* X_{ki}^*(t) + B_k^T P_{ki}^*(t) X_{ki}^*(t) \end{aligned} \quad (15)$$

Equations (13), (14), and (15) are the conditions stated in Theorem 1.

Note that for the non-singular X_{ki} , the optimal control gain is uniquely determined by the solutions of the Riccati-like differential equations $P_{ki}(t)$. In other words, the Riccati-like and the covariance-like differential equations are not coupled through K_{ki} . Therefore, this case is not a boundary value problem.

3.2 Only FDI Process Accessible

This is the second scenario where only the FDI process is available for control reconfiguration. It is well known that, errors in failure detection may not only lead to a degraded system performance but also to a complete loss of system stability. To reduce the risk of losing stability, the initial probability distribution for the actuator under study is considered. Let the actuator failure process $\eta(t)$ have a Markovian transition probabilities defined in (2) and a probability distribution as

$$v_k(t) = \text{Prob}\{\eta(t) = k\} \quad (16)$$

where

$$v_k(t) \geq 0 \quad \forall k \in S, \quad \sum_{k=1}^s v_k(t) = 1 \quad (17)$$

The initial state is assumed to be a random variable with

$$\mathcal{E}\{x(t_0)\} = 0, \quad \mathcal{E}\{x(t_0)x(t_0)^T\} = X(t_0) \quad (18)$$

By (16), the covariance-like matrices will be

$$X_i(t) = \sum_{k=1}^s v_k(t) X_{ki}(t) \quad (19)$$

Theorem 2 A necessary condition for the control law $u_i(t) = -K_i x(t)$, $i \in R$ to be optimal is that the gains K_i satisfy the following matrix differential equations:

$$\begin{aligned} \dot{P}_{ki}(t) &+ \tilde{A}_{ki}^T P_{ki}(t) + P_{ki}(t) \tilde{A}_{ki} + \sum_{j \in S, j \neq k} \alpha_{jk} P_{ji}(t) + \\ &\sum_{j \in R, j \neq i} q_{ij}^k P_{kj}(t) + K_i^T R_{ki} K_i + Q_{ki} = 0, \quad \forall P_{ki}(t_f) = 0 \end{aligned}$$

with

$$\begin{aligned} \dot{X}_{ki}(t) &= \tilde{A}_{ki} X_{ki}(t) + X_{ki}(t) \tilde{A}_{ki}^T + \sum_{j \in S, j \neq k} \alpha_{jk} X_{ji}(t) + \sum_{j \in R, j \neq i} q_{ji}^k X_{kj}(t) \\ \forall X_{ki}(t_0) &= \sum_{k=1}^s v_k(t_0) X_o \end{aligned}$$

and the optimal control gain is obtained from

$$\sum_{k=1}^s [R_{ki} K_i v_k(t) X_{ki}(t) - B_k^T v_k(t) P_{ki}(t) X_{ki}(t)] = 0$$

Proof: The proof can be obtained using similar arguments to that in Theorem 1. Therefore, only the cost function to be minimized is stated.

Lemma 2: Let the actuator failure process $\eta(t)$ have a probability distribution $v = [v_1, v_2, \dots, v_s]$. Assume that $\eta(t_0)$ is independent from $x(t_0)$. If the current state of the FDI process is $i \in R$, then for some matrices $(Q_{ki} + K_i^T R_{ki} K_i)$, the cost function is

$$J_{d2} = \int_{t_0}^{t_f} \text{tr}\left\{\sum_{k=1}^s v_k(t) X_{ki}(t) (Q_{ki} + K_i^T R_{ki} K_i)\right\} dt$$

The Riccati-like differential equations are non-linearly coupled with the covariance-like differential equations through the gain matrices, K_i . This case requires the solution of a boundary value problem to calculate the optimal control gain matrices. Fortunately, these calculations can be done off-line and only control reconfiguration is completed on-line.

3.3 Both Failure and FDI Processes Not Accessible

There are cases where the decision of the FDI process is not available. One possible case arise due to large time delays needed by the FDI algorithm to converge to a decision. This issue was emphasized by several works in the area of FTCS [4, 7, 8]. Another possible case is the physical malfunction

to the data acquisition hardware of the FDI algorithm due to aging, wearing, or any extreme environmental changes. To synthesize an optimal control law, the actuator failure process $\eta(t)$ is assumed to have the probability distribution in (16), whereas the conditional probability distribution for the FDI process is

$$w_i^k(t) = Prob\{\Psi(t) = i \mid \eta(t) = k\} \quad (20)$$

where

$$w_i^k(t) \geq 0, \quad \sum_{i=1}^r w_i^k(t) = 1, \quad \forall k \in S, i \in R$$

Then, the covariance-like matrix becomes

$$X(t) = \sum_{i=1}^r \sum_{k=1}^s w_i^k(t) v_k(t) X_{ki}(t) \quad (21)$$

The cost function for this scenario is defined in Lemma 3.

Lemma 3: *Let the actuator failure process $\eta(t)$ have a probability distribution $v(t_o) = [v_1(t_o), v_2(t_o), \dots, v_s(t_o)]$, and the FDI process has a conditional probability distribution $w_i^k(t_o) = [w_1^k(t_o), w_2^k(t_o), \dots, w_r^k(t_o)]$. Assume that $\eta(t_o)$, and $\Psi(t_o)$ are independent from $x(t_o)$. Then for some matrices $(Q_{ki} + K_i^T R_{ki} K_i)$, the cost function to be minimized is*

$$J_{d3} = \int_{t_o}^{t_f} tr\left\{ \sum_{i=1}^r \sum_{k=1}^s w_i^k(t) v_k(t) X_{ki}(t) (Q_{ki} + K_i^T R_{ki} K_i) \right\} dt \quad (22)$$

Theorem 3 states the conditions for which an optimal control law exists.

Theorem 3 *A necessary condition for the control law $u(t) = -Kx(t)$ to be optimal is that the constant gain K satisfies the following matrix differential equations:*

$$\begin{aligned} \dot{P}_{ki}(t) + \bar{A}_{ki}^T P_{ki}(t) + P_{ki}(t) \bar{A}_{ki} + \sum_{\substack{j \in S \\ j \neq k}} \alpha_{kj} P_{ji}(t) + \\ \sum_{\substack{j \in R \\ j \neq i}} q_{ij}^k P_{kj}(t) + K^T R_{ki} K + Q_{ki} = 0, \quad \forall P_{ki}(t_f) = 0 \end{aligned}$$

with

$$\begin{aligned} \dot{X}_{ki}(t) &= \bar{A}_{ki} X_{ki}(t) + X_{ki}(t) \bar{A}_{ki}^T + \sum_{\substack{j \in S \\ j \neq k}} \alpha_{jk} X_{ji}(t) + \sum_{\substack{j \in R \\ j \neq i}} q_{ji}^k X_{kj}(t) \\ \forall X_{ki}(t_o) &= \sum_{i=1}^r \sum_{k=1}^s w_i^k(t_o) v_k(t_o) X_o \end{aligned}$$

and the optimal control gain is obtained from

$$\sum_{i=1}^r \sum_{k=1}^s [R_{ki} K w_i^k(t) v_k(t) X_{ki}(t) - B_k^T w_i^k(t) v_k(t) P_{ki}(t) X_{ki}(t)] = 0$$

Similar to the previous theorems, the three optimality conditions define the covariance-like, Riccati-like, and the gain equations.

The results of Theorem 3 are similar to those of Theorem 2. That is, to obtain an optimal control law, a boundary value problem must be solved. The coupling between the Riccati-like and the covariance-like differential equations contains nonlinearities of higher order. The increased non-linearity will result in increased computation time. However, the gain calculation is completed off-line.

4 OPTIMAL CONTROL ON AN INFINITE TIME-HORIZON

As the time interval approaches infinity, the behavior of the optimization problem needs to be carefully examined. The cost function itself may become infinite, therefore, the problem will be unsolvable. The limiting behavior for the solutions of the differential equations is not well defined, therefore, the existence of an optimal control law is in doubt. In this section, these difficulties will be attacked and the conditions for which an optimal control law exists for the infinite time-horizon are derived. Since the three scenarios considered in this work have a unified approach of solution, the developments of this section is not restricted to a particular scenario.

4.1 The Cost Function for Infinite Time-Interval

For the optimization problem to be solvable as $t \rightarrow \infty$, the cost function must be finite for some control and associated state trajectories. Under certain conditions, this finiteness can be guaranteed as stated in the following lemma.

Lemma 4: *If the system (1) is exponentially stable in the mean square, a non-empty set of control and trajectory pairs $u(t)$, $x(t)$ exists for which (4) is finite for each $x(t_o)$.*

Proof: If the system (1) is exponentially stable in the mean square, then for some $a, b > 0$ [2, 3, 4]

$$\mathcal{E}\{\|x(t)\|^2\} \leq a \|x_o\|^2 \exp\{-b(t - t_o)\} \quad (23)$$

Let

$$\begin{aligned} Q(\eta(t), \Psi(t)) &= F^T(\eta(t), \Psi(t)) F(\eta(t), \Psi(t)) \\ R(\eta(t), \Psi(t)) &= W^T(\eta(t), \Psi(t)) W(\eta(t), \Psi(t)) \end{aligned} \quad (24)$$

Then,

$$\begin{aligned} \mathcal{E}\{x^T(t) Q_{ki} x(t)\} &\leq a \|x_o\|^2 \|F\|^2 \exp\{-b(t - t_o)\} \\ \mathcal{E}\{u^T(t) R_{ki} u(t)\} &\leq a \|x_o\|^2 \|K_{ki}\|^2 \|W\|^2 \exp\{-b(t - t_o)\} \end{aligned}$$

Therefore, the cost function to be minimized is

$$J \leq \int_{t_o}^{\infty} a \|x_o\|^2 [\|F\|^2 + \|K_{ki}\|^2 \|W\|^2] \exp\{-bt\} dt < \infty$$

The proof is completed.

4.2 Steady-State Solutions for P_{ki}

It was proved that if the FTCSMP (1) is exponentially stable in the mean square, then there exist steady-state solutions for the Riccati-like differential equations [4]. The interested reader may refer to the mentioned citation.

Lemma 5: *If the system (1) is exponentially stable in the mean square, then the Riccati-like differential equations have unique positive-definite steady-state solutions, \bar{P}_{ki} , obtained from the following algebraic matrix equations*

$$\tilde{A}_{ki}^T \bar{P}_{ki} + \bar{P}_{ki} \tilde{A}_{ki} + \sum_{\substack{j \in S \\ j \neq k}} \alpha_{kj} \bar{P}_{ji} + \sum_{\substack{j \in R \\ j \neq i}} q_{ij}^k \bar{P}_{kj} + K_i^T R_{ki} K_i + Q_{ki} = 0 \quad (25)$$

4.3 Steady-State Solutions for X_{ki}

The limiting behavior for the covariance-like matrices is the key factor for the cost function to be finite. In the following, the existence of steady-state solutions for the covariance-like differential equations will be studied. Theorem 4 gives the condition under which these solutions exist.

Theorem 4 *If the system (1) is deterministically stable, then the covariance-like differential equations have steady-state solutions, \bar{X}_{ki} , obtained from the following algebraic matrix equations*

$$\tilde{A}_{ki} \bar{X}_{ki} + \bar{X}_{ki} \tilde{A}_{ki}^T + \sum_{\substack{j \in S \\ j \neq k}} \alpha_{jk} \bar{X}_{ji} + \sum_{\substack{j \in R \\ j \neq i}} q_{ji}^k \bar{X}_{kj} + X_{ki}(t_0) = 0 \quad (26)$$

Proof: The FTCSMP (1) under state feedback is

$$\dot{x}(t) = (A - B_k K_i) x(t) = \hat{A}_{ki} x(t) \quad (27)$$

Let the fundamental transition matrix associated with \hat{A}_{ki} be

$$\Phi_{ki}(t, t_0) = \exp(\hat{A}_{ki}(t - t_0)) \quad (28)$$

Then, the state trajectory is

$$x(t) = \hat{\Phi}_{ki}(t, t_0) x(t_0) \quad (29)$$

and

$$X_{ki}(t) = \hat{\Phi}_{ki}(t, t_0) X_{ki}(t_0) \hat{\Phi}_{ki}^T(t, t_0) \quad (30)$$

Substituting in (9), we have

$$\begin{aligned} \dot{X}_{ki}(t) &= \tilde{A}_{ki} \hat{\Phi}_{ki}(t, t_0) X_{ki}(t_0) \hat{\Phi}_{ki}^T(t, t_0) + \\ &\hat{\Phi}_{ki}(t, t_0) X_{ki}(t_0) \hat{\Phi}_{ki}^T(t, t_0) \tilde{A}_{ki}^T + \sum_{\substack{j \in S \\ j \neq k}} \alpha_{jk} \hat{\Phi}_{ji}(t, t_0) X_{ji}(t_0) \hat{\Phi}_{ji}^T(t, t_0) \\ &+ \sum_{\substack{j \in R \\ j \neq i}} q_{ji}^k \hat{\Phi}_{kj}(t, t_0) X_{kj}(t_0) \hat{\Phi}_{kj}^T(t, t_0) \end{aligned} \quad (31)$$

If the gain matrix, K_i , is chosen such that the system is deterministically stable, we have

$$\lim_{t \rightarrow \infty} \Phi_{ki}(t, t_0) = 0 \quad \forall k \in S, i \in R \quad (32)$$

Taking the limit of (31), we have

$$\lim_{t \rightarrow \infty} \dot{X}_{ki}(t) = 0 \quad (33)$$

That is the steady-state solutions for the covariance differential equations exist as $t \rightarrow \infty$.

4.4 Optimal Controller for Steady-State Solutions

The existence of steady-state solutions for the Riccati-like and covariance-like differential equations leads to constant gain matrices K_i . The optimal control gains in the infinite time horizon are obtained by Theorem 5.

Theorem 5 *The optimal control gains K that minimize (4) is obtained from the solution of one of the following equations*

$$\left\{ \begin{array}{l} R_{ki} K_{ki} \bar{X}_{ki} - B_k^T \bar{P}_{ki} \bar{X}_{ki} = 0 \quad \text{Scenario 1} \\ \sum_{k=1}^s R_{ki} K_i v_k \bar{X}_{ki} - \sum_{k=1}^s B_k^T v_k \bar{P}_{ki} \bar{X}_{ki} = 0 \quad \text{Scenario 2} \\ \sum_{i=1}^r \sum_{k=1}^s [R_{ki} K w_i^k v_k \bar{X}_{ki} - B_k^T w_i^k v_k \bar{P}_{ki} \bar{X}_{ki}] = 0 \quad \text{Scenario 3} \end{array} \right.$$

where \bar{P}_{ki} and \bar{X}_{ki} are the solutions of the algebraic equations (25) and (26), respectively.

5 COMPUTATIONAL ALGORITHM

To find an optimal control law for the FTCSMP (1), the Riccati-like and covariance-like differential equations coupled by the gain equation must be solved. The complexity of the solution is different for the different scenarios considered in this paper. However, a unified computational algorithm can be defined. In the following, the stochastic stability of a system is used to denote the exponential stability in the mean square for that system.

Algorithm

- 1) Choose an initial guess for the control gain matrices K_i^o . The chosen gain matrices must guarantee both the deterministic and the stochastic stability.
- 2) For the selected initial gains, solve the Riccati-like and the covariance-like differential/algebraic equations.
- 3) Substitute the calculated $P_{ki}^o(t)$, and $X_{ki}^o(t)$ in the control law gain equation and compute K_i^1 .
- 4) Use K_i^1 in step 2 to calculate the new $P_{ki}^1(t)$, and $X_{ki}^1(t)$.
- 5) Iterate to generate a plot or a table for K_i^{itr} , $P_{ki}^{itr}(t)$, and $X_{ki}^{itr}(t)$.

6 NUMERICAL EXAMPLE

Consider the following system with one possible failure in the actuator. The system and other design parameters are given as:

$$A = [0.4], \quad B_1 = [1.5], \quad B_2 = [0.2].$$

The actuator failure rates are assumed to be

$$\alpha_{12} = 0.005, \quad \alpha_{21} = 0.001.$$

The conditional transition rates of the FDI process are

$$q_{ij}^1 = \begin{bmatrix} -.18 & .18 \\ .79 & -.79 \end{bmatrix}, \quad q_{ij}^2 = \begin{bmatrix} -1.06 & 1.06 \\ .12 & -.12 \end{bmatrix}$$

The weighting matrices are

$$\{Q_{11}, Q_{12}, Q_{21}, Q_{22}\} = \{1.0, 0.5, 1.75, 1.0\},$$

$$R_{ki} = 1.0 \quad \forall i, j \in \{1, 2\}$$

The objective is to synthesize an optimal control law for the given FTCSMP. The designed control gain must be able to stabilize the unstable open-loop system, and to maintain the closed-loop stochastic stability when the system is fault-free or when the actuator failure has occurred. The three scenarios presented in this paper are considered. The performance of the proposed algorithm is tested by its ability to calculate an optimal control law for the infinite time-horizon. The steady state Riccati-like, covariance-like and optimal control laws after 30 iterations are given in Tables 1, 2 and 3.

Table 1. Optimal fault tolerant control law, Scenario 1

P_{ki}	X_{ki}	K_{ki}
$P_{11} = 1.0060$	$X_{11} = .2511$	$K_{11} = 1.5092$
$P_{12} = 0.8953$	$X_{12} = .0703$	$K_{12} = 1.3436$
$P_{21} = 21.5648$	$X_{21} = .1090$	$K_{21} = 4.3130$
$P_{22} = 21.2062$	$X_{22} = .1101$	$K_{22} = 4.2412$

Table 2. Optimal fault tolerant control law, Scenario 2

P_{ki}	X_{ki}	K_i
$P_{11} = 1.7838$	$X_{11} = .1406$	$K_1 = 2.5488$
$P_{12} = 2.0815$	$X_{12} = .0191$	
$P_{21} = 24.8717$	$X_{21} = .3991$	$K_2 = 3.6019$
$P_{22} = 22.2648$	$X_{22} = .3067$	

Table 3. Optimal fault tolerant control law, Scenario 3

P_{ki}	X_{ki}	K
$P_{11} = 2.1403$	$X_{11} = .1030$	$K = 3.4741$
$P_{12} = 2.0701$	$X_{12} = .0188$	
$P_{21} = 22.6470$	$X_{21} = .1690$	
$P_{22} = 22.2171$	$X_{22} = .1688$	

7 SUMMARY

In this paper, an optimal control law has been synthesized for the FTCSMP. The design is completed using equivalent deterministic cost function for the stochastic optimization

problem. The matrix maximum principle was employed. Three scenarios were considered: the first assumes that both the failure process and the FDI process are available for control reconfiguration. Unfortunately, this scenario is not practical. In the second scenario, the controller is reconfigured upon the decisions of the FDI process only and does not need direct accessibility to the failure process which is closer to practical FTCS. In practice, there are situations where the FDI process itself is not able to give any decision due to physical problems or long convergence time for the statistical tests. Although these situations are rarely happened but they may lead to a complete loss of system stability. For these cases the third scenario has been considered, and an optimal control law has been developed. The conditions which guarantee the finiteness of the cost function and the existence of steady-state solutions for both the Riccati-like and the covariance-like differential equations were addressed. Particularly, if the FTCSMP is deterministically and stochastically stabilizable, an optimal control law for the infinite time-horizon exist. A numerical example has been used to illustrate the computational algorithm and to design a steady-state optimal control law for the three scenarios.

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