Optimal Hankel norm approximation for the Pritchard-Salamon class of non-exponentially stable infinite-dimensional systems

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Abstract. The optimal Hankel norm approximation problem is solved for a class of infinite-dimensional systems without assuming exponential stability.

1 Introduction.

The optimal Hankel norm approximation problem has received a lot of attention, both in the mathematical and engineering literature (see Adamjan et al. [1], Ball and Helton [2], Ball and Ran [3], Curtain and Ran [4], Glover [7] and Ran [13]). Its importance in control theory is due to its connections with the model reduction problem (see, for instance, Glover [7], Glover et al. [8] and Young [18]).

Given a transfer function $G(s) \in L_\infty(\mathbb{C}^{p\times m})$, we suppose that $G(s)$ has a compact Hankel operator $\Gamma : L_2([0,\infty),\mathbb{C}^m) \to L_2([0,\infty),\mathbb{C}^p)$ which is defined by

$$(\Gamma u)(t) = \int_0^\infty h(t+s)u(s)ds \quad \forall u \in L_2(0,\infty;\mathbb{C}^m),$$

where $h(\cdot)$ denotes the impulse response of the system. Γ then has countably many singular values $\sigma_1 \geq \sigma_2 \geq$ ... and these are also called the Hankel singular values of G. The suboptimal Hankel norm approximation problem associated with G is now defined as follows:

$$\begin{array}{l} \text{Find all } K(-s) \in H_{\infty,l}(\mathbb{C}^{p\times m}) \text{ such that} \\ ||G+K||_\infty \leq \sigma \text{ for } \sigma_l > \sigma > \sigma_{l+1}, \\ \text{in the } L_\infty-\text{norm}, \end{array}$$

where $H_{\infty,l}(\mathbb{C}^{p\times m})$ denotes the set of complex $p\times m$ matrix valued functions $X(\cdot)$ of a complex variable with a decomposition $X = \hat{G} + F$, where $\hat{G}$ is the matrix transfer function of a system of MacMillan degree at most equal to l, with all its poles in the open right half-plane, and $F \in H_\infty(\mathbb{C}^{p\times m})$.

It is well-known that (see Adamjan et al. [1])

$$\inf_{K(-s)\in H_{\infty,l}(\mathbb{C}^{p\times m})} ||G+K||_\infty = \sigma_{l+1}.$$

In the same paper the optimal Hankel norm approximation problem is solved for the scalar case, and solutions for other classes of functions G have been obtained in Ball and Helton [2] and Nikol'skii [11].

In this paper we consider the state-space solution to the optimal Hankel norm approximation problem in terms of the system parameters A, B, C. This problem has also been studied in literature assuming that A is the infinitesimal generator of an exponentially stable C_0-semigroup and B and C are linear operators. Glover et al. [8] was the first paper to do this. In Sasane and Curtain [16], B and C are bounded operators. Extensions to the case in which B and C are unbounded can be found in Sasane and Curtain [17] or Curtain and Ran [4]. In all these papers, it is assumed that A generates an **exponentially** stable C_0-semigroup. However, there exists an important class of systems with a transfer function $G \in H_\infty(\mathbb{C}^{p\times m})$, for which A does not generate an exponentially stable C_0-semigroup. In [8], approximating solutions to the optimal Hankel norm approximation problem were obtained without assuming exponential stability, but only for the case that the Hankel operator is nuclear; this is a strong assumption. It is the aim of this paper to find solutions to the suboptimal Hankel norm approximation problem in terms of the A, B, C operators for the Pritchard-Salamon class of non-exponentially stable systems.

The specific class of systems we consider in this paper is defined below:

Definition 1.1 *Let V and W be separable Hilbert spaces with continuous, dense injections and which satisfy*

$$W \hookrightarrow Z \hookrightarrow V.$$

Suppose that A is the infinitesimal generator of strongly continuous semigroups $T^W(t)$, $T^Z(t)$ and $T^V(t)$ on W, Z and V, respectively, such that $T^V(t)|_Z = T^Z(t)$ and $T^Z(t)|_W = T^W(t)$. Since these semigroups are consistent, we shall simply use the notation $T(t)$. Assume further that U and Y are separable Hilbert spaces (the input and output spaces), respectively.

1. *$B \in \mathcal{L}(U,V)$ is an admissible control operator for $T(t)$ if there exists a constant $\beta > 0$ such that*

$$\left\| \int_0^t T(t-s)Bu(s)ds \right\|_W \leq \beta \|u\|_{L_2([0,t),U)}$$

for all finite $t > 0$.

2. *$C \in \mathcal{L}(W,Y)$ is an admissible observation operator for $T(t)$ if there exists a constant $\gamma > 0$ and a $t > 0$ such that*

$$\|CT(\cdot)z\|_{L_2([0,t),Y)} \leq \gamma \|z\|_V$$

for all $z \in W$.

Under the above assumptions, the state linear system $\Sigma(A,B,C,D)$ is called a **Pritchard-Salamon system** *for any $D \in \mathcal{L}(U,Y)$. If, in addition, $D(A^V) \hookrightarrow W$, $\Sigma(A,B,C,D)$ is called a regular Pritchard-Salamon system.*

Moreover, we assume that the following assumptions are satisfied:

A1. $\sigma(A) \cap \mathbb{C}_0^+$ is empty and A satisfies the spectrum determined growth assumption where $\mathbb{C}_0^+ := \{s \in \mathbb{C} | \text{Re}(s) > 0\}$.

A2. $U = \mathbb{C}^m$, $Y = \mathbb{C}^p$.

A3. $\Sigma(A,B,C)$ is input stable, i.e., the controllability map $\mathcal{B}$ from $L_2([0,\infty),U)$ to Z defined by

$$\mathcal{B}u = \int_0^\infty T(t)Bu(t)dt, \quad u \in L_2([0,\infty),U),$$

is bounded.

A4. $\Sigma(A,B,C)$ is output stable, i.e., the observability map $\mathcal{C}$ from Z to $L_2([0,\infty),Y)$ defined by

$$(\mathcal{C}z)(t) = CT(t)z, \quad z \in Z,$$

is bounded.

A5. $G \in H_\infty(\mathbb{C}^{p\times m})$ is such that $\omega \mapsto G(j\omega) : \mathbb{R} \to \mathbb{C}^{p\times m}$ is continuous[1] and has a (unique) limit $G_\infty \in \mathbb{C}^{p\times m}$ at $\pm\infty$.

We outline the contents of the following sections. In section 2 we first develop the mathematical tools which we need for the proof of our main results. In section 3 we prove a few properties of the class of systems that we consider which will be used in the proofs in the subsequent sections. Finally, in section 4, we prove our main result.

[1] This means that the boundary function $G(j\cdot) : \mathbb{R} \to \mathbb{C}^{p\times m} \in \mathcal{L}_\infty$ is equal almost everywhere to a continuous function, which we denote by the same symbol $G(j\cdot)$. This convention is used throughout this paper.

2 Mathematical Preliminaries

The key to the proof of our new result is Corollary 2.2 which is an easy consequence of the following lemma.

Lemma 2.1 *If $G \in H_\infty(\mathbb{C}^{p\times m})$ and $\omega \mapsto G(j\omega)$: $\mathbb{R} \to \mathbb{C}^{p\times m}$ is uniformly continuous, then given any $\epsilon > 0$, $\exists\ \delta > 0$ such that $\sup_{\omega\in\mathbb{R}} \|G(j\omega) - G(\alpha + j\omega)\| < \epsilon$ whenever $0 \leq \alpha \leq \delta$.*

Proof: It follows from Theorem 5.18 (page 96, M. Rosenblum and J. Rovnyak [14]) that

$$G(\alpha + j\omega) = \frac{\alpha}{\pi}\int_{-\infty}^{\infty} \frac{G(jt)}{(t-\omega)^2 + \alpha^2}dt, \quad \alpha > 0.$$

Since for $\alpha > 0$, $\frac{\alpha}{\pi}\int_{-\infty}^{\infty}\frac{1}{(t-\omega)^2+\alpha^2}dt = 1$, we have

$$\begin{aligned} & \|G(j\omega) - G(\alpha + j\omega)\| \\ = & \left\| \frac{\alpha}{\pi}\int_{-\infty}^{\infty} \frac{G(jt)}{(t-\omega)^2 + \alpha^2}dt - G(j\omega) \right\| \\ = & \left\| \frac{\alpha}{\pi}\int_{-\infty}^{\infty} \frac{G(jt) - G(j\omega)}{(t-\omega)^2 + \alpha^2}dt \right\|. \end{aligned}$$

Choose a $\zeta > 0$ such that $\|G(jt) - G(j\omega)\| < \frac{\epsilon}{2}$ for every t and ω satisfying $|t-\omega| < \zeta$. Now choose a $\delta > 0$ such that for any α satisfying $0 \leq \alpha \leq \delta$, we have

$$\left\| \frac{\alpha}{\pi}\int_{\mathbb{R}\setminus[\omega-\zeta,\omega+\zeta]} \frac{1}{(t-\omega)^2 + \alpha^2}dt \right\| < \frac{\epsilon}{4\,\|G(j\cdot)\|_\infty}.$$

Thus

$$\begin{aligned} & \|G(j\omega) - G(\alpha + j\omega)\| \\ = & \left\| \frac{\alpha}{\pi}\int_{-\infty}^{\infty} \frac{G(jt) - G(j\omega)}{(t-\omega)^2 + \alpha^2}dt \right\| \\ \leq & \frac{\alpha}{\pi}\int_{\omega-\zeta}^{\omega+\zeta} \frac{1}{(t-\omega)^2 + \alpha^2}\|G(jt) - G(j\omega)\|dt \\ + & \frac{\alpha}{\pi}\int_{\mathbb{R}\setminus[\omega-\zeta,\omega+\zeta]} \frac{1}{(t-\omega)^2 + \alpha^2}\|G(jt) - G(j\omega)\|dt \\ \leq & \frac{\epsilon}{2}\left(\frac{\alpha}{\pi}\int_{\omega-\zeta}^{\omega+\zeta} \frac{1}{(t-\omega)^2 + \alpha^2}dt \right) \\ + & 2\|G(j\cdot)\|_\infty \left(\frac{\alpha}{\pi}\int_{\mathbb{R}\setminus[\omega-\zeta,\omega+\zeta]} \frac{1}{(t-\omega)^2 + \alpha^2}dt \right) \\ \leq & \frac{\epsilon}{2}.\ 1 + 2\|G(j\cdot)\|_\infty.\ \frac{\epsilon}{4\,\|G(j\cdot)\|_\infty} \\ = & \epsilon. \end{aligned}$$

Since the choice of ω is arbitrary, this completes the proof. ∎

Corollary 2.2 *If $G \in H_\infty(\mathbb{C}^{p\times m})$ and $\omega \mapsto G(j\omega)$: $\mathbb{R} \to \mathbb{C}^{p\times m}$ is continuous and has limits $G(\pm j\infty)$ at $\pm\infty$, then given any $\epsilon > 0$, $\exists\ \delta > 0$ such that $\sup_{\omega\in\mathbb{R}}$ $\|G(j\omega) - G(\alpha + j\omega)\| < \epsilon$ whenever $0 \leq \alpha \leq \delta$.*

Proof: Given any $\epsilon > 0$, $M_1 > 0$ and $M_2 > 0$ such that

$$\sup_{\omega\in[M_2,\infty)} \|G(j\omega) - G(j\infty)\| < \frac{\epsilon}{2}, \text{ and} \quad (1)$$

$$\sup_{\omega\in(-\infty,M_1]} \|G(j\omega) - G(-j\infty)\| < \frac{\epsilon}{2}. \quad (2)$$

Moreover, since $\omega \mapsto G(j\omega)$: $\mathbb{R} \to \mathbb{C}^{p\times m}$ is continuous, it is uniformly continuous in $[M_1 - 1, M_2 + 1]$, and so given any $\epsilon > 0$, $\exists\ \delta$ such that $1 > \delta > 0$ and whenever ω_1, $\omega_2 \in [M_1 - 1, M_2 + 1]$ and $|\omega_1 - \omega_2| < \delta$,

$$\|G(j\omega_1) - G(j\omega_2)\| < \epsilon. \quad (3)$$

Thus it follows from (1), (2) and (3) that whenever ω_1, $\omega_2 \in \mathbb{R}$ and $|\omega_1 - \omega_2| < \delta$, then $\|G(j\omega_1) - G(j\omega_2)\| < \epsilon$. Hence $\omega \mapsto G(j\omega)$: $\mathbb{R} \to \mathbb{C}^{p\times m}$ is uniformly continuous, and so the result follows from Lemma 2.1. ■

3 Properties of the System.

Next we prove a few properties of our class of systems which will be used in the sequel.

Lemma 3.1 *If $\sigma(A) \cap \mathbb{C}_0^+$ is empty, A satisfies the spectrum determined growth assumption and $\alpha \in \mathbb{C}$, and $Re(\alpha) > 0$, then $A - \alpha I$ is the infinitesimal generator of the exponentially stable semigroup $\{e^{-\alpha t}T(t)\}_{t\geq 0}$ on Z.*

Proof: From Exercise 2.4 (Curtain and Zwart [5]), $A - \alpha I$ is the infinitesimal generator of the C_0-semigroup $\{e^{-\alpha t}T(t)\}_{t\geq 0}$ on Z with a growth bound equal to the sum of the growth bound of $\{T(t)\}_{t\geq 0}$ and $-\mathrm{Re}(\alpha)$. Since $\sigma(A) \cap \mathbb{C}_0^+$ is empty and A satisfies the spectrum determined growth assumption, it follows that the growth bound of $\{T(t)\}_{t\geq 0}$ is non-positive and so $A - \alpha I$ generates the exponentially stable semigroup $\{e^{-\alpha t}T(t)\}_{t\geq 0}$ on Z. ■

Lemma 3.2 *If the regular Pritchard-Salamon system $\Sigma(A,\ B,\ C)$ satisfies the assumptions* **A2-4**, *then*

1. *$L_B := \mathcal{B}\mathcal{B}^* \in \mathcal{L}(Z)$ is a self-adjoint, nonnegative solution of the operator Lyapunov equation*

   $$AL_Bz + L_BA^*z = -BB^*z \quad \forall z \in D(A^*), \quad (4)$$

   and $L_C := \mathcal{C}^*\mathcal{C} \in \mathcal{L}(Z)$ is a self-adjoint, nonnegative solution of the operator Lyapunov equation

   $$A^*L_Cz + L_CAz = -C^*Cz \quad \forall z \in D(A). \quad (5)$$

2. *$\Gamma = \mathcal{C}\mathcal{B}$.*
3. *In addition, if the assumption* **A5** *is satisfied, then Γ and L_CL_B are compact, and the nonzero Hankel singular values of $\Sigma(A, B, C)$ are equal to the square roots of the nonzero eigenvalues of L_CL_B.*

   If $\sigma_{l+1} < \sigma < \sigma_l$, then $I - \sigma^{-2}L_BL_C$ is invertible.

Proof:

1. This follows from Theorem 3.1, page 10, Hansen and Weiss [10] and its dual statement.
2. Since

   $$\begin{aligned}\mathcal{C}(\mathcal{B}u)(t) &= CT(t)\int_0^\infty T(s)Bu(s)ds \\ &= \int_0^\infty CT(t+s)Bu(s)ds \\ &= (\Gamma u)(t),\end{aligned}$$

   it follows that $\Gamma = \mathcal{C}\mathcal{B}$.
3. It follows from Corollary 4.10 (Hartman's theorem for the half-plane, page 46, Partington [12]), that Γ is compact. Now the proof is analogous to the proof of Lemma 8.2.9 (page 401, Curtain and Zwart [5]).

■

Notation. It is easy to see that $\Sigma(A - \alpha I,\ B,\ C)$ is also a regular Pritchard-Salamon system for any $\alpha \in \mathbb{C}$. If A satisfies the assumption **A1** and $\alpha > 0$, then $A - \alpha I$ generates the exponentially stable semigroup $\{e^{-\alpha t}T(t)\}_{t\geq 0}$ on Z, and $\Sigma(A - \alpha I,\ B,\ C)$ is an exponentially stable, regular Pritchard-Salamon system. Denote the controllability map of this system by $\mathcal{B}^{[\alpha]}$, the observability map by $\mathcal{C}^{[\alpha]}$, the Hankel operator by $\Gamma^{[\alpha]}$, and the l^{th} Hankel singular value by $\sigma_l^{[\alpha]}$. If $\sigma_{l+1}^{[\alpha]} < \sigma < \sigma_l^{[\alpha]}$, let $N_\sigma^{[\alpha]} := (I - \sigma^{-2}\ L_B^{[\alpha]}L_C^{[\alpha]})^{-1}$, where $L_B^{[\alpha]} := \mathcal{B}^{[\alpha]}\mathcal{B}^{[\alpha]\,*}$, $L_C^{[\alpha]} := \mathcal{C}^{[\alpha]\,*}\mathcal{C}^{[\alpha]}$.

Lemma 3.3 *If the regular Pritchard-Salamon system $\Sigma(A, B, C)$ satisfies the assumptions* **A1-5**, *then*

1. *$\Gamma^{[\alpha]\,*}\Gamma^{[\alpha]} \to \Gamma^*\Gamma$ uniformly as $\alpha \to 0$,*
2. *$\sigma_l^{[\alpha]} \to \sigma_l$ as $\alpha \to 0$ for every $l \in \mathbb{N}$.*

Proof:

1. Let $S_1^{[\alpha]}$: $L_2([0,\infty),\mathbb{C}^m) \to L_2([0,\infty),\mathbb{C}^m)$ be the multiplication operator by $e^{-\alpha t}$: $(S_1^{[\alpha]}u)(t) = e^{-\alpha t}u(t)$, and $S_2^{[\alpha]}$: $L_2([0,\infty),\mathbb{C}^p) \to$

$L_2([0,\infty),\mathbb{C}^p)$ be the multiplication operator by $e^{-\alpha t}$: $(S_2^{[\alpha]}y)(t) = e^{-\alpha t}y(t)$. Then $\Gamma^{[\alpha]} = S_2^{[\alpha]}\Gamma S_1^{[\alpha]}$.

Sublemma 3.4 *$S_1^{[\alpha]} = S_1^{[\alpha]\,*} \to I_m$ strongly as $\alpha \to 0$, and $S_2^{[\alpha]} = S_2^{[\alpha]\,*} \to I_p$ strongly as $\alpha \to 0$.*

Proof: Let $u \in L_2([0,\infty),\mathbb{C}^m)$. Given any $\epsilon > 0$, choose a $M > 0$ such that $\int_M^\infty \|u(t)\|^2 dt < \frac{\epsilon^2}{2}$. Now choose a $\delta > 0$ such that $0 \le \alpha < \delta$ implies that

$$\sup_{t\in[0,M]} |e^{-\alpha t} - 1| < \frac{\epsilon}{\sqrt{2}\left(\int_0^\infty \|u(t)\|^2 dt\right)^{\frac{1}{2}}}.$$

Thus

$$\begin{aligned}
& \|S_1^{[\alpha]}u - u\|^2 \\
= & \int_0^\infty \|(e^{-\alpha t} - 1)u(t)\|^2 dt \\
\le & \left(\sup_{t\in[0,M]} |e^{-\alpha t} - 1|\right)^2 \int_0^M \|u(t)\|^2 dt \\
& + \int_M^\infty |e^{-\alpha t} - 1|^2 \|u(t)\|^2 dt \\
\le & \frac{\epsilon^2}{2} + \int_M^\infty \|u(t)\|^2 dt \le \frac{\epsilon^2}{2} + \frac{\epsilon^2}{2} \\
= & \epsilon^2.
\end{aligned}$$

Thus $S_1^{[\alpha]} = S_1^{[\alpha]\,*} \to I_m$ strongly as $\alpha \to 0$. Similarly, $S_2^{[\alpha]} = S_2^{[\alpha]\,*} \to I_p$ strongly as $\alpha \to 0$ strongly. ∎

Since $\Gamma^{[\alpha]\,*}\Gamma^{[\alpha]} = S_1^{[\alpha]}\,\Gamma^* S_2^{[\alpha]} S_2^{[\alpha]}\,\Gamma S_1^{[\alpha]}$, we have

$$\begin{aligned}
\Gamma^{[\alpha]\,*}\Gamma^{[\alpha]} &= S_1^{[\alpha]}\left[\Gamma^* S_2^{[\alpha]} S_2^{[\alpha]}\Gamma - \Gamma^*\Gamma\right] S_1^{[\alpha]} \\
&\quad + S_1^{[\alpha]}\Gamma^*\Gamma S_1^{[\alpha]}.
\end{aligned}$$

Defining $K = (\Gamma^*\Gamma)^{\frac{1}{2}}$, we have that K is compact, since K^2 is compact and K is self-adjoint (using Exercise 18, page 127, Gohberg and Goldberg [9]). We will use the following (Exercise 6.6', page 136, Weidmann [?]).

Sublemma 3.5 *Suppose that T_n, $T \in \mathcal{L}(\mathcal{V},\mathcal{W})$ and that $T_n \to T$ strongly. Let $S \in \mathcal{L}(\mathcal{U},\mathcal{V})$ be compact. Then $T_n S \to TS$ uniformly.*

$S_1^{[\alpha]}K \to K$ uniformly, using the Sublemmas 3.4 and 3.5. But $KS_1^{[\alpha]} = (S_1^{[\alpha]}K)^*$, and $(S_1^{[\alpha]}K)^* \to K^*$ $(= K)$ uniformly, and so $KS_1^{[\alpha]} \to K$ uniformly. Consequently, $S_1^{[\alpha]}\Gamma^*\Gamma S_1^{[\alpha]} \to \Gamma^*\Gamma$ uniformly.

Using the Sublemmas 3.4 and 3.5, we obtain that $S_2^{[\alpha]}\Gamma \to \Gamma$ uniformly. Thus $\Gamma^* S_2^{[\alpha]} = (S_2^{[\alpha]}\Gamma)^* \to \Gamma^*$ uniformly. As a result, $\Gamma^* S_2^{[\alpha]} S_2^{[\alpha]}\Gamma \to \Gamma^*\Gamma$ uniformly, and since $\|S_1^{[\alpha]}\| \le 1$, we have $S_1^{[\alpha]}\left[\Gamma^* S_2^{[\alpha]} S_2^{[\alpha]}\Gamma - \Gamma^*\Gamma\right] S_1^{[\alpha]} \to 0$ uniformly. Hence it follows that $\Gamma^{[\alpha]\,*}\Gamma^{[\alpha]} \to \Gamma^*\Gamma$ uniformly as $\alpha \to 0$.

2. Since $\Gamma^{[\alpha]\,*}\Gamma^{[\alpha]} \to \Gamma^*\Gamma$ uniformly as $\alpha \to 0$, $\lambda_l(\Gamma^{[\alpha]\,*}\Gamma^{[\alpha]}) \to \lambda_l(\Gamma^*\Gamma)$ (using Corollary 4.(a), page 1090, Dunford and Schwartz [6]).

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4 Optimal Hankel Norm Approximation.

In this section, we will prove our main result about the existence of solutions to the optimal Hankel norm approximation problem.

We will use $H_\infty^c(\mathbb{C}^{p\times m})$ to denote the set of $p\times m$ matrix valued functions defined in the closed right half-plane, which are bounded and holomorphic in $\mathbb{C}_0^+$, and continuous in $\overline{\mathbb{C}_0^+}$. $H_{\infty,[l]}^c(\mathbb{C}^{p\times m})$ denotes the set of $p\times m$ matrix valued functions $X(\cdot)$ of a complex variable with a decomposition $X = \hat{G} + F$ where $\hat{G}$ is the matrix transfer function of a system of MacMillan degree equal to l, with all its poles in the open right half-plane, and $F \in H_\infty^c(\mathbb{C}^{p\times m})$.

We quote the following theorem from Sasane and Curtain [17].

Theorem 4.1 *Suppose that $\Sigma(A,B,C)$ is an exponentially stable, regular Pritchard-Salamon unfinite-dimensional system with finite-dimensional input space $\mathbb{C}^m$, output space $\mathbb{C}^p$ and $\sigma_{l+1} < \sigma < \sigma_l$. Let $X(\cdot)$ be given by*

$$\begin{aligned}
X(s) &= \begin{bmatrix} I_p & 0 \\ 0 & \sigma I_m \end{bmatrix} + \sigma^{-2}\begin{bmatrix} -CL_B \\ \sigma B' \end{bmatrix} \cdot \\
&\quad N_\sigma'(sI + A')^{-1}\begin{bmatrix} C' & L_C B \end{bmatrix}
\end{aligned}$$

where $N_\sigma = (I - \sigma^{-2}L_B L_C)^{-1}$.

$K(-s) \in H_{\infty,[l]}^c(\mathbb{C}^{p\times m})$ and $\|G + K\|_\infty \le \sigma$ iff $K(-s) := R_1(-s)R_2(-s)^{-1}$, where

$$\begin{bmatrix} R_1(-s) \\ R_2(-s) \end{bmatrix} := X^{-1}(-s)\begin{bmatrix} Q(-s) \\ I_m \end{bmatrix},$$

for some $Q(-s) \in H_\infty^c(\mathbb{C}^{p\times m})$ satisfying $\|Q\|_\infty \le 1$.

We now state our main result:

Theorem 4.2 *Suppose that the regular Pritchard-Salamon system $\Sigma(A,\,B,\,C)$ satisfies the assumptions* **A1-5** *and let $\sigma_{l+1} < \sigma < \sigma_l$.*

If $Q(-s) \in H_\infty(\mathbb{C}^{p\times m})$, and $\|Q\|_\infty \leq 1$, then there exists a $\delta > 0$ such that for every α satisfying $0 < \alpha < \delta$,

$$K^{[\alpha]}(-s) = R_1^{[\alpha]}(-s)R_2^{[\alpha]}(-s)^{-1},$$

where

$$\begin{bmatrix} R_1^{[\alpha]}(-s) \\ R_2^{[\alpha]}(-s) \end{bmatrix} = X^{[\alpha]^{-1}}(-s) \begin{bmatrix} Q(-s-\alpha) \\ I_m \end{bmatrix}$$

and

$$X^{[\alpha]}(s) = \begin{bmatrix} I_p & 0 \\ 0 & \sigma I_m \end{bmatrix} + \sigma^{-2} \begin{bmatrix} -CL_B^{[\alpha]} \\ \sigma B' \end{bmatrix} \cdot N_\sigma^{[\alpha]'}(sI + A' - \alpha I)^{-1} \begin{bmatrix} C' & L_C^{[\alpha]}B \end{bmatrix},$$

is such that $K^{[\alpha]}(-s) \in H^c_{\infty,[l]}(\mathbb{C}^{p\times m})$ and $\|G+K^{[\alpha]}\|_\infty \leq \sigma$.

Proof:

1. For any $\alpha > 0$, consider the system $\Sigma(A-\alpha I,\,B,\,C)$. From Lemma 3.1, $A-\alpha I$ is the infinitesimal generator of the exponentially stable semigroup $\{e^{-\alpha t}T(t)\}_{t\geq 0}$ on Z. Thus $\Sigma(A-\alpha I,\,B,\,C)$ is an exponentially stable, regular Pritchard-Salamon system.

2. Moreover, for any $\alpha > 0$, $Q(-\cdot-\alpha) \in H^c_\infty(\mathbb{C}^{p\times m})$, and $\|Q(-\cdot-\alpha)\|_\infty \leq 1$.

3. Let $\epsilon := \frac{\sigma-\sigma_{l+1}}{2} > 0$. Choose a $\delta_1 > 0$ small enough so that whenever $0 < \alpha < \delta_1$, $\sup_{j\omega\in\mathbb{R}} \|G(j\omega) - G(\alpha + j\omega)\| < \epsilon$. This can be done, owing to assumption **A5** and Corollary 2.2.

4. Next choose a $\delta_2 > 0$, such that whenever $0 < \alpha < \delta_2$, we have (see Lemma 3.3.2)
$$\sigma_{l+1}^{[\alpha]} < \frac{\sigma_{l+1}+\sigma}{2} < \sigma_l^{[\alpha]}.$$

5. Let $\delta := \min\{\delta_1, \delta_2\}$ and consider any α satisfying $0 < \alpha < \delta$. Applying Theorem 4.1, we have that $K^{[\alpha]}(-s) \in H^c_{\infty,[l]}(\mathbb{C}^{p\times m})$ and $\sup_{\omega\in\mathbb{R}} \|G(j\omega+\alpha) + K^{[\alpha]}(j\omega)\| \leq \frac{\sigma+\sigma_l}{2}$. Thus,
$$\begin{aligned} & \|G(j\omega) + K^{[\alpha]}(j\omega)\| \\ = & \|G(j\omega) - G(j\omega+\alpha) \\ & +G(j\omega+\alpha) + K^{[\alpha]}(j\omega)\| \\ \leq & \|G(j\omega) - G(j\omega+\alpha)\| \\ & +\|G(j\omega+\alpha) + K^{[\alpha]}(j\omega)\| \\ \leq & \frac{\sigma-\sigma_{l+1}}{2} + \frac{\sigma_{l+1}+\sigma}{2} \\ = & \sigma. \end{aligned}$$

   This completes the proof.

■

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