Stability results for some classes of cooperative systems[1]

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Abstract

This paper deals with the constant control problem for homogeneous cooperative and irreducible systems. These systems serve as models for positive systems. A necessary and sufficient condition for global asymptotic stability of the zero solution of this class of systems is known. Adding a constant control allows to shift the equilibrium point from zero to a point in the first orthant. We prove that for every nontrivial nonnegative control vector a unique nontrivial equilibrium point is achieved which is globally asymptotically stable if the zero solution of the uncontrolled system is globally asymptotically stable. Additionally a stability result for a particular class of Kolmogorov systems is established. We compare our main results to those in the literature.

1 Introduction

A dynamical system is said to be positive if when initiated in the first orthant of $\mathbb{R}^n$, its state remains in this orthant for future times. Examples of these systems are found in a variety of applied areas such as biology, chemistry and sociology ([3], [13] and [6]). Typical issues arising in the study of positive systems are boundedness of solutions, permanence or persistence and (asymptotic) stability of equilibrium points.

In [8] we considered homogeneous cooperative and irreducible systems and we have characterized the stability behavior of the zero solution. (A preliminary version can be found in [9]) In particular we have shown that the zero solution is globally asymptotically stable (GAS) if and only if there exists a unique invariant ray in the interior of the first orthant such that the vector field on this ray points towards the origin.

In applications it is often undesirable that the zero solution is GAS. Indeed, in a biological system for example, this implies that all species die out. On the other hand a (asymptotically) stable *nontrivial* equilibrium implies coexistence of several species.

The aim of this paper is twofold:

1. To investigate the effect of a constant control on a particular class of positive systems with a GAS zero solution. The controlled system remains *positive* if and only if the control vector is nonnegative. We prove that the controlled system possesses a unique nontrivial GAS equilibrium point.
2. To establish a stability result for a particular class of so-called Kolmogorov systems.

This paper is organized as follows. In Section 2 we review some known results on homogeneous cooperative and irreducible systems. A control problem is stated and proved in Section 3. In Section 4 we examine the stability behavior of a particular class of Kolmogorov systems. We conclude in Section 5 with a comparison between our main results and those in the literature.

2 Preliminaries

2.1 Notation

Let $\mathbb{R}$ be the set of real numbers and $\mathbb{R}^n$ the set of n-tuples for which all components belong to $\mathbb{R}$. $\mathbb{R}^+ = [0,+\infty)$ and $\mathbb{R}^+_0 = (0,+\infty)$, while $\mathbb{R}^n_+$ ($\text{int}(\mathbb{R}^n_+)$) is the set of n-tuples for which all components belong to $\mathbb{R}^+$ ($\mathbb{R}^+_0$). Finally, the boundary of $\mathbb{R}^n_+$, $\mathbb{R}^n_+ \setminus \text{int}(\mathbb{R}^n_+)$, is denoted as $\text{bd}(\mathbb{R}^n_+)$.

When $x, y \in \mathbb{R}^n_+$, then $x \leq y$ means $x_i \leq y_i$, $\forall i = 1,...,n$. Furthermore $x < y$ if and only if $x \leq y$ and $x \neq y$ and $x << y$ if and only if $x_i < y_i$, $\forall i = 1,...,n$.

Let I be a nonempty and proper subset of $\{1,2,...,n\}$. The set $F_I := \{x \in \mathbb{R}^n_+ | x_i = 0 \text{ for } i \in I\}$ is a *face* of $\mathbb{R}^n_+$. The *dimension* of F_I equals $\#I$, the cardinality of the set I.

When $x \in \mathbb{R}^n_+$ we define $[0,x] = \{z \in \mathbb{R}^n_+ | 0 \leq z \leq x\}$ and $(0,x) = \{z \in \mathbb{R}^n_+ | 0 << z << x\}$.

Given a vector $x \in \mathbb{R}^n$, $\text{diag}(x)$ is a real $n \times n$ diagonal

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matrix where the i-th diagonal entry equals x_i, the i-th component of the vector x. A real $n \times n$ matrix $A = (a_{ij})$ is *Metzler* if and only if its off-diagonal entries a_{ij}, $\forall i \neq j$ belong to $\mathbb{R}^+$.

A is *irreducible* if and only if for every nonempty proper subset K of $N := \{1, ..., n\}$, there exists an $i \in K$ and a $j \in N \setminus K$ such that $a_{ij} \neq 0$. When A is not irreducible, it is called *reducible*. It can be shown that A is reducible if and only if there exists a permutation matrix P such that

$$PAP^T = \begin{pmatrix} B & 0 \\ C & D \end{pmatrix}$$

where B and D are square matrices.

Consider the system

$$\dot{x} = f(x) \tag{1}$$

where $x \in \mathbb{R}^n$ and $f(x)$ is a continuously differentiable vector field.

The *forward solution* of system (1) with initial condition $x_0 \in \mathbb{R}^n$ at $t = 0$ is denoted as $x(t, x_0)$ and is defined on the *maximal forward interval of existence* $\mathcal{I}^+_{x_0} := [0, T_{\max}(x_0))$. A set $D \subset \mathbb{R}^n$ is called *forward invariant* if and only if for all $x_0 \in D$, $x(t, x_0) \in D$ for all $t \in \mathcal{I}^+_{x_0}$. A system is called *positive* if and only if $\mathbb{R}^n_+$ is forward invariant.

The flow of system (1) is *monotone in D* if and only if for all x_0, $y_0 \in D$ with $x_0 \leq (<, <<) y_0$ holds that $x(t, x_0) \leq (<, <<) x(t, y_0)$ for all $t \in (\mathcal{I}^+_{x_0} \cap \mathcal{I}^+_{y_0}) \setminus \{0\}$. The flow of system (1) is *strongly monotone in D* if and only if it is monotone in D and for all x_0, $y_0 \in D$ with $x_0 < y_0$ holds that $x(t, x_0) << x(t, y_0)$ for all $t \in (\mathcal{I}^+_{x_0} \cap \mathcal{I}^+_{y_0}) \setminus \{0\}$.

2.2 Homogeneous systems

Now we introduce the concept of a homogeneous vector field. We adopt the following definition from [5]:

Definition 1. *A vector field $f(x)$, $x \in \mathbb{R}^n$ is said to be* homogeneous *of order $\tau \in \mathbb{R}$ with respect to the dilation map $\delta^r_\lambda(x) := (\lambda^{r_1} x_1, ..., \lambda^{r_n} x_n)^T (\forall i = 1, ..., n : r_i \in \mathbb{R}^+_0)$ if:*

$$\forall x \in \mathbb{R}^n, \forall \lambda \in \mathbb{R}^+ : f(\delta^r_\lambda(x)) = \lambda^\tau \delta^r_\lambda(f(x)) \tag{2}$$

Assume

H1 $f(x)$ is a homogeneous vector field of order $\tau \in \mathbb{R}^+$ with respect to the dilation map $\delta^r_\lambda(x)$.

System (1) is called homogeneous if **H1** holds. Notice that if **H1** holds, then $f(0) = 0$ and thus $x = 0$ is an equilibrium point of system (1).

Suppose **H1** holds. If there exists a point $\bar{x} \in \mathbb{R}^n$, $\bar{x} \neq 0$ such that $f(\bar{x}) = \gamma_{\bar{x}} \operatorname{diag}(r) \bar{x}$, $\gamma_{\bar{x}} \in \mathbb{R}$, then it can be shown, see e.g. [8], that the vector field at each point of the *ray* $R_{\bar{x}} = \{\delta^r_\lambda(\bar{x}) | \lambda \in \mathbb{R}^+_0\}$ *through* $\bar{x}$ is tangent to $R_{\bar{x}}$. As a consequence the forward (and backward) solution of system (1) starting in a point of the ray through $\bar{x}$ will remain on this ray for all future (and past) times for which this solution is defined. We call such a ray an *invariant ray* for system (1).

2.3 Cooperative systems

Next we call on the concept of a cooperative vector field, which has been widely studied [2], [11].

Definition 2. *A vector field $f(x)$, $x \in \mathbb{R}^n$ is said to be* cooperative *in $W \subset \mathbb{R}^n$ if the Jacobian matrix $\frac{\partial f}{\partial x}$ is Metzler for all $x \in W$.*

Assume

H2 $f(x)$ is cooperative in $\mathbb{R}^n_+$.

System (1) is called cooperative if **H2** holds.

The meaning of the term cooperative is best explained in a biological context. Suppose that the state of system (1) consists of a n interacting species. If the system is cooperative this implies that the presence of species i induces the growth of species j, for all $j \neq i$.

2.4 Irreducible systems

Finally we introduce the concept of an irreducible vector field.

Assume

H3 For $x \in \operatorname{int}(\mathbb{R}^n_+)$, the Jacobian matrix $\frac{\partial f}{\partial x}$ is irreducible.

For $x \in \operatorname{bd}(\mathbb{R}^n_+) \setminus \{0\}$ holds that

$$\exists i \in N \text{ with } x_i = 0 \text{ such that } f_i(x) > 0. \tag{3}$$

System (1) is called irreducible if **H3** holds.

2.5 A stability result for homogeneous cooperative and irreducible systems

The following result was proved in [8].

Theorem 1. *If* **H1**, **H2** *and* **H3** *hold, then system* (1) *is positive. In addition the zero solution of system* (1) *is GAS with respect to initial conditions in $\mathbb{R}^n_+$ if and only if there exists a unique invariant ray $R_{\bar{x}}$ in $\mathbb{R}^n_+$ for system* (1) *such that $R_{\bar{x}} \subset \operatorname{int}(\mathbb{R}^n_+)$ and $f(\bar{x}) = \gamma_{\bar{x}} \operatorname{diag}(r) \bar{x}$ with $\gamma_{\bar{x}} < 0$.*

Theorem 1 gives a criterion for GAS of the zero solution of system (1). From a practical point of view a GAS zero solution is not very interesting. Indeed in a biological system for example this implies that all the species die out. One of the goals of this paper is to examine the effect of applying a nonnegative constant control to system (1). It may be expected that this control will shift the equilibrium point from $x = 0$ to a nontrivial equilibrium point in $\mathbb{R}^n_+$. We will show that this is indeed the case if the uncontrolled system is GAS with respect to initial conditions in $\mathbb{R}^n_+$. Moreover we shall prove that this equilibrium point is unique and that it is GAS with respect to initial conditions in $\mathbb{R}^n_+$.

In the sequel we will frequently consider systems satisfying the conditions of Theorem 1. To avoid a cumbersome notation we introduce the following hypothesis:

H Hypotheses **H1**, **H2** and **H3** hold and the zero solution of system (1) is GAS with respect to initial conditions in $\mathbb{R}^n_+$. The unique invariant ray in $\text{int}(\mathbb{R}^n_+)$ of system (1) is denoted as $R_{\bar{x}}$ and has the property that $f(\bar{x}) = \gamma_{\bar{x}}\text{diag}(r)\bar{x}$ with $\gamma_{\bar{x}} < 0$.

3 Constant control

In this section we consider the following controlled system.

$$\dot{x} = f(x) + b \tag{4}$$

where $x \in \mathbb{R}^n$, $b \in \mathbb{R}^n$ and $f(x)$ satisfies **H1** and **H2**.

3.1 A positive controlled system

Proposition 1. *If* **H1** *and* **H2** *hold, then system* (4) *is positive if and only if* $b \in \mathbb{R}^n_+$.

Proof
The proof is omitted. □

3.2 Equilibrium points and their properties

When **H** holds and $b \in \mathbb{R}^n_+$, system (4) possesses at least one equilibrium point as can be seen from the following result.

Proposition 2. *If* **H** *holds and if* $b \in \mathbb{R}^n_+$, *then there exists* $y_b \in R_{\bar{x}}$ *such that for all* $y \in R_{\bar{x}}$ *with* $y \geq y_b$, *the set* $[0, y]$ *is forward invariant for system* (4). *For all* $y \in R_{\bar{x}}$ *with* $y \geq y_b$, *the set* $[0, y]$ *contains at least one equilibrium point of system* (4).

Proof
The proof is omitted. □

Next we show that equilibrium points of system (4) never belong to $\text{bd}(\mathbb{R}^n_+)$.

Proposition 3. *If* **H** *holds and if* $b \in \mathbb{R}^n_+ \setminus \{0\}$, *then every equilibrium point of system* (4) *in* $\mathbb{R}^n_+$, *belongs to* $\text{int}(\mathbb{R}^n_+)$.

Proof
We omit the proof. □

Next we establish the local stability behavior of an arbitrary equilibrium point of system (4). It turns out that every equilibrium point is locally asymptotically stable.

Proposition 4. *If* **H** *holds and if* $b \in \mathbb{R}^n_+ \setminus \{0\}$, *then the Jacobian matrix of* $f(x) + b$, *evaluated at an equilibrium point of system* (4) *in* $\mathbb{R}^n_+$, *is a Hurwitz matrix.*

Proof
We omit the proof. □

Now we are ready to state the main result of this section.

Theorem 2. *If* **H** *holds and if* $b \in \mathbb{R}^n_+ \setminus \{0\}$, *then there exists a unique equilibrium point* z^* *in* $\mathbb{R}^n_+$ *for system* (4). *This equilibrium point belongs to* $\text{int}(\mathbb{R}^n_+)$ *and is GAS for system* (4) *with respect to initial conditions in* $\mathbb{R}^n_+$.

Proof
Let us first prove that there exists a unique equilibrium point for system (4).
Invoking Proposition 2, we shall establish that for all $y \in R_{\bar{x}}$ with $y \geq y_b$ the sets $[0, y]$ contain *exactly one* equilibrium point of system (4). This implies uniqueness of the equilibrium point of system (4) in $\mathbb{R}^n_+$ (which, as we know from Proposition 3, must belong to $\text{int}(\mathbb{R}^n_+)$).

To prove that all the mentioned sets $[0, y]$ contain exactly one equilibrium point, we introduce the concept of degree of $f(x) + b$ relative to $(0, y)$ (see e.g. [12], p. 283):

$$\deg(f(x) + b, (0, y)) = \sum_{f(z)+b=0} \text{sign}\ \det\left(\frac{\partial f}{\partial x}(z)\right) \tag{5}$$

where sign denotes the sign of a real number ($0, +1$ or -1) and $\det(\frac{\partial f}{\partial x}(z))$ stands for the determinant of the Jacobian matrix $\frac{\partial f}{\partial x}(z)$.
To define the concept of degree, $f(x) + b$ and $(0, y)$ should satisfy the following conditions:

1. $f(x) + b$ is C^1 on the open set $(0, y)$ and C^0 on the closure of $(0, y)$ (This holds because of **H2**).
2. $f(x) + b$ is has no zeros on the boundary of $(0, y)$ (This is the case as can be seen from Proposition 3).
3. The Jacobian matrices of all zeros of $f(x) + b$ in $(0, y)$ are nonsingular (This is the case as can be seen from Proposition 4).

The degree has the property that it is a homotopy invariant. We will show that $f(x) + b$ is homotopic to the vector field $g(x) = -(x - z)$ where z is any equilibrium point of system (4) in the set $[0, y]$. Define the function $h(x,t) = -t(x-z) + (1-t)(f(x)+b)$. Then it is clear that

$$\begin{aligned} h(x,0) &= f(x) + b \\ h(x,1) &= -(x-z) \end{aligned}$$

and that $h(x,t)$ is C^0 in $[0,y] \times [0,1]$. Also it can be checked that $h(x,t)$ does not vanish on the boundary of $[0,y] \times [0,1]$, implying that $h(x,t)$ is a homotopy as claimed.
This leads to

$$\deg(f(x)+b,(0,y)) = \deg(-(x-z),(0,y)) = (-1)^n \tag{6}$$

Proposition 4 implies that sign $\det(\frac{\partial f}{\partial x}(z)) = (-1)^n$ when z is an equilibrium point of (4) in $[0,y]$. Then it follows from (5) and (6) that there exists exactly one equilibrium point of (4) in $[0,y]$.

So far we have shown that system (4) contains a unique equilibrium point in $\mathbb{R}^n_+$. Let us denote this equilibrium point as z^*. It remains to be shown that z^* is GAS for system (4) with respect to initial conditions in $\mathbb{R}^n_+$. (Local asymptotic) Stability of z^* is clear from Proposition 4, while convergence of all trajectories in $\mathbb{R}^n_+$ to z^* follows from Theorem 3.1 in [11]. Indeed, this result is applicable since

1. The flow of system (4) is strongly monotone in $\mathbb{R}^n_+$. This follows from Kamke's Theorem which can for be found in e.g. [11].
2. All the solutions of system (4) starting in $\mathbb{R}^n_+$ have compact forward orbit closure in $\mathbb{R}^n_+$. This follows from the fact that all the compact sets $[0,y]$ with $y \in R_{\bar{x}}$ and $y \geq y_b$ are known to be forward invariant sets (see Proposition 2).
3. The equilibrium point z^* of system (4) is unique in $\mathbb{R}^n_+$.

□

4 Example: Constant control of a reversible chemical reaction with dissipation

Consider the following reversible chemical reaction taking place inside a chemical reactor at a constant temperature with two chemicals X_1 and X_2.

$$X_1 \rightleftarrows X_2 \tag{7}$$

We assume that both reactions take place following the so-called *mass action principle* and that the rate constants of both reactions are equal to 1 (this last assumption is not necessary and can easily be relaxed but we assume it because it simplifies the notation). For more on modeling of chemical reactions we refer to [4]. Then it turns out that the concentrations of X_1 and X_2, denoted as x_1 and x_2, are described by the following differential equation

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1^\alpha \\ x_2^\alpha \end{pmatrix} \tag{8}$$

where α is real number with $\alpha \geq 1$. Notice that system (8) satisfies **H1**, **H2** and **H3**.
It follows from Theorem 1 that the zero solution is stable for system (8) but not asymptotically stable. Indeed, it is easily seen that the ray through $(1\ 1)^T$ is a stable invariant ray (but not asymptotically stable).
We will add a particular dissipation term to the vector field of system (8) in such a way that the new system is still homogeneous, cooperative and irreducible but such that the zero solution is asymptotically stable.
Consider the following system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1^\alpha \\ x_2^\alpha \end{pmatrix} + \begin{pmatrix} -x_1^\alpha \\ 0 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \tag{9}$$

where b_1 and $b_2 \in \mathbb{R}^+$ and at least one of them is different from zero.
We can interpret the addition of the dissipation term as follows. The chemical reactor corresponding to system (8) is *closed* in the sense that no chemicals are inserted or removed from it. The reactor corresponding to system (9) on the contrary, is *open*. Indeed, X_1 is removed from the reactor with a rate x_1^α (Notice that the power α is the *same* as the power used in the modeling of the rates of both reactions. This implies that system (9) with $b_1 = b_2 = 0$ is homogeneous. It is also easy to see that this system satisfies **H2** and **H3**.) In addition the reactor is fed at constant rates b_1 and b_2 with the chemicals X_1 and X_2. Invoking Theorem 1 again, it is possible to prove that the zero solution of system (9) with $b_1 = b_2 = 0$ is globally asymptotically stable (because we can show that there exists an asymptotically stable ray in $\mathrm{int}(\mathbb{R}^2_+)$, although we do not explicitly have to determine this ray). As a result we can invoke Theorem 2 to obtain that system (9) contains a unique equilibrium point in $\mathbb{R}^2_+$ that belongs to $\mathrm{int}(\mathbb{R}^2_+)$ and is globally asymptotically stable with respect to initial conditions in $\mathbb{R}^2_+$.
This result for a chemical reaction described by a two-dimensional system can be extended to particular chemical reactions described by n-dimensional systems with $n > 2$.

5 Kolmogorov systems

In this Section we consider a particular class of Kolmogorov systems. Kolmogorov systems are described by the following differential equation

$$\dot{x} = \mathrm{diag}(x)F(x) \tag{10}$$

where $F(x)$ is C^1 on $\mathbb{R}^n$.
Notice that the well-known Volterra-Lotka systems are examples of Kolmogorov systems where $F(x)$ is an affine map. The (biological) interpretation for the map $F(x)$ in a Kolmogorov system is the following: a component $F_i(x)$ of the map $F(x)$ is the per-capita growth-rate of species i.

We shall restrict ourselves here to the study of the following particular class of Kolmogorov systems

$$\dot{x} = \text{diag}(x)(f(x) + b) \tag{11}$$

where it will be assumed that **H** holds and that $b \in \text{int}(\mathbb{R}^n_+)$ (Notice the slightly stronger restriction on b compared to the one in the previous Section). Hypothesis **H** implies in particular that system (11) is cooperative in $\mathbb{R}^n_+$ and irreducible in $\text{int}(\mathbb{R}^n_+)$.

5.1 A positive system with invariant faces

It can be established (without proof) that

Proposition 5. *The sets $\mathbb{R}^n_+$, $\text{bd}(\mathbb{R}^n_+)$, $\text{int}(\mathbb{R}^n_+)$ and all the faces of $\mathbb{R}^n_+$ are invariant sets for system (10). In particular system (10) is a positive system.*

Notice that Proposition 5 holds for system (11) since it is a particular Kolmogorov system.

5.2 Stability properties of the interior equilibrium point

It is clear that every equilibrium point of system (11) is also an equilibrium point of system (4). In addition, the equilibrium points of system (4) and (11) in $\text{int}(\mathbb{R}^n_+)$ are the same. On the other hand system (11) may have equilibrium points on $\text{bd}(\mathbb{R}^n_+)$ which are not equilibrium points of system (4).

When we assume that **H** holds for system (4) and that $b \in \mathbb{R}^n_+ \setminus \{0\}$, it follows from Theorem 2 that system (4) possesses an equilibrium point z^* in $\text{int}(\mathbb{R}^n_+)$ which is unique in $\mathbb{R}^n_+$. The previous discussion implies that z^* is also an equilibrium point of system (11) and that it is the unique equilibrium point in $\text{int}(\mathbb{R}^n_+)$ of system (11). On the other hand system (11) may possess equilibrium points on $\text{bd}(\mathbb{R}^n_+)$, while there are no boundary equilibrium points for system (4).
From Proposition 5 we have that $\text{int}(\mathbb{R}^n_+)$ and $\text{bd}(\mathbb{R}^n_+)$ are forward invariant sets for system (11). This implies that z^* can *at best* be a GAS equilibrium point of system (11) *with respect to initial conditions in* $\text{int}(\mathbb{R}^n_+)$. We will show that this is indeed the case, but before doing so we need an auxiliary result

Proposition 6. *If **H** holds and if $b \in \mathbb{R}^n_+ \setminus \{0\}$, then there exists $y_b \in R_{\bar{x}}$ such that for all $y \in R_{\bar{x}}$ with $y \geq y_b$, the set $[0, y]$ is forward invariant for system (11).*

Proof
The proof is omitted. □

Theorem 3. *If **H** holds and if $b \in \text{int}(\mathbb{R}^n_+)$, then there exists an equilibrium point z^* of system (11) which is unique in $\text{int}(\mathbb{R}^n_+)$ and GAS with respect to initial conditions in $\text{int}(\mathbb{R}^n_+)$.*

Proof
Existence of the equilibrium point z^* and its uniqueness in $\text{int}(\mathbb{R}^n_+)$ has been shown already. We are left with proving that z^* is GAS for system (11) with respect to initial conditions in $\text{int}(\mathbb{R}^n_+)$.

(Local asymptotic) Stability of z^* follows from the fact that the Jacobian of $\text{diag}(x)(f(x) + b)$ evaluated at z^* equals $\text{diag}(z^*)\frac{\partial f}{\partial x}(z^*)$. We know from Proposition 4 that $\frac{\partial f}{\partial x}(z^*)$ is a Hurwitz matrix, implying that $\text{diag}(z^*)\frac{\partial f}{\partial x}(z^*)$ is also a Hurwitz matrix (This follows from the fact -which we don't prove here- that if M is a Hurwitz Metzler matrix then DM is also a Hurwitz Metzler matrix for all diagonal matrices D having strictly positive diagonal elements).
Convergence of all trajectories in $\text{int}(\mathbb{R}^n_+)$ to z^* follows from Theorem 3.1 in [11]. This result can be invoked for reasons which are similar to the reasons mentioned in the proof of Theorem 2. But some extra care has to be taken since there is a slight difference as will become clear in the following (more specifically in the second item).

1. The flow of system (11) is strongly monotone in $\text{int}(\mathbb{R}^n_+)$. This follows from Kamke's Theorem which can be found in e.g. [11].

2. All the solutions of system (11) in $\text{int}(\mathbb{R}^n_+)$ have compact forward orbit closure in $\text{int}(\mathbb{R}^n_+)$. (We stress that they must have a compact closure in $\text{int}(\mathbb{R}^n_+)$ and not in $\mathbb{R}^n_+$.)
   Indeed, the compact sets $[0, y]$ with $y \in R_{\bar{x}}$ and $y \geq y_b$ are known to be forward invariant for system (11) (see Proposition 6), implying that all the forward solutions of system (11) starting in $\mathbb{R}^n_+$ are bounded. We still need to establish that the closure of every forward orbit is *compact in* $\text{int}(\mathbb{R}^n_+)$. For this purpose we use the same reasoning as in [10]:
   For all $y \in \text{int}(\mathbb{R}^n_+)$, sufficiently close to the origin, holds that $\text{diag}(y)(f(y) + b) >> 0$ by continuity of f and since $f(0) = 0$ and $b \in \text{int}(\mathbb{R}^n_+)$. This implies that for all these y, the sets $\{x \in \text{int}(\mathbb{R}^n_+) | x \geq y\}$ are forward invariant sets for system (11) as this system is cooperative on $\mathbb{R}^n_+$. Then it follows that the closure of the forward orbit of solutions of system (11) starting in $\text{int}(\mathbb{R}^n_+)$, belongs to $\text{int}(\mathbb{R}^n_+)$.

3. The equilibrium point z^* of system (11) is unique in $\text{int}(\mathbb{R}^n_+)$. □

6 Discussion of the main results

In this Section we discuss the main results of this paper (Theorem 2 and Theorem 3) and compare them to some known results.

Let us first discuss Theorem 2. Consider the following affine system

$$\dot{x} = Ax + b \tag{12}$$

where A is an irreducible Metzler matrix and $b \in \mathbb{R}^n_+ \setminus \{0\}$. It can be shown that system (12) is a positive system. A classical result as proved for example in [6] states that if A is a Hurwitz matrix, then there exists a unique equilibrium point of system (12) in $\mathbb{R}^n_+$, which belongs to $\mathrm{int}(\mathbb{R}^n_+)$ and which is GAS. Theorem 2 can be interpreted as a generalization of this result to a particular class of nonlinear systems. The role of the Hurwitz matrix is now played by the GAS system $\dot{x} = f(x)$, while the linear vector field Ax is replaced by a homogeneous (and therefore nonlinear) one.

To conclude we compare Theorem 3 with a result from [10]. In that paper a particular class of Kolmogorov systems is studied and the following Theorem is proved.

Theorem 4. *Consider system* (10) *and suppose that the following holds:*

1. *$F(x)$ is cooperative in $\mathbb{R}^n_+$.*
2. *$F(0) >> 0$.*
3. *$\frac{\partial F}{\partial x}(y) \geq \frac{\partial F}{\partial x}(z)$ (where the inequality is to be interpreted entry-wise) when $z \geq y \geq 0$.*

If system (10) *possesses an equilibrium point in* $\mathrm{int}(\mathbb{R}^n_+)$*, then this equilibrium point is unique in* $\mathrm{int}(\mathbb{R}^n_+)$ *and it is GAS with respect to initial conditions in* $\mathrm{int}(\mathbb{R}^n_+)$*.*

The third condition is a concavity condition and we refer to it as such in what follows.
This result and Theorem 3 are dealing with the same problem: determining the stability properties of an interior equilibrium point for particular Kolmogorov systems. Both results have in common that a cooperativity condition holds and the fact that the vectors $f(0) + b$ and $F(0)$ point towards the interior of the first orthant. More importantly we point out the differences between both Theorems: In our result there is no concavity condition (and typically this concavity condition is not satisfied for systems for which our result applies). On the other hand our systems are subject to a homogeneity condition and an irreducibility condition, both absent in the result of [10].

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