Accessibility properties of controlled Lotka-Volterra systems[1]

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Abstract

Although the dynamical behavior of Lotka-Volterra systems has been studied thoroughly, there exist few results on control related aspects for these systems. We introduce controlled Lotka-Volterra systems and deal with their accessibility properties. Accessibility can be seen as a first step towards controllability. In particular both a necessary condition and a sufficient condition are proposed. Both conditions amount to checking the rank of a particular matrix.

1 Introduction

A particular class of state-constrained systems is the class of positive systems. A system is said to be positive if when initiated in the first orthant of $\mathbb{R}^n$, its state remains in this orthant for future times. Within the class of positive systems, Lotka-Volterra systems are well known and have been studied extensively [3], [5]. The largest part of the literature is devoted to questions concerning analysis of Lotka-Volterra systems. Indeed, issues such as stability of equilibrium points, permanence or persistence dominate this literature.

On the other hand few results exist on control related issues for these systems. In this paper we introduce controlled Lotka-Volterra systems and study their accessibility properties. Accessibility is a basic concept when dealing with control systems [4], [6]. In addition, accessibility properties provide information on the controllability properties of the system. In particular a system which is not accessible, is not controllable either. Some results on accessibility are available for particular classes of systems: results for chemical reaction systems (which form a particular class of positive systems) are found in [2]; accessibility and controllability of recurrent neural networks is discussed in [1] and [7]. However, to our best of knowledge, no results exist for Lotka-Volterra systems.

In this paper we provide a necessary condition for accessibility of controlled Lotka-Volterra systems. We also present a sufficient condition for accessibility. Both conditions amount to checking whether a matrix has full rank.

This paper is organized as follows. In Section 2 we review some results from the literature on Lotka-Volterra systems and control systems. In addition we define the class of controlled Lotka-Volterra systems. The main results are stated and proved in Section 3.

2 Preliminaries

2.1 Notation

Let $\mathbb{R}$ be the set of real numbers and $\mathbb{R}^n$ the set of n-tuples for which all components belong to $\mathbb{R}$. $\mathbb{R}^+ = [0, +\infty)$ ($\mathbb{R}^- = (-\infty, 0]$) and $\mathbb{R}^+_0 = (0, +\infty)$, while $\mathbb{R}^n_+$ ($\text{int}(\mathbb{R}^n_+)$, $\mathbb{R}^n_-$) is the set of n-tuples for which all components belong to $\mathbb{R}^+$ ($\mathbb{R}^+_0$, $\mathbb{R}^-$). Finally, the boundary of $\mathbb{R}^n_+$, $\mathbb{R}^n_+ \setminus \text{int}(\mathbb{R}^n_+)$, is denoted as $\text{bd}(\mathbb{R}^n_+)$.

Let I be a nonempty and proper subset of the index set $N := \{1, 2, ..., n\}$. The set $F_I := \{x \in \mathbb{R}^n_+ | x_i = 0 \text{ for } i \in I\}$ is a *face* of $\mathbb{R}^n_+$. The *dimension* of F_I equals $\#I$, the cardinality of the set I.

Given a vector $x \in \mathbb{R}^n$, $\text{diag}(x)$ is a real $n \times n$ diagonal matrix where the i-th diagonal entry equals x_i, the i-th component of the vector x and when j is a natural number, $\text{diag}^j(x)$ stands for $(\text{diag}(x))^j$. When B is a real $n \times m$ matrix we denote the rank of B as $\text{rg}B$ and $\text{col}B$ stands for the set of columns of matrix B.

2.2 Lotka-Volterra systems

Consider the following differential equation

$$\dot{x} = f(x) \tag{1}$$

where $x \in \mathbb{R}^n$ and where we assume that $f(x)$ is such that existence and uniqueness of solutions of system (1) is guaranteed (for example if f satisfies a local Lipschitz property for all $x \in \mathbb{R}^n$). The solution (forward

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solution) starting in x_0 is denoted as $x(t, x_0), t \in \mathcal{I}_{x_0}$ ($t \in \mathcal{I}_{x_0}^+$), where $\mathcal{I}_{x_0}$ ($\mathcal{I}_{x_0}^+$) stands for the maximal interval (maximal forward interval) of existence of the solution.
A set $S \subset \mathbb{R}^n$ is *invariant* (*forward invariant*) for system (1) if for all $x_0 \in S$ holds that $x(t, x_0) \in S$, for all $t \in \mathcal{I}_{x_0}$ (for all $t \in \mathcal{I}_{x_0}^+$). If $\mathbb{R}_+^n$ is forward invariant for system (1), then the system is said to be *positive.*

A Lotka-Volterra system is given by the following differential equation

$$\dot{x} = \text{diag}(x)(Ax + b) \tag{2}$$

where A is a real $n \times n$-matrix and $b, x \in \mathbb{R}^n$.
Existence and uniqueness of solutions for system (2) is guaranteed since the vector field $\text{diag}(x)(Ax + b)$ satisfies a local Lipschitz property for all $x \in \mathbb{R}^n$.

It can be established (we do not include a proof here) that

Proposition 1. *The sets* $\mathbb{R}_+^n$, $\text{bd}(\mathbb{R}_+^n)$, $\text{int}(\mathbb{R}_+^n)$ *and all the faces of* $\mathbb{R}_+^n$ *are invariant sets for system* (2).

An obvious consequence of Proposition 1 is the following corollary.

Corollary 1. *System* (2) *is positive.*

Finally we would like to point out a striking feature of Lotka-Volterra systems: A Lotka-Volterra system, restricted to an arbitrary invariant face of $\mathbb{R}_+^n$ is also a Lotka-Volterra system and its dimension equals the dimension of the corresponding invariant face.
Indeed, it makes sense to consider the dynamical systems that result when restricting system (2) to the faces of $\mathbb{R}_+^n$ since these faces are invariant by Proposition 1. Suppose that one is interested in the dynamics of system (2), restricted to some face F_I. Denote by A_I the principal sub-matrix of matrix A which is obtained by deleting both rows and columns corresponding to elements of I (i.e. the k-th row and column of A are deleted when $k \in I$). Likewise, b_I is the vector which is obtained when deleting those entries of b corresponding to elements of I. The dynamics of system (2) restricted to face F_I is then given by

$$\dot{x}_I = \text{diag}(x_I)(A_I x_I + b_I) \tag{3}$$

where $x_I \in \mathbb{R}^{\#I}$. It is clear that system (3) is a Lotka-Volterra system.

2.3 Accessibility for control systems

Consider the following control system

$$\dot{x} = f(x) + g(x)u \tag{4}$$

where $x \in U$ and U is an open subset of $\mathbb{R}^n$, $u \in \mathbb{R}$ and $f(x)$ and $g(x)$ are smooth (i.e. C^∞) vector fields on $\mathbb{R}^n$. Notice that this system is affine in the control variable u.

Next we give an informal review of a number of concepts needed to state our main results. No originality is claimed here and the interested reader is referred to e.g. [6] and [4].
$R(x_0, T)$ denotes the reachable set from x_0 at time $T \in \mathbb{R}_0^+$. Thus a point z belongs to $R(x_0, T)$ if there exists an input function $u : [0, T] \to \mathbb{R}$ such that the solution of system (4) starting from x_0 and driven by the input signal $u(t)$ satisfies $x(T, x_0) = z$. The set of points reachable from x_0 in time $T \in \mathbb{R}_0^+$ or less is denoted as $R_T(x_0) = \cup_{\tau \leq T} R(x_0, \tau)$.

Definition 1. *System* (4) *is locally accessible at* x_0 *if* $R_T(x_0)$ *contains a nonempty open set for all* $T \in \mathbb{R}_0^+$. *System* (4) *is said to be locally accessible when it is locally accessible for all* $x \in U$.

In the study of the accessibility properties of system (4) the Lie algebra $\mathcal{C}$, called the accessibility algebra, plays a prominent role. Every element of $\mathcal{C}$ is a linear combination of Lie brackets of the form

$$[h_1, [h_2, ...[h_{k-1}, h_k]...]] \tag{5}$$

where k is a strictly positive natural number and h_j equals f or g for all $j = 1, ..., k$.
The *accessibility distribution* $C(x)$ is generated by the accessibility algebra in the following way:

$$C(x) = \text{span}\{h(x) | h \text{ is a vector field in } \mathcal{C}\} \tag{6}$$

The role of the accessibility algebra when dealing with the accessibility properties of system (4) becomes clear in the following Theorems.

Theorem 1. *If* $\dim(C(x)) = n$, *then system* (4) *is locally accessible at* x. *If this holds for all* $x \in U$, *then the system is locally accessible.*

The distribution $C(x)$ is *nonsingular at* $x \in \mathbb{R}^n$ if there exists an open neighborhood of x such that the dimension of the distribution is constant on this neighborhood.
The full-dimension condition for $C(x)$ is close to necessary, as can be seen from the following result. This result follows from Frobenius' Theorem since $C(x)$ is an involutive distribution.

Theorem 2. *If* $\dim(C(x)) = n_1 < n$ *and if* $C(x)$ *is nonsingular at* x, *then there exists a smooth local change of coordinates in some neighborhood of* x

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \phi(x)$$

where $z_1 \in \mathbb{R}^{n_1}$ *and such that in* z*-coordinates the dynamics are*

$$\begin{aligned} \dot{z}_1 &= H_1(z, u) \\ \dot{z}_2 &= 0 \end{aligned}$$

2.4 Controlled Lotka-Volterra systems

In this section we would like to introduce a particular class of control systems. We demand that these control systems are controlled versions of the classical Lotka-Volterra systems described by (2). At the same time these controlled systems should remain positive systems. It will be shown that a simple and structurally appealing class satisfying these constraints can indeed be introduced.

Consider the following control system

$$\dot{x} = \mathrm{diag}(x)(Ax + bu) \tag{7}$$

where $x, b \in \mathbb{R}^n$, A is a $n \times n$ matrix and $u \in \mathbb{R}$. Notice that this system has the structure of system (4) with $f(x) = \mathrm{diag}(x)Ax$ and $g(x) = \mathrm{diag}(x)b$. Henceforth we call system (7) a *controlled Lotka-Volterra system.*

As mentioned in the beginning of this subsection, the effect of the introduction of a control value, should not affect the positivity character of the system. Before we show that this is the case for system (7), we generalize the definition of (forward) invariant sets to control systems (4) in a straightforward way.
A set $S \subset \mathbb{R}^n$ is *invariant* (*forward invariant*) for system (4) if for all $x_0 \in S$ and for all input signals $u(t)$ holds that the solution of system (4) starting in x_0 and driven by $u(t)$ remains in S for all (future) times for which this solution is defined. If $\mathbb{R}^n_+$ is forward invariant for system (4), then the system is said to be *positive.*

We have (without proof) that

Proposition 2. *The sets* $\mathbb{R}^n_+$, $\mathrm{bd}(\mathbb{R}^n_+)$, $\mathrm{int}(\mathbb{R}^n_+)$ *and all the faces of* $\mathbb{R}^n_+$ *are invariant sets for system* (7) *and therefore system* (7) *is positive.*

We notice that the dynamics of the controlled Lotka-Volterra system restricted to an invariant face of $\mathbb{R}^n_+$ are also controlled Lotka-Volterra systems. For example the restriction to the face F_I is given by (using the same notation as in subsection 2.2)

$$\dot{x}_I = \mathrm{diag}(x_I)(A_I x + b_I u) \tag{8}$$

Remark 1. There are different ways to introduce control in classical Lotka-Volterra systems. For example the system

$$\dot{x} = \mathrm{diag}(x)(Ax + b) + b'v \tag{9}$$

where $b' \in \mathbb{R}^n_+$ and $v \in \mathbb{R}^+$ is a positive control system. Let us comment on the constraints on both b' and v. The purpose of these constraints is to guarantee that system (9) is positive.

1. Suppose that two distinct components of b' have different signs. The solution of system (9) starting in 0 leaves $\mathbb{R}^n_+$ when $v(t) \neq 0$. This implies that b' should belong to $\mathbb{R}^n_+$ or to $\mathbb{R}^n_-$ in order that system (9) is positive. Henceforth, we assume without loss of generality that $b' \in \mathbb{R}^n_+$.
2. To see that $v \in \mathbb{R}^+$ is necessary to guarantee that (9) is positive, simply consider the solution starting in 0 and driven by an input signal which takes values in $\mathbb{R}^-$. Indeed, this solution leaves $\mathbb{R}^n_+$ (remember that it is assumed that $b' \in \mathbb{R}^n_+$).

When comparing system (7) and system (9), the absence of a sign constraint for the former implies for example that the system can be steered approximately in both the positive and negative direction of the control vector field $\mathrm{diag}(x)b$. For the latter system it is only possible to approximately steer the system in the positive direction of the control vector field b'. Although we do not address controllability questions in this paper this suggests that this extra (directional) degree of freedom in the former model may yield stronger controllability properties than for the latter model.

Another difference between (7) and (9) is that the faces of $\mathbb{R}^n_+$ are invariant for the former, but in general not for the latter system. This invariance property allowed us to show that the striking structural feature of Lotka-Volterra systems, namely that their restrictions to the faces of $\mathbb{R}^n_+$ are also Lotka-Volterra systems, was inherited by an appropriately defined controlled Lotka-Volterra version as in (7). This feature does not hold for system (9).

3 Main Results

In this section we deal with the accessibility properties of system (7) restricted to the set $\mathrm{int}(\mathbb{R}^n_+)$, which is, as we know from Proposition 2, invariant for system (7). Let us comment on why we restrict to the set $\mathrm{int}(\mathbb{R}^n_+)$.

1. It follows from physical considerations that system (7) is only meaningful for states in $\mathbb{R}^n_+$. Indeed, the state of systems (7) will typically consists of concentrations of chemical products in chemical systems or of concentrations of species in ecological systems, which are nonnegative. This intuition has been given a sound mathematical basis in Proposition 2 where we have shown that $\mathbb{R}^n_+$ is an invariant set.
2. As shown in Proposition 2, the faces of $\mathbb{R}^n_+$ are also invariant sets for (7) and have a dimension which is less than n. Therefore, the system cannot be locally accessible from initial states in a face of $\mathbb{R}^n_+$. This implies that the maximal subset of $\mathbb{R}^n_+$ in which the system may be accessible is

int($\mathbb{R}^n_+$), explaining why we restrict the investigation to int($\mathbb{R}^n_+$).

3. We point out that the accessibility properties of the system restricted to the faces of $\mathbb{R}^n_+$ can be easily determined once we know how to determine them for the system on int($\mathbb{R}^n_+$). Indeed, we have shown in the previous section that a controlled Lotka-Volterra system, restricted to a face of $\mathbb{R}^n_+$ is also a controlled Lotka-Volterra system.

3.1 A necessary condition

The real $n \times (n+1)$ matrix consisting of b and the columns of A is denoted as $[b\ A]$. Furthermore, the columns of A are denoted as $a_1, a_2, ..., a_n$. Finally, $f(x)$ stands for $\mathrm{diag}(x)Ax$, $g(x)$ for $\mathrm{diag}(x)b$ and $C(x)$ is the accessibility distribution associated to the Lie algebra resulting from f, g and all repeated brackets of f and g.

Proposition 3. *For all* $x \in \mathbb{R}^n$ *holds that* $\dim C(x) \leq \mathrm{rg}[b\ A]$.

Proof

We will show that for all $x \in \mathbb{R}^n$ holds that

$$C(x) \subset \text{span col}(\mathrm{diag}(x)[b\ A]) \qquad (10)$$

This implies that for all $x \in \mathbb{R}^n$, $\dim C(x) \leq \mathrm{rg}[b\ A]$ which proves the Theorem.

To prove (10) it suffices to show that

$$f,\ g,\ [f,h],\ [g,h] \in \text{span col}(\mathrm{diag}(x)[b\ A]) \qquad (11)$$

for all $h \in \text{span col}(\mathrm{diag}(x)[b\ A])$.
We have that

1. $f,\ g \in \text{span col}(\mathrm{diag}(x)[b\ A])$ is clear from the definition of f and g.
2. To show that $[f,h] \in \text{span col}(\mathrm{diag}(x)[b\ A])$ for all $h \in \text{span col}(\mathrm{diag}(x)[b\ A])$, it is sufficient to prove that $[f, \mathrm{diag}(x)b]$, $[f, \mathrm{diag}(x)a_i] \in \text{span col}(\mathrm{diag}(x)[b\ A])$ for all $i \in N$.
We have that

$$\begin{aligned}[f, \mathrm{diag}(x)b] =& \mathrm{diag}(b)\mathrm{diag}(x)Ax \\ & - (\mathrm{diag}(Ax) + \mathrm{diag}(x)A)\mathrm{diag}(x)b \\ =& - \mathrm{diag}(x)A\mathrm{diag}(x)b \\ & \in \text{span col}(\mathrm{diag}(x)[b\ A])\end{aligned}$$

and similarly that

$$\begin{aligned}[f, \mathrm{diag}(x)a_i] =& \mathrm{diag}(a_i)\mathrm{diag}(x)Ax \\ & - (\mathrm{diag}(Ax) + \mathrm{diag}(x)A)\mathrm{diag}(x)a_i \\ =& - \mathrm{diag}(x)A\mathrm{diag}(x)a_i \\ & \in \text{span col}(\mathrm{diag}(x)[b\ A])\end{aligned}$$

for all $i \in N$.

3. Finally, to show that $[g,h] \in \text{spancol}(\mathrm{diag}(x)[b\ A])$ for all $h \in \text{span col}(\mathrm{diag}(x)[b\ A])$, it is sufficient to prove that $[g, \mathrm{diag}(x)b]$, $[g, \mathrm{diag}(x)a_i] \in \text{span col}(\mathrm{diag}(x)[b\ A])$ for all $i \in N$.
But all these brackets turn out to be zero. Indeed, $[g, \mathrm{diag}(x)b] = [g,g] = 0$ and

$$\begin{aligned}[g, \mathrm{diag}(x)a_i] =& \mathrm{diag}(a_i)\mathrm{diag}(x)b - \mathrm{diag}(b)\mathrm{diag}(x)a_i \\ =& 0\end{aligned}$$

for all $i \in N$.

□

From Proposition 3 and Theorem 2 readily follows that

Theorem 3. *If* $\mathrm{rg}[b\ A] < n$ *and if* $C(x)$ *is nonsingular for some* $x \in \mathrm{int}(\mathbb{R}^n_+)$, *then system* (7) *is not locally accessible at* x.

Let us define the set $V := \{z \in \mathrm{int}(\mathbb{R}^n_+) | C(z)$ is nonsingular$\}$. This set is open and dense in int($\mathbb{R}^n_+$) [4] and thus we obtain that

Theorem 4. *If* $\mathrm{rg}[b\ A] < n$ *then there exists an open and dense subset* V *in* int($\mathbb{R}^n_+$) *such that system* (7) *is not locally accessible at the points of* V.

3.2 A sufficient condition

In this subsection we present a sufficient condition for accessibility of (7), restricted to int($\mathbb{R}^n_+$).

Proposition 4. *If* $\mathrm{rg}[b\ A] = n$ *and if* $b_i \neq b_j$ *for all* $i \neq j$, *then* $\dim C(x) = n$ *for all* $x \in \mathrm{int}(\mathbb{R}^n_+)$.

Proof

Case 1: $\mathrm{rg}A = n$.
Consider f and the brackets $[g,f]$, $[g,[g,f]]$, ... , $[g,[g,...,[g,f]...]$ (each bracket contains f once). Then we obtain the following set of vector fields

$$\mathrm{diag}(x)A\{x, \mathrm{diag}(b)x, \mathrm{diag}^2(b)x, ..., \mathrm{diag}^{n-1}(b)x\} \quad (12)$$

or written more compactly,

$$\mathrm{col}(\mathrm{diag}(x)A\mathrm{diag}(x)M) \qquad (13)$$

with

$$M := \begin{pmatrix} 1 & b_1 & b_1^2 & \dots & b_1^{n-1} \\ 1 & b_2 & b_2^2 & \dots & b_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & b_n & b_n^2 & \dots & b_n^{n-1} \end{pmatrix}, \qquad (14)$$

known as Van der Monde's matrix. M is nonsingular if and only if $b_i \neq b_j$ for all $i \neq j$. If M is nonsingular and since $\mathrm{rg}A = n$ we obtain that $\dim C(x) = n$ for all

$x \in \mathrm{int}(\mathbb{R}^n_+)$.

Case 2: $\mathrm{rg}A < n$.
By assumption $\mathrm{rg}[b\ A] = n$, and this implies that $\mathrm{rg}A = n-1$ and that $b \notin \mathrm{span\ col}A$.
Clearly the set of vector fields considered in Case 1 do not generate a distribution with dimension n when $\mathrm{rg}A < n$. Consider the same set of vector fields as in Case 1 (f, $[g,f]$, $[g,[g,f]]$,...) and add to this set vector field g. Then we obtain the following set of vector fields

$$\mathrm{col}(\mathrm{diag}(x)[b\ A\mathrm{diag}(x)M]) \qquad (15)$$

Since $\mathrm{rg}[b\ A] = n$ we obtain again that $\dim C(x) = n$ for all $x \in \mathrm{int}(\mathbb{R}^n_+)$. $\square$
From Proposition 4 and Theorem 1 readily follows

Theorem 5. *If* $\mathrm{rg}[b\ A] = n$ *and if* $b_i \neq b_j$ *for all* $i \neq j$ *then system* (7) *is locally accessible for all* $x \in \mathrm{int}(\mathbb{R}^n_+)$ *and thus system* (7)*, restricted to* $\mathrm{int}(\mathbb{R}^n_+)$*, is locally accessible.*

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