

Linearization by prolongations of two-input driftless systems¹

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Abstract

This communication deals with the problem of linearization by prolongations of two-input driftless systems. For general two-input systems, the number of computations needed to check if a system is linearizable by prolongations is quite large. However, for driftless systems, the conditions presented in this paper require very few computations. The methodology is illustrated for some engineering systems which fulfill these conditions, e.g., a unicycle, a planar robot, and a hopping robot.

1 Introduction

Feedback linearization of nonlinear control systems has been an active area of research over the last twenty years. Some salient results in this area have been summarized in recent books on this subject ([5], [8]). The significance of this concept is twofold: (i) it transfers the properties of a linear system to a nonlinear one; (ii) it allows us to propose simple solutions to an important problem in automatica, i.e., stabilization of the system around a given trajectory.

Dynamic feedback linearization is in general an open problem. A particular class of dynamic feedback linearization is “linearization by prolongations”. It was first studied in [1]. Some contributions to the complete solution of the problem of linearization by prolongations appeared in [2], [3] and [9]. These solutions require carrying out a large number of computations in order to decide whether a system is linearizable by prolongations. However, for driftless systems, this problem is easier and is described in this communication.

This paper is organized as follows: Section 3 establishes the main results proved after some background results

given in Section 2. Some examples are worked out in section 4, which are then followed by conclusions.

2 Background Results

Definition 1 *Let*

$$\dot{x} = f(x) + \sum_{i=1}^m g_i(x)u_i \quad (1)$$

be a nonlinear system with m inputs. The new system

$$\begin{aligned} \dot{x} &= f(x) + \sum_{i=1}^m g_i(x)u_i^0 \\ \dot{u}_i^0 &= u_i^1 & \forall i = 1 \dots m \\ &\vdots \\ \dot{u}_i^{k_i-1} &= v_i \end{aligned}$$

is defined to be a prolongation of system (1). Variables u_i^j , $\forall i = 1 \dots m$, $j = 0 \dots k_i - 1$, defined as u_i^j , are together with x , the new state variables. The new inputs are v_i .

Definition 2 *Let*

$$\dot{x} = f(x) + \sum_{i=1}^m g_i(x)u_i$$

be a nonlinear system with m inputs. This system is said to be linearizable by prolongations if there exists a prolongation of the original system which is static feedback linearizable.

Proposition 1 *If the system*

$$\dot{x} = f(x) + g_1(x)u_1 + g_2(x)u_2 \quad x \in R^n$$

is linearizable by derivation of u_1 n_1 times and u_2 n_2 times (with $n_1 \geq n_2 \geq 1$), then the system is also linearizable by derivation of u_1 $n_1 - 1$ times and u_2 $n_2 - 1$ times.

Proof.

The proof of this proposition can be found in [3] or in [9]. ■

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This proposition implies that when considering prolongations, derivatives of just one input need to be considered.

3 Linearization of Driftless Systems

This section deals with necessary and sufficient conditions for the two input driftless system

$$\dot{x} = g_1(x)u_1 + g_2(x)u_2 \quad x \in R^n \quad (2)$$

be linearizable by prolongations.

For simplicity in the statement of the main result, let us define

$$\begin{aligned} \Delta_{(k,1)} &= \langle g_1, ad_{g_2}g_1, \dots, ad_{g_2}^k g_1 \rangle \\ \Delta_{(k,2)} &= \langle g_2, ad_{g_1}g_2, \dots, ad_{g_1}^k g_2 \rangle \end{aligned}$$

Theorem 1 *The two input driftless system (2) is linearizable by prolongations if, and only if*

i) $\forall k \in \{1, \dots, n-3\}$, $ad_{(ad_{g_2}g_1)}^2 g_2 \in \Delta_{(k,1)}$.

ii) $dim \langle g_1, ad_{g_2}g_1, \dots, ad_{g_2}^{n-2} g_1, g_2 \rangle = n$

or

i) $\forall k \in \{1, \dots, n-3\}$, $ad_{(ad_{g_1}g_2)}^2 g_1 \in \Delta_{(k,2)}$.

ii) $dim \langle g_2, ad_{g_1}g_2, \dots, ad_{g_1}^{n-2} g_2, g_1 \rangle = n$

Proof

As it has been previously indicated, it is sufficient to prolong only one of the two inputs. Let us consider prolongation of u_2 . The prolonged system is given by

$$\begin{aligned} \dot{x} &= g_1(x)v_1 + g_2(x)y_1 \\ \dot{y}_1 &= y_2 \\ &\vdots \\ \dot{y}_{r-1} &= y_r \\ \dot{y}_r &= v_2 \end{aligned} \quad (3)$$

where $x \in R^n$ stand for the state variables of the original system, $y_1 = u_2$, $y_2 = \dot{u}_2$, \dots , $y_r = u_2^{(r-1)}$, $v_1 = u_1$ and $v_2 = u_2^{(r)}$. Thus, the drift vector field and the inputs of the prolonged system read as follows:

$$F = y_1 g_2 + y_2 \frac{\partial}{\partial y_1} + \dots + y_r \frac{\partial}{\partial y_{r-1}} \quad (4)$$

$$G_1 = g_1 \quad (5)$$

$$G_2 = \frac{\partial}{\partial y_r} \quad (6)$$

where, $x \in R^n$ stand for the state variables of the original system and y_j 's represent the additional state variables obtained by prolongation.

Let us define D_i by

$$D_0 = \langle G_1, G_2 \rangle \quad (7)$$

$$D_i = \langle D_{i-1}, [F, X] \mid \forall X \in D_{i-1} \rangle \quad (8)$$

As shown in [8], the prolonged system (3) is linearizable if, and only if distributions D_i are involutives and

$$dim D_{n+r} = n + r \quad (9)$$

Therefore, let us compute D_i , for $i = 0, 1, \dots$ and check the involutivity conditions. Namely,

- $i = 0$,

$$D_0 = \langle G_1, G_2 \rangle = \langle g_1, \frac{\partial}{\partial y_r} \rangle$$

Since $g_1 = g_1(x)$, D_0 is involutive.

- $i = 1$,

$$\begin{aligned} D_1 &= \langle D_0, [F, G_1], [F, G_2] \rangle = \\ &= \langle g_1, ad_{g_2}g_1, \frac{\partial}{\partial y_{r-1}}, \frac{\partial}{\partial y_r} \rangle \end{aligned}$$

Note that D_1 is involutive if and only if

$$[g_1, ad_{g_2}g_1] \in \langle g_1, ad_{g_2}g_1 \rangle = \Delta_{(1,1)}$$

Since

$$[g_1, ad_{g_2}g_1] = -ad_{g_1}^2 g_2$$

D_1 is involutive if and only if

$$ad_{g_1}^2 g_2 \in \Delta_{(1,1)}$$

- $i = 2$,

$$\begin{aligned} D_2 &= \langle g_1, ad_{g_2}g_1, ad_{g_2}^2 g_1, \\ &\quad \frac{\partial}{\partial y_{r-2}}, \frac{\partial}{\partial y_{r-1}}, \frac{\partial}{\partial y_r} \rangle \end{aligned}$$

Note that D_2 is involutive if, and only if

1. $[g_1, ad_{g_2}g_1] \in D_2$,
2. $[g_1, ad_{g_2}^2 g_1] \in D_2$, and
3. $[ad_{g_2}g_1, ad_{g_2}^2 g_1] \in D_2$.

At this step, since we can assume the involutivity of the previous distributions,

$$[g_1, ad_{g_2}g_1] \in \Delta_{(1,1)} \subset \Delta_{(2,1)}$$

and condition 1 holds. Moreover

$$ad_{g_1}^2 g_2 = -[g_1, ad_{g_2}g_1] = \alpha g_1 + \beta ad_{g_2}g_1$$

On the other hand, the Jacobi identity applied to $[g_1, [g_2, [g_2, g_1]]]$ yields

$$[g_1, ad_{g_2}^2 g_1] = [g_2, ad_{g_1}^2 g_2]$$

thus,

$$\begin{aligned} [g_1, ad_{g_2}^2 g_1] &= [g_2, ad_{g_1}^2 g_2] = \\ &= [g_2, \alpha g_1 + \beta ad_{g_2} g_1] = \\ &= (g_2 \alpha) g_1 + (g_2 \beta) ad_{g_2} g_1 + \\ &+ \alpha ad_{g_2} g_1 + \beta ad_{g_2}^2 g_1 \in D_2 \end{aligned}$$

and therefore, condition 2 holds too.

As far as condition 3 is concerned, again from the Jacobi identity,

$$\begin{aligned} [ad_{g_2} g_1, ad_{g_2}^2 g_1] &= -[g_2, [ad_{g_2} g_1, ad_{g_2} g_1]] - \\ &- ad_{(ad_{g_2} g_1)}^2 g_2 \end{aligned}$$

But $[ad_{g_2} g_1, ad_{g_2} g_1] = 0$, thus, assuming the involutivity of D_1, D_2 is involutive if and only if

$$ad_{ad_{g_2} g_1}^2 g_2 \in \langle g_1, ad_{g_2} g_1, ad_{g_2}^2 g_1 \rangle = \Delta_{(2,1)}$$

The corresponding statements for $D_i, i = 3, \dots, r-1$ can be obtained proceeding by induction. Let us point out that

$$\begin{aligned} D_{r-1} &= \langle g_1, ad_{g_2} g_1, \dots, ad_{g_2}^{r-1} g_1, \\ &\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \dots, \frac{\partial}{\partial y_r} \rangle \end{aligned}$$

$$\begin{aligned} D_r &= \langle g_1, ad_{g_2} g_1, \dots, ad_{g_2}^r g_1, \\ &\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \dots, \frac{\partial}{\partial y_r}, g_2 \rangle \end{aligned}$$

- D_r is involutive if D_{r-1} is so.
- Since the Lie bracket $[F, g_2] = 0$, the distributions' flag is stationary from the level r , namely

$$\begin{aligned} D_0 \subset D_1 \subset \dots \subset D_i \subset \dots \subset D_r &= \\ &= D_{r+1} = \dots = D_{n+r} \end{aligned}$$

- Finally, condition (9) and the later statement implies

$$2r + 2 = \dim D_r = \dim D_{n+r} = n + r$$

which, in turn, allow us to conclude $r = n - 2$.

Notice that a necessary condition for this equality to hold is

$$\langle g_1, ad_{g_2} g_1, \dots, ad_{g_2}^{n-2} g_1, g_2 \rangle$$

be linearly independent. ■

Corollary 1 *The two input driftless system (2) is linearizable by prolongations if, and only if $\Delta_{(k,1)}$ or $\Delta_{(k,2)}$ for $k = 1, \dots, n - 2$ are involutives and $\langle g_1, ad_{g_2} g_1, \dots, ad_{g_2}^{n-2} g_1, g_1 \rangle = R^n$, respectively $\langle g_2, ad_{g_1} g_2, \dots, ad_{g_1}^{n-2} g_2, g_1 \rangle = R^n$.*

4 Examples

The above results apply to a number of examples, some of which are worked out below.

1. Example 1 considers the problem of steering a unicycle [6]. The controls are the driving speed and the steering speed. The equations are

$$\begin{cases} \dot{x}_1 = \cos x_3 u_1 \\ \dot{x}_2 = \sin x_3 u_1 \\ \dot{x}_3 = u_2 \end{cases}$$

Therefore, the control fields are

$$\begin{aligned} g_1 &= \cos(x_3) \frac{\partial}{\partial x_1} + \sin(x_3) \frac{\partial}{\partial x_2} \\ g_2 &= \frac{\partial}{\partial x_3} \end{aligned}$$

So, for the system to be linearizable by prolongations, the distributions that need to be involutive are:

$$\begin{aligned} \Delta_1 &= \langle g_1, [g_2, g_1] \rangle = \\ &= \langle \cos(x_3) \frac{\partial}{\partial x_1} + \sin(x_3) \frac{\partial}{\partial x_2}, \\ &- \sin(x_3) \frac{\partial}{\partial x_1} + \cos(x_3) \frac{\partial}{\partial x_2} \rangle \end{aligned}$$

$$\Delta_2 = \langle g_1, [g_2, g_1], g_2 \rangle = R^3$$

Δ_2 is trivially involutive. Δ_1 is also involutive since the Lie bracket between its two vector fields vanishes. Therefore, this system is linearizable by prolongations.

2. The second example consists of the following family of systems [4]:

$$\begin{cases} \dot{x} &= u \\ \dot{y} &= f(x)u \end{cases} \quad (10)$$

where x and u belong to R^2 , $y \in R^k$, and $f(x) = (f_1(x), \dots, f_k(x))$. The control vector fields are

$$g_1 = \frac{\partial}{\partial x_1} + \sum_{i=1}^k f_i^1(x) \frac{\partial}{\partial y_i} \quad (11)$$

$$g_2 = \frac{\partial}{\partial x_2} + \sum_{i=1}^k f_i^2(x) \frac{\partial}{\partial y_i} \quad (12)$$

The distributions that must be considered are:

$$\begin{aligned} \Delta_1 &= \langle g_1, [g_2, g_1] \rangle \\ \Delta_2 &= \langle g_1, [g_2, g_1], ad_{g_2}^2 g_1 \rangle \\ \Delta_l &= \langle g_1, [g_2, g_1], ad_{g_2}^2 g_1, \dots, ad_{g_2}^l g_1 \rangle \end{aligned}$$

Since

$$[g_2, g_1] = \sum_{i=1}^k \left(\frac{\partial f_i^1(x)}{\partial x_2} - \frac{\partial f_i^2(x)}{\partial x_1} \right) \frac{\partial}{\partial y_i}$$

the involutivity of Δ_1 means that

$$\frac{\partial}{\partial x_1} ([g_2, g_1]) \in \Delta_1$$

or, in other words, is equivalent to

$$\frac{\partial}{\partial x_1} ([g_2, g_1]) = h(x)[g_2, g_1] \quad (13)$$

for a certain function h .

$$\Delta_2 = \langle g_1, [g_2, g_1], ad_{g_2}^2 g_1 \rangle$$

Remember that there is only one involutivity condition to check (see theorem 1):

$$[[g_2, g_1], ad_{g_2}^2 g_1] \in \Delta_2$$

But this Lie bracket is zero because

$$ad_{g_2}^2 g_1 = \frac{\partial}{\partial x_2} ([g_2, g_1])$$

And, in general, from

$$[ad_{g_2}^{l-1}, ad_{g_2}^l] = 0 \in D_l$$

and Theorem 1,

$$\Delta_l = \langle g_1, [g_2, g_1], ad_{g_2}^2 g_1, \dots, ad_{g_2}^l g_1 \rangle$$

is involutive if Δ_1 is so.

Second condition from Theorem 1 can be written as

$$dim \langle ad_{g_2} g_1, \dots, ad_{g_2}^k g_1 \rangle = k \quad (14)$$

Summarizing, the system (10) is linearizable by prolongations if, and only if (13) and (14) or the respective conditions obtained by permutation of subindices hold. Particular cases of this family are the hopping robot and the planar robot [7].

3. A one-chain system [7] has the form

$$\begin{cases} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_2 u_1 \\ &\vdots \\ \dot{x}_n &= x_{n-1} u_1 \end{cases}$$

So, the control fields are

$$\begin{aligned} g_1 &= \frac{\partial}{\partial x_1} + \sum_{i=2}^{n-1} x_i \frac{\partial}{\partial x_{i+1}} \\ g_2 &= \frac{\partial}{\partial x_2} \end{aligned}$$

It is easy to see by induction that

$$ad_{g_1}^k g_2 = (-1)^k \frac{\partial}{\partial x_{k+2}}$$

Hence, $\forall k = 0, \dots, n-2$,

$$\Delta_k = \langle \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \dots, \frac{\partial}{\partial x_{k+2}} \rangle$$

is involutive, and

$$\Delta_{n-1} = R^n$$

So, one concludes that a one-chain system is always linearizable by prolongations.

5 Conclusions

This paper presented results of dynamic feedback linearization by prolongations applied to driftless dynamic systems. It was shown that the necessary and sufficient conditions for linearization of driftless systems is computationally quite easy to verify. These conditions were illustrated using examples of some simple, yet meaningful engineering systems, such as a unicycle, a planar robot, and a hopping robot.

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