# A Predictive Approach to Discrete-time Time-Optimal Servomechanism

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**Abstract**

This paper presents a novel approach to discrete-time time-optimal servomechanism. Differently from existing approaches to discrete-time time-optimal control, we attempt to find the control law minimizing point-wise the distance between the one-step-ahead prediction of the system state and a given reference velocity trajectory. The performance of the proposed controller is shown to be mathematically well supported and, moreover, its practical significance is demonstrated through realistic simulation based on the head-positioning servo in a hard disk drive.

## 1 Introduction

Fast point-to-point positioning is ubiquitous in industrial servo applications. It is usually understood theoretically as the time-optimal control with some constraint on control input. As widely known, the solutions to the time-optimal control problems usually lead to the so-called "bang-bang" controller. Because of its infinite-gain nature, however, the ideal realization of the bang-bang controller is practically impossible. For this reason, a variety of near time-optimal controllers, employing mainly the finite-gain approximations, have been developed and popularly used in industrial practice [1, 2]. While finite-gain control have advantages such as increased robustness to model uncertainties, they usually fail to make the system state follow exactly the optimal velocity trajectories given in phase plane, leading to some oscillation near target position. In the well-known proximate time-optimal servomechanism (PTOS) [1], the acceleration discount factor was introduced to give some conservatism to the reference velocity trajectories such that it is followed with some margin the finite-gain controller.

Recently, the positioning servo controllers are implemented digitally using microprocessors. For example, the high-end disk storage devices such as hard disk drives (HDD) or optical disk drives requires sophisticated servo algotirhms for fast positioning of the data head. When implemented digitally, however, the original time-optimal controller inevitably introduces chattering or limit cycling due to finite sampling rate, even in the absence of any uncertainty. Moreover, since the effect of finite sampling rate is not fully considered in the existing near time-optimal approaches, their performance tends to depend on initial conditions, or equivalently seek distances. One widely-used method to overcome this limitation is to divide the whole operating region into several regions at each of which the design parameters, e.g. control gains and/or velocity-trajectory parameters, are calibrated to get acceptable performance. Nonetheless, this is usually time-consuming and disgraces the definiteness of the original time-optimal solution.

In this context, this paper presents a novel approach to discrete-time time-optimal servomechanism. Differently from other approaches, we formulate the problem as that of discrete-time tracking of a given phase-plane trajectory. Exploiting the inherently discrete-time nature of this formulation, we then attempt to derive a control law that minimizes point-wise the distance between the reachable state and the reference velocity profile. The main advantages of the present work is as follows: Firstly, the proposed controller makes the best of plant information to extend the original time-optimal performance to discrete time, while removing naturally the undesirable features, e.g. chattering, inherent in direct discretization of the original time-optimal control. Secondly, the proposed method attains uniform performance regardless of initial conditions since it utilizes efficiently the one-step-ahead information of the system state. This feature can make simple and rapid otherwise time-consuming design procedure of the time-optimal servomechanism.

## 2 Problem Statement

Consider a system whose dynamics are given by

$$\dot{y}(t) = v(t) \tag{1a}$$

$$\dot{v}(t) = au(t) \tag{1b}$$

where $y(t)$ is the position, $v(t)$ is the velocity, $u(t)$ is the control input (usually current), and $a > 0$ is the coefficient of acceleration. We are interested in steering $y$ from a given position to another, say $y_r$, in a time-optimal fashion. As well-known from the literature, this can be accomplished by the following procedure; The system is accelerated with the maximum input available until the state $(y, v)$ is sufficiently close to the optimal deceleration profile. Then, the system is controlled (via feedback) to follow closely the deceleration profile leading to the target state $(y_r, 0)$.

We consider the discrete version of the problem mentioned above. Here, the bound of the input magnitude is assumed to be a function of the current velocity $v(t)$. By doing so, we are able to deal with a near time-optimal control in a slightly more general system than (1) as explained in the end of this section. Let us be more precise. Let $T_s > 0$ be the sampling period and let $s_k$, $k \in \mathcal{Z}_+ \triangleq \{0, 1, 2, \ldots\}$, denote the value of a continuous-time signal $s$ at a $k$th sampling instant, i.e., $s_k = s(kT_s)$. Assuming that the control $u$ is applied in the zero-order-hold form, the system (1a) can then be represented by the following difference equations:

$$x_{k+1} = \Phi x_k + \Gamma u_k \tag{2a}$$

$$y_k = H x_k \tag{2b}$$

where $x_k \triangleq [y_k \; v_k]^T$,

$$\Phi \triangleq \begin{bmatrix} 1 & T_s \\ 0 & 1 \end{bmatrix}, \quad \Gamma \triangleq \begin{bmatrix} aT_s^2/2 \\ aT_s \end{bmatrix}.$$

The control input $u_k$ in (2) is subject to the magnitude constraint represented by

$$u_{min}(v_k) \le u_k \le u_{max}(v_k), \quad \forall k \in \mathcal{Z}_+ \tag{3}$$

where $u_{min}(v) \triangleq -\bar{I} - K_e v$ and $u_{max}(v) \triangleq \bar{I} - K_e v$ for some constants $\bar{I} > 0$ and $K_e \ge 0$. Assume without loss of generality that $y_r = 0$. Then, the (deceleration) velocity profile for (2) is assumed to be given and represented by a function $v = f(-y)$ where function $f : \Re \to (-\bar{I}/K_e, \bar{I}/K_e)$ is a continuous function satisfying

(P.1) $f(0) = 0$ and $f(e)e > 0$, $\forall e \neq 0$
(P.2) $f$ is continuous, monotonically increasing, and piece-wise $C^1$.

Then, the problem considered in this paper is to find a control law that minimizes the deviation of the state $x_k$ from the velocity profile subject to the control input constraint given in (3).

Note finally that the velocity dependent magnitude constraint given in (3) makes our formulation more versatile. For $K_e = 0$, our problem reduces to the time-optimal control problem for the double integrator plant with constant bound on input magnitude. On the other hand, consider a dynamic model for the dc motor including the current dynamics:

$$J\ddot{\theta} + B\dot{\theta} = K_t i \tag{4a}$$

$$L\dot{i} + Ri + K_t\dot{\theta} = e. \tag{4b}$$

Here the actual control input is the applied voltage $e$, which is limited to power supply voltage, i.e., $|e(t)| \le V_{max}$. Then, it can be shown through some singular-perturbation argument that the time-optimal control problem for this third-order system reduces to our problem with $(y, v) = (\theta, \dot{\theta})$, $a \triangleq K_t/J$, $\bar{I} \triangleq V_{max}/R$, and $K_e \triangleq (B + K_t^2/R)/J \neq 0$.

## 3 Controller Design

As addressed in the previous section, our approach focuses on the tracking of a prescribed velocity profile, that is, having $(y_k, v_k)$ as near as possible the phase-plane curve $v = f(-y)$. In this regards we define the deviation of $v$ from the velocity profile by, for $x = (y, v)$,

$$\epsilon(x, u) \triangleq f(-y^+(x, u)) - v^+(x, u) \tag{5}$$

and define a cost function by

$$J(x, u) \triangleq |\epsilon(x, u)| \tag{6}$$

where $x^+(x, u) \triangleq [y^+(x, u) \; v^+(x, u)]^T \triangleq \Phi x + \Gamma u$ . Then the control $u$ is chosen so as to minimize pointwise the cost function in (6), that is

$$u_k = \operatorname*{Arg\,min}_{u^*} \{ J(x_k, u^*) \mid u_{min}(v_k) \le u^* \le u_{max}(v_k) \} \tag{7}$$

It is obvious from (P.2), (2), and (5) that, for each fixed $x \in \Re^2$,

$$\epsilon(x, \cdot) \text{ decreasing}, \qquad \lim_{u \to \pm\infty} \epsilon(x, u) = \mp\infty. \tag{8}$$

Therefore, the optimization problem in (7) over the compact interval $[u_{min}(v_k), u_{max}(v_k)]$ has a unique solution for each $k \in \mathcal{Z}_+$ and hence the control law (7) is well-defined. In fact, the solution to (7) can be further expressed as follows: Let $u^*$ is the unique solution of

$$\epsilon(x_k, u^*) = 0. \tag{9a}$$

Then the control law is given by

$$u_k = \sigma(u^*, v_k) \tag{9b}$$

where

$$\sigma(u^*, v) \triangleq \begin{cases} -\bar{I} - K_e v, & \text{if } u^* < -\bar{I} - K_e v \\ \bar{I} - K_e v, & \text{if } u^* > \bar{I} - K_e v \\ u^*, & \text{otherwise.} \end{cases} \tag{9c}$$

The physical meaning of the control law in (7) is clear; At each $k$, $u_k$ is chosen, among the admissible input values, as the one that can steer the state to the closest point to the curve $v = f(-y)$. In fact, this interpretation is only valid when $f$ is decreasing as assumed in (P.2). On the other hand, it is interesting to see how our controller behaves as $T_s \to 0$. Let $\delta y$, $\delta v$, $\delta t$ be the infinitesimal variations of $y$, $v$, and $t$, respectively. Note from (1) that $\delta y = \delta v$ and $\delta v = (au)\delta t$. Since $f(-y-\delta y) = f(-y) - f'(-y)\delta y$, it then follows that the solution of $\epsilon(y+\delta y, v+\delta v, u^*) = 0$ becomes $u^* = \left[f(-y) - v - f'(-y)v\delta t\right]/(a\delta t)$. Thus, in view of the constraint (9c), letting $T_s \to 0$, i.e., $\delta t = 0$ leads to

$$u(t) = \begin{cases} -\bar{I} - K_e v(t) & \text{if } f(-y(t)) < v(t), \\ \bar{I} - K_e v(t) & \text{if } f(-y(t)) > v(t), \end{cases}$$

that is, the bang-bang controller. Furthermore, the "equivalent control" $u(t)$ when $(y(t), v(t))$ slides on the curve $v = f(-y)$ are directly computed to be $u(t) = -f'(-y(t))v(t)/a$. In this context, the proposed controller (7) may be thought of as the "correct" discrete-time version of the time-optimal controller.

Unfortunately the closed-form expression for $u_k$ in (7) cannot be obtained since $f$ in (9a) is nonlinear. Nonetheless, the approximate solution to (9a) can be derived under a very reasonable assumption, as described in what follows.

Let $k$ be fixed and define

$$\check{x}_k = \begin{bmatrix} \check{y}_k \\ \check{v}_k \end{bmatrix} \triangleq \Phi x_k + \Gamma u_{k-1}. \tag{10}$$

Note that $\check{x}_k$ can be interpreted as a one-step-ahead prediction of $x_k$ when $u_k = u_{k-1}$. Let $u^*$ be the solution of (9a) and denote $\Delta u \triangleq u^* - u_{k-1}$. Then it is obvious that $x^+(x_k, u^*) = \check{x}_k + \Gamma\Delta u$. Under the quite reasonable assumption that $|aT_s^2\Delta u/2|^2 \ll 1$ and $f$ is piece-wise smooth, we have the following approximation:

$$\begin{aligned} f\left(-y^+(x_k, u^*)\right) &= f\left(-\check{y}_k - \frac{aT_s^2}{2}\Delta u\right) \\ &= f(-\check{y}_k) + f'(-\check{y}_k)\frac{aT_s^2\Delta u}{2} \\ &\quad + o(\frac{aT_s^2\Delta u}{2}) \\ &\approx f(-\check{y}_k) - f'(-\check{y}_k)\frac{aT_s^2\Delta u}{2} \end{aligned} \tag{11}$$

Note here that (P.2) assures that $f'$ exists almost everywhere on $\Re$ [3]. Now, solving $\epsilon(x_k, u^*) = 0$ for $u^*$ based on the above approximation yields

$$u_k = \sigma(u^*, v_k) \tag{12a}$$

where

$$u^* \triangleq \frac{1}{aT_s\left(1 + T_s f'(-\check{y}_k)/2\right)} \times \left[f(-\check{y}_k) - v_k + \frac{aT_s^2 f'(-\check{y}_k)}{2} u_{k-1}\right] \tag{12b}$$

In fact we can further simplify the controller by neglecting the second term in the right-hand side (RHS) of (11):

$$u_k = \sigma(u^*, v_k) \tag{13a}$$

where $u^*$ is as simple as

$$u^* \triangleq \left[f(-\check{y}_k) - v_k\right]/(aT_s) \tag{13b}$$

In a lot of practical systems including disk drives, the current state estimator [1] is popularly used to reduce the computation delay. Since the current estimator employs the state prediction part, we can obtain $\check{y}_k$ required for the implementation of the controllers (12) and (13) quite naturally. Meanwhile, the slight increase in computation delay is inevitable.

## 4 Performance Analysis

The closed-loop performance of the proposed controller (7) is analyzed in this section. To do this, some sets need to be defined. We denote $x = (y, v)$.

$$\Sigma = \left\{x \in \Re^2 \;\middle|\; v \in (-\bar{I}/K_e, \bar{I}/K_e)\right\} \tag{14a}$$

$$\mathcal{G} \triangleq \left\{x \in \Re^2 \mid v = f(-y)\right\} \tag{14b}$$

$$U \triangleq \left\{x \in \Sigma \mid \exists u \in [u_{min}(v), u_{max}(v)] : \epsilon(x, u^*) = 0\right\} \tag{14c}$$

$$\Sigma^- \triangleq \left\{x \in \Sigma \mid \epsilon(x, u^*) = 0 \Rightarrow u^* < u_{min}(v)\right\} \tag{14d}$$

$$\Sigma^+ \triangleq \left\{x \in \Sigma \mid \epsilon(x, u^*) = 0 \Rightarrow u^* > u_{max}(v)\right\}. \tag{14e}$$

The set $\Sigma$ will be considered as the state space for the closed-loop system and $\mathcal{G} \subset \Sigma$ is the graph of the function $y \mapsto f(-y)$. On the other hand, a geometric interpretation of the sets $U$, $\Sigma^+$, and $\Sigma^-$ is not straightforward and will be explored in the upcoming analysis.

On the other hand, let $v_r(y) \triangleq f(-y)$. Then, in view of (P.2), the function $g \triangleq v_r^{-1} : (-\bar{I}/K_e, \bar{I}/K_e) \to \Re$ is always well defined. Furthermore, it will turn out that our analysis can be presented more efficiently in terms of this function $g$ rather than of $f$. We are now ready to state the following theorem.

**Theorem 1** *Suppose that $T_s$ is chosen so as to satisfy*

$$aT_sK_e < 1. \tag{15}$$

*Suppose further that, in addition to (P.1) and (P.2), the function $f$ is chosen such that the inverse function $g$ of the function $y \mapsto f(-y)$ satisfies, for all $v_1, v_2, v \in (-\bar{I}/K_e, \bar{I}/K_e)$,*

$$(v_1 - v_2)\left(g(v_1) - g(v_2)\right) \le -\frac{T_s}{2}\left(v_1^2 - v_2^2\right) \tag{16}$$

$$\begin{aligned} g(\beta v + \alpha) - \frac{T_s}{2}\left((\beta+1)v + \alpha\right) &\le g(v) \\ &\le g(\beta v - \alpha) - \frac{T_s}{2}\left((\beta+1)v - \alpha\right) \end{aligned} \tag{17}$$

*where $\alpha \triangleq aT_s\bar{I}$ and $\beta \triangleq 1 - aT_sK_e$. Then, the controller (7) assures that for any $x_0 = (y_0, v_0) \in \Sigma$, the following properties hold:*

*i)* $x_k \in \Sigma$ *for all* $k \in \mathcal{Z}_+$
*ii)* $\exists k_0 \in \mathcal{Z}_+$ *such that* $x_k \in \mathcal{G},\ \forall k \geq k_0$
*iii)* $x_k \to (0,0)$ *as* $k \to \infty$.

**Proof:** Note first that the condition (15) implies $0 < \beta \leq 1$ where the equality holds for $K_e = 0$. Now we start the main proof. The property (i) follows easily from the fact that $\Sigma$ is a positively invariant set for the closed-loop system given by (2) and (7). To show this fact, let $x_k = (y_k, v_k) \in \Sigma$. Then, we have the following inclusion, which is obvious from the construction of the control law in (3) and the condition in (15). $v_{k+1} = v_k + aT_su_k \in [\beta v_k - aT_s\bar{I}, \beta v_k + \alpha] \subset (-\bar{I}/K_e, \bar{I}/K_e)$. Hence, the property (i) is proved.

To show the property (ii), we first prove the following claims:

*Claim 1:* $\mathcal{G} \subset U$
*Claim 2: For any* $x_0 \in \Sigma$, $\exists l \in \mathcal{Z}_+$ *such that* $x_l \in U$.

If the above claims are true, then it follows easily from the definition of the controller (7) that (ii) holds with $k_0 = l + 1$. The proofs of the claims are based on the following observation: Let

$$\bar{\epsilon}(x, \Delta v) \triangleq g(v + \Delta v) - y - T_sv - \frac{Ts}{2}\Delta v. \tag{18}$$

Then, by (P.2) and the definition of $g$, the function $\bar{\epsilon}(x, \cdot)$ is monotonically decreasing and

$$\epsilon(x, u^*) = 0 \quad \Longleftrightarrow \quad \bar{\epsilon}(x, aT_su^*) = 0$$

for each fixed $x$. Therefore, we can replace $\epsilon(x, u)$ in (6) with $\bar{\epsilon}(x, aT_su)$. Furthermore, the sets $U$, $\Sigma^+$, and $\Sigma^-$ can be characterized in terms of $\bar{\epsilon}$ as follows:

$$x \in \Sigma^+ \Longleftrightarrow 0 < \bar{\epsilon}(x, \Delta v_{max}) < \bar{\epsilon}(x, \Delta v_{min}) \tag{19a}$$
$$x \in \Sigma^- \Longleftrightarrow \bar{\epsilon}(x, \Delta v_{max}) < \bar{\epsilon}(x, \Delta v_{min}) < 0 \tag{19b}$$
$$x \in U \Longleftrightarrow x \notin \Sigma^+ \text{ and } x \notin \Sigma^- \tag{19c}$$

where $\Delta v_{max}(v) \triangleq aT_su_{max}(v)$ and $\Delta v_{min}(v) \triangleq aT_su_{min}(v)$. Finally, the following identity will be helpful:

$$\bar{\epsilon}(x, \Delta v) = g(\beta v + \hat{\alpha}) - y - \frac{T_s}{2}\left((\beta + 1)v + \hat{\alpha}\right) \tag{20}$$

where $\hat{\alpha} = \alpha$ for $\Delta v = \Delta v_{max}$ and $\hat{\alpha} = -\alpha$ for $\Delta v = \Delta v_{min}$.

*Proof of Claim 1:* By (19) and (20), the claim follows directly from the condition (17). ■

*Proof of Claim 2:* Observing that the state space $\Sigma$ is partitioned into three disjoint subsets as follows:

$$\Sigma = \Sigma^+ \cup U \cup \Sigma^-.$$

Hence, Claim 2 is proved if we can show that (a1) any solution trajectory can not be trapped in either of the sets $\Sigma^+$, $\Sigma^-$ and that (a2) the state cannot jump from $\Sigma^+$ ($\Sigma^-$) to $\Sigma^-$ ($\Sigma^+$).

Suppose that $x_k \in \Sigma^+(\Sigma^-)$ for all $k \in \mathcal{Z}_+$. Solving the difference equations (2) in this case is straightforward and leads to

$$\begin{aligned} v_k &= \beta^k v_0 + \bar{\alpha}\rho_k \\ y_{k+1} &= y_k + \frac{T_s}{2}(1+\beta)v_k + \frac{T_s}{2}\bar{\alpha} \end{aligned}$$

where $\bar{\alpha} = \alpha$ ($= -\alpha$ for $x \in \Sigma^-$) and $\rho_k \triangleq \frac{1-\beta^k}{1-\beta}$ if $0 < \beta < 1$ and $\rho_k \triangleq k$ if $\beta = 1$. By (18), this implies that, as $k \to \infty$, $\bar{\epsilon}(x_k, \Delta v_{min}) \to -\infty$ ($\bar{\epsilon}(x_k, \Delta v_{max}) \to +\infty$). In view of (19), this contradicts the supposition above and hence the assertion (a1) is true.

To prove (a2), let $x_k \in \Sigma^+$ ($\Sigma^-$) for some $k$ and denote $\bar{\epsilon}^k_{min} = \bar{\epsilon}(x_k, \Delta v_{min}(v_k))$ and $\bar{\epsilon}^k_{max} = \bar{\epsilon}(x_k, \Delta v_{max}(v_k))$. Then, by (7) and (19), there holds

$$\bar{\epsilon}^k_{max} = g(v_{k+1}) - y_{k+1} > 0 \ \ (\bar{\epsilon}^k_{min} = g(v_{k+1}) - y_{k+1} < 0).$$

In view of the identity (20), this implies that

$$\bar{\epsilon}^{k+1}_{min} > g(\beta v_{k+1} + \alpha) - g(v_{k+1}) - \frac{T_s}{2}\left[(\beta+1)v_{k+1} + \alpha\right] > 0$$
$$\left(\bar{\epsilon}^{k+1}_{max} < g(\beta v_{k+1} - \alpha) - g(v_{k+1}) - \frac{T_s}{2}\left[(\beta+1)v_{k+1} - \alpha\right] < 0\right).$$

By (19), we have $x_{k+1} \notin \Sigma^-$ ($x_{k+1} \notin \Sigma^+$) and therefore (a2) is true. This completes the proof of Claim 2 and, in turn, the property (ii).

Finally, we prove the property (iii). Let $k_0$ be such that the property (ii) holds. Then, the following holds for all $k \geq k_0$:

$$\begin{aligned} g(v_{k+1}) &= g(v_k + aT_su_k) \\ &= g(v_k) + \frac{T_s}{2}v_k + \frac{T_s}{2}\left(v_k + aT_su_k\right). \end{aligned} \tag{21}$$

Furthermore, the sign of $v_k$ is not altered by the control $u_k$; To see this, multiply (21) by $v_k$ and use the condition in (16) to obtain

$$v_kg(v_{k+1}) < \frac{T_s}{2}v_kv_{k+1}.$$

This implies that if $v_k \geq 0$ ($v_k \leq 0$) then $v_{k+1} \geq 0$ ($v_{k+1} \leq 0$, resp.).

On the other hand, again using the condition in (16), we have

$$aT_su_k\left[g(v_{k+1}) - g(v_k)\right] \leq -\frac{T_s}{2}|aT_su_k|^2.$$

Substituting (21) into the above inequality yields $aT_su_kv_k \leq -2|aT_su_k|^2$, from which it follows

$$|v_{k+1}|^2 - |v_k|^2 \leq |aT_su_k|^2 \leq 0.$$

By using the fact that the numbers $v_k$, $k \geq k_0$, have the same sign, it can then be easily shown that $v_k \to v_0$ for some $v_0$ and further that $u_k \to 0$ as $k \to \infty$. Finally, from the continuity of $g$ and the equality (21), it follows that $(y_k, v_k) \to (0, 0)$ as $k \to \infty$. This completes the proof. ■

The conditions in (16) and (17) are quite easy to check once $f$ is given. For (16) to hold, it is sufficient that $g'(v) \leq -T_s/2$ (or $f'(e) \geq 2/T_s$) for almost all $v$. Even though (17) is not as straightforward to verify as (16), it is still readily verifiable with the help of the simplest graphic software. In fact, it can be shown that the well-known square-root-type optimal trajectory for a double integrator plant [4] always satisfies (17). Because of the finite slope condition (16), however, it is not directly applicable to the proposed approach. It is worth noting that as $T_s \to 0$, (16) and (17) reduces to none other than the conditions (P.1) and (P.2).

On the other hand, in contrast with most of the existing near time-optimal control algorithms, the proposed one does not have free (design) parameter. That is, the controller parameters are completely determined by the plant parameters and the function $f$. This feature of the proposed controller is of practical importance for the following reason: In actual implementation of the existing algorithms, the design parameters usually have to be calibrated with respect to seek distances (or equivalently, the initial conditions) because of the performance variation (especially in settling near the target position). By contrast, in the proposed approach, the control parameters assuring uniform performance regardless of seek distances are systematically determined, whenever the plant parameters are known fairly accurately.

## 5 Application to Head Positioning Servo in Hard Disk Drives

In this section, we demonstrate the practical significance of the proposed control scheme through application to head-positioning servomechanism in HDD's. Because of the limited space, we omit the description on the disk-drive servo system. Interested readers are referred to [1].

The schematic diagram of the simulation setup is show in Fig. 1. For realistic simulation, the real parameters of a HDD (Model No. WINNER-II, Samsung Electronics, Co. Ltd) with the track density of 6000 tracks per inch (TPI) was used in simulation. This model employs the 86 sectors-per-track disk with rotational frequency 4500 [rev/min], which limits the sampling interval to $T_s = 60/4500/86 = 155.04$ [$\mu$sec]. The mechanical parameters of the voice coil motor (VCM) actuating the

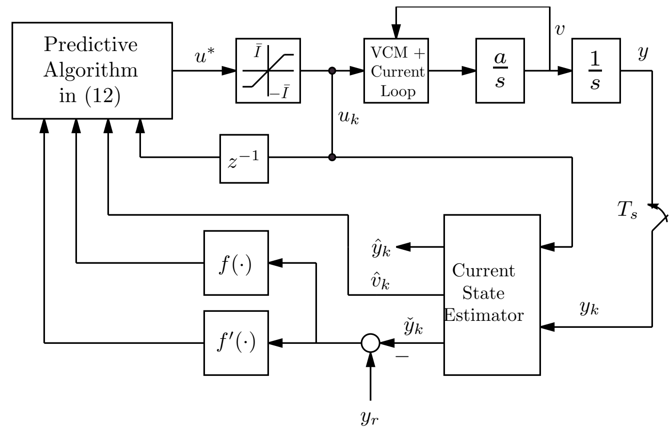



**Figure 1:** Schematic diagram of the simulation setup

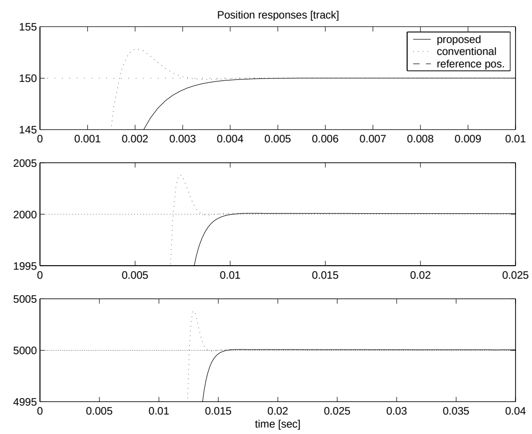



**Figure 2:** The position responses for three different track commands ($y_r = 150, 2000, 5000$)

data head give $a = 4.4928 \times 10^4$ [inch/sec$^2$/A]. The electrical parameters are given by $L = 1.080$ [mH] and $R = 14.00$ [Ω]. A high-gain proportional-integral controller is used to achieve fast current-loop dynamics. As a result, the VCM current can be regarded as our control input $u$. It is assumed that $\bar{I} = 0.635$ [A].

The reference velocity profile was chosen to be the well-known square root profile with extended linear region near the origin [1], that is, $f(e) \triangleq (k_1/k_2)e$ if $|e| \leq e_l$ and $f(e) \triangleq \mathrm{sgn}(e)\left(2a\bar{I}|e|\right)^{1/2} - v_l$, otherwise. With the profile parameters $k_1 = 143.16$ [A/inch], $k_2 = 0.08$ [A·sec/inch], $e_l = 27$ [track], and $v_l = 7.98$ [inch/sec], it is easy to verify that all the conditions given in Theorem 1 are met. Note here that the acceleration discount factor $\alpha$ in [5] is chosen to be unity, which amounts to the most aggressive case. The derivative of $f$ used in the proposed algorithm is easily obtained analytically from the definition of $f$. This is also the case in actual implementation where the polynomial-fitted version of $f$ is used instead. As shown in Fig. 1, the approximate version (12) of the proposed controller is used in simulation.

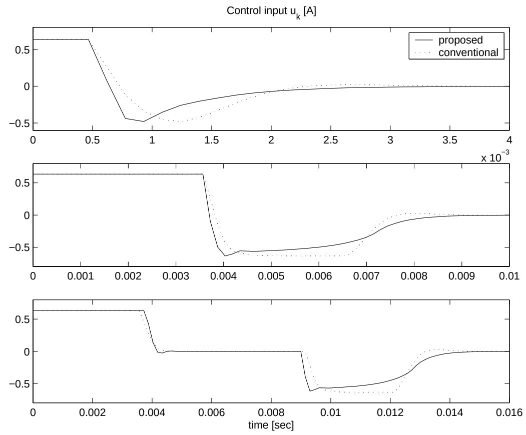
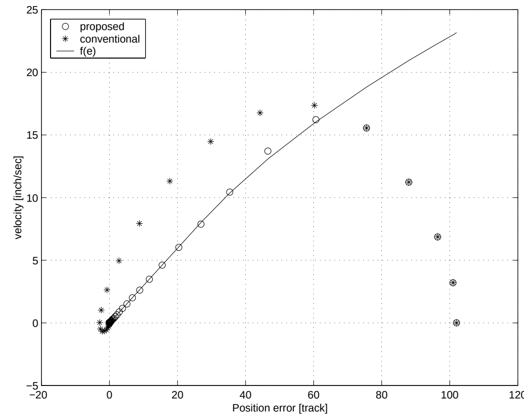
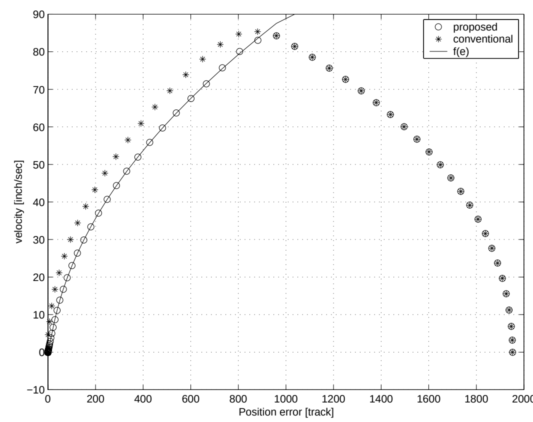



**Figure 3:** Comparison of the control inputs for each track commands




(a) Phase-plane trajectories for $y_r = 150$ (track)




(b) Phase-plane trajectories for $y_r = 2000$ (track)

**Figure 4:** Comparison of reference trajectory-following performance in phase plane

Our simulation results are presented in Fig. 2–4. In Fig. 2, the position responses of the proposed controller for three different track commands ($y_r = 150, 2000, 5000$) are compared to those of a conventional near time-optimal controller [5] under exactly the same setting. It is shown that the proposed controller ensures the overshoot-free responses for all seek range. Time histories of the corresponding control input in Fig. 3 shows that our controller decelerates more rapidly and hence introduces almost no overshoot near the target track. This clearly results from the good tracking of the reference velocity trajectory in phase plane as shown in Fig. 4. The figure reveals that the conventional approach fails to track closely the velocity trajectory, while the proposed one does. Observe that the velocity hits the reference trajectory and then slides along it. This is in close agreement with our theoretical analysis given in Theorem 1, despite the presence of the state estimator and current dynamics that were not considered therein.

## 6 Concluding Remark

A novel approach to discrete-time time-optimal servomechanism is presented in this paper. In short, the time-optimal controller is derived as the one minimizing, at each sampling instance, the distance between the given optimal velocity trajectory and the state at the next sampling instance. Even though the controller is given in an implicit form, its approximate version admits a simple form that allows easy implementation. Simulation results supports very well that theoretical performance analysis for the original solution is still valid for its approximate version even with the discrete-time current state estimator.

## References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