Improvability of Feedback Systems[1]

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Abstract

This paper introduces the notion of improvability and bottlenecks of feedback systems in the context of instrumentation cost. Specifically, a feedback system is improvable if its performance can be enhanced by re-allocating sensor and actuator costs under budget constraints. We derive a criterion which determines when the system is improvable, using the LQG performance index. In addition, we introduce and analyze the notion of instrumentation bottleneck and provide a criterion for bottleneck identification. An important feature of the results derived is that both improvability and BN indicators can be evaluated using on-line measurements in the feedback loop, without requiring precise knowledge of the plant data. Examples illustrating results are provided.

1. MOTIVATION AND PROBLEM FORMULATION

Instrumentation in feedback systems, i.e., sensors and actuators, cost money. For large-volume consumer products, such as automobiles, household appliances, video equipment, etc., these costs are often limited by economic considerations. This can be conceptualized by a constraint

$$c_A + c_S = c_I^*, \tag{1.1}$$

where c_A and c_S are the costs of the actuator and sensor, respectively, and c_I^* is a fixed budget for the instrumentation. In this situation, two questions are of interest:

1. Is it possible to re-distribute c_I^* so that system performance is improved?
2. If re-distribution is impossible, in which device - sensor or actuator - should additional funds be invested so that system performance is improved in the best possible manner?

These are the questions addressed in this paper. For the sake of simplicity, we limit the consideration to linear SISO feedback systems described by: plant equations

$$\begin{aligned} \dot{x} &= Ax + Bu + D_1 w_1, \\ y &= Cx + D_2 w_2, \\ z &= E_1 x + E_2 u; \end{aligned} \tag{1.2}$$

performance index

$$J = \lim_{t\to\infty} E[z(t)^T z(t)]; \tag{1.3}$$

LQG controller equations

$$\begin{aligned} \dot{x}_c &= A_c x_c + B_c y, \\ u &= C_c x_c, \end{aligned} \tag{1.4}$$

where $x \in R^n$ and $x_c \in R^n$ are state vectors of the plant and the controller, respectively, $u \in R$ is the control, $y \in R$ and $z \in R^{n+1}$ are the measured and controlled outputs, respectively, w_1 and w_2 are uncorrelated standard white noise processes, and A, B, C, D_1, D_2, E_1, E_2, A_c, B_c, and C_c are matrices of appropriate dimensionality.

The effect of costs on the efficacy of the actuator and sensor is modeled by

$$B = B(c_A),\ C = C(c_S), \tag{1.5}$$

where $||B(\cdot)||$ and $||C(\cdot)||$ are monotonically increasing functions with $||B(0)|| = ||C(0)|| = 0$. Obviously, this implies that larger c_A and c_S result in larger signal-to-noise ratio for both control signal and measured output. Again, for the sake of simplicity, functions $B(\cdot)$ and $C(\cdot)$ are assumed to be linear with respect to the device cost:

$$B = c_A B_0,\ C = c_S C_0, \tag{1.6}$$

where B_0 and C_0 are given matrices. Physically, B_0 and C_0 represent the efficacy of actuators and sensors per unit cost. We now formally define the notion of improvability.

Definition 1.1 *The feedback system (1.2)-(1.5) is improvable under constraints if there exist c_A^* and c_S^* satisfying (1.1), such that the resulting LQG controller, designed with $B^* = B(c_A^*)$ and $C^* = C(c_S^*)$, results in*

$$J(c_A^*, c_S^*) < J(c_A, c_S). \tag{1.7}$$

Otherwise, the system is called unimprovable under constraints.

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The first problem addressed in this paper is:

Problem 1.1 *Derive a rule for determining whether a given system is improvable under constraints or not.*

We refer to this rule as Indicator of Improvability. Such an Indicator is derived in Section 2 below.

Assume now that re-distribution is impossible, but a reduction of $J(c_A, c_S)$ is required by performance specifications. Then, the only way to accomplish this is to increase c_I^*. Will it be more efficient to invest additional funds in the sensor or the actuator? To formalize this question, introduce

Definition 1.2 *The actuator is the bottleneck (BN) of the closed loop system (1.2)-(1.5) if*

$$\left|\frac{\partial J(c_A, c_S)}{\partial c_A}\right| > \left|\frac{\partial J(c_A, c_S)}{\partial c_S}\right|. \qquad (1.8)$$

Otherwise, the BN is the sensor.

This definition leads to posing the following problem.

Problem 1.2 *Derive a rule for identifying the bottleneck of a feedback system.*

We refer to this a rule as BN Indicator. Such an Indicator is derived in Section 3.

It should be pointed out that the effect of sensors and actuators on the LQG performance index was studied extensively in [1],[2] in the context of instrumentation positioning for control of structures. In the present work, we study similar questions but in the context of budget constraints.

The notion of improvability was introduced in [3] in the context of manufacturing production lines. Here we extend this ideology to feedback systems.

As it was mentioned above, section 2 and 3 below address the conditions of improvability and bottlenecks respectively. All proofs are given in the Appendix.

2. IMPROVABILITY UNDER CONSTRAINTS

2.1 Improvability indicator

Let matrices X and Y denote

$$\begin{aligned} X(c_A) &= B(c_A)(E_2^T E_2)^{-1} B(c_A)^T, && (2.9)\\ Y(c_S) &= C(c_S)^T (D_2 D_2^T)^{-1} C(c_S), && (2.10)\end{aligned}$$

Let matrices P and Q be the positive definite solutions of the following Riccati equations:

$$\begin{aligned} 0 &= P(c_A)A + A^T P(c_A) + E_1^T E_1 \\ &\quad -P(c_A)X(c_A)P(c_A), && (2.11)\\ 0 &= AQ(c_S) + Q(c_S)A^T + D_1 D_1^T \\ &\quad -Q(c_S)Y(c_S)Q(c_S). && (2.12)\end{aligned}$$

These solutions exist if (1.2) is controllable and observable for all $B(c_A)$ and $C(c_S)$ of (1.5). In the case of (1.6), controllability and observability of (A, B_0, C_0) along with those of (A, D_1, E_1), $E_2^T E_2 > 0$, and $D_2 D_2^T > 0$ imply the existence of such P and Q. Let, finally, $\Lambda_1(c_A, c_S)$ and $\Lambda_2(c_A, c_S)$ be the positive definite solutions of the following Lyapunov equations:

$$\begin{aligned} 0 &= (A - X(c_A)P(c_A))\Lambda_1 + \Lambda_1(A^T - P(c_A)X(c_A)) \\ &\quad +Q(c_S)Y(c_S)Q(c_S), && (2.13)\\ 0 &= (A^T - Y(c_S)Q(c_S))\Lambda_2 + \Lambda_2(A - Q(c_S)Y(c_S)) \\ &\quad +P(c_A)X(c_A)P(c_A). && (2.14)\end{aligned}$$

These matrices exist if (2.11) and (2.12) have positive definite solutions P and Q.

Theorem 2.1 *The feedback system (1.2)-(1.5) is unimprovable under constraints only if*

$$\begin{aligned} &\mathrm{Tr}\{\Lambda_1(c_A, c_S)P(c_A)\frac{\partial X(c_A)}{\partial c_A}P(c_A)\} \\ &= \mathrm{Tr}\{\Lambda_2(c_A, c_S)Q(c_S)\frac{\partial Y(c_S)}{\partial c_S}Q(c_S)\}. \quad (2.15)\end{aligned}$$

Proof: *See the Appendix.*

Remark 2.1 *Note that condition (2.15) is invariant under similarity transformation.*

Corollary 2.1 *The feedback system (1.2)-(1.4), (1.6) is unimprovable under constraints only if*

$$\begin{aligned} &\frac{\mathrm{Tr}\{\Lambda_1(c_A, c_S)P(c_A)X(c_A)P(c_A)\}}{c_A} \\ &= \frac{\mathrm{Tr}\{\Lambda_2(c_A, c_S)Q(c_S)Y(c_S)Q(c_S)\}}{c_S}. \quad (2.16)\end{aligned}$$

Moreover, the system (1.2)-(1.4), (1.6) is improvable by re-allocating funds from the sensor to the actuator if

$$\begin{aligned} &\frac{\mathrm{Tr}\{\Lambda_1(c_A, c_S)P(c_A)X(c_A)P(c_A)\}}{c_A} \\ &> \frac{\mathrm{Tr}\{\Lambda_2(c_A, c_S)Q(c_S)Y(c_S)Q(c_S)\}}{c_S}. \quad (2.17)\end{aligned}$$

If the inequality is reversed, improvement can be accomplished by re-allocating funds form the actuator to the sensor.

Proof: *See the Appendix.*

Although, Theorem 2.1 and Corollary 2.1 provide conditions of optimality (or "unimprovability"), their practical value may be limited since in reality the problem data, i.e., all matrices involved in (1.2)-(1.6) may not be precisely known. Therefore, determining if the system is improvable and in which direction changes must be made, without knowing the matrices, may be more important in practice. This can be accomplished using the following.

Theorem 2.2 *For the closed loop system (1.2)-(1.4), (1.6), the numerators of (2.16), have the following interpretations*

$$||T_{z\ w_2}||_2^2 = \mathrm{Tr}\{\Lambda_2(c_A, c_S)Q(c_S)Y(c_S)Q(c_S)\}, \quad (2.18)$$

$$||T_{z_2\ w}||_2^2 = \mathrm{Tr}\{\Lambda_1(c_A, c_S)P(c_A)X(c_A)P(c_A)\}, \quad (2.19)$$

where $z_1 = E_1 x$, $z_2 = E_2 u$, $z = z_1 + z_2$, $T_{z\ w_2}$ is the closed loop transfer function from w_2 to z, and $T_{z_2\ w}$ is the closed loop transfer function from w to z_2.
Proof: *See the Appendix.*

If the quantities $||T_{z\ w_2}||_2^2$, and $||T_{z_2\ w}||_2^2$ can be measured experimentally during the closed loop system operation, Corollary 2.1 would provide a possibility of determining if the system is improvable without the exact knowledge of the problem data. To make $||T_{z\ w_2}||_2^2$ and $||T_{z_2\ w}||_2^2$ measurable, we assume the followings:

(1) Signals $z(t)$ and $u(t)$ can be measured.

(2) Matrices D_2 and E_2 are known.

(3) Additional measurement noise, w_3, can be introduced in the system.

Assumption (1) and (2) imply that the followings may be evaluated during normal system operation.

$$J_0 = \lim_{t\to\infty} E[z(t)^T z(t)],$$

$$||T_{z_2\ w}||_2^2 = \lim_{t\to\infty} E[z_2(t)^T z_2(t)] = \lim_{t\to\infty} E[u(t)^T E_2^T E_2 u(t)]. \quad (2.20)$$

This allows the evaluation of the quantity defined in (2.19). Assumptions (1)-(3) imply that the performance index J can be evaluated for system (1.2)-(1.4) when y is replaced by $\hat{y} = y + D_2 w_3$. To evaluate the quantity defined in (18), we state the following.

Theorem 2.3 *Consider the feedback system (1.2)-(1.4) and (1.6). Suppose the output y is changed to $\hat{y} = y + D_2 w_3$, where w_3 is a standard white noise process, uncorrelated with w_1, and w_2. Suppose that, as a result, the performance index is changed from J_0 to J. Then,*

$$||T_{z\ w_2}||_2^2 = J - J_0. \quad (2.21)$$

Proof: *See the Appendix.*

Equations (2.20)and (2.21) imply that both $||T_{z\ w_2}||_2^2$ and $||T_{z_2\ w}||_2^2$ can be evaluated using experimental measurements.

Based on the above, Corollary 2.1 and Theorem 2.2 we have the following improvability indicator:

Improvability Indicator *A closed loop system is improvable by re-allocating funds from the sensor to the actuator if*

$$\frac{||T_{z_2\ w}||_2^2}{c_A} > \frac{||T_{z\ w_2}||_2^2}{c_S}. \quad (2.22)$$

If the inequality is reversed, the improvement is accomplished by re-allocating funds form the actuator to the sensor.

2.2 Examples

Example 2.1 Suppose the following system is given:

$$A = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix}, \ B_0 = \begin{bmatrix} 0 \\ 0.01 \end{bmatrix}, \ C_0 = [0.1\ 0],$$

$$D_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \ D_2 = 1, \ E_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad (2.23)$$

Assume that sensor and actuator cost effectiveness can be modeled as (1.6). It is known that the actuator, represented by B, costs \$100 (c_A=\$100), and the sensor, represented by C, costs \$10 (c_S=\$10). This implies that $B = [0\ 1]^T$ and $C = [1\ 0]$. The resulting closed loop system yields J=31992. Calculating the quantities involved in the Improvability Indicator, we obtain,

$$\frac{||T_{z_2\ w}||_2^2}{c_{A_0}} = 311.9, \ \frac{||T_{z\ w_2}||_2^2}{c_{S_0}} = 3180.3. \quad (2.24)$$

Therefore, this system is improvable by re-allocating instrumentation cost from the actuator to the sensor. Indeed, the optimal allocation of the instrumentation cost is c_A^*=\$58, c_S^*=\$52, and the resulting performance index is J^*=3926.7.

Example 2.2 In order to characterize the effect of various parameters of the plant on the optimal instrumentation cost allocation, consider the system defined by (1.2)-(1.4), (1.6) and

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ B_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ C_0 = [1\ 0],$$

$$D_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \ D_2 = 1, \ E_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad (2.25)$$

Assume the total instrumentation budget is \$110. Two cases of instrumentation cost allocations are analyzed along with the optimal allocation. Since the LQG performance index consists of two components- the state cost and the control cost- not only the performance index J, but also the state cost J_s, and the control cost J_c, are shown in Table 2.1. The optimal cost distribution is \$55 for sensor and actuator.

Table 2.1: Instrumentation Cost Allocation and Performance Indices

	c_A, c_S 10, 100	c_A^*, c_S^* 55, 55	c_A, c_S 100, 10
J	0.08336	0.02080	0.08336
J_s	0.03344	0.01300	0.07076
J_c	0.04991	0.00780	0.01260

Similar observations are made when $B_0 = [0\ \frac{1}{10}]^T$ and summarized in Table 2.2. Due to the smaller value of B_0, the actuator efficiency cannot be improved as much as in the case of $B_0 = [0\ 1]^T$ using the same amount of money. The optimal distribution is now \$77 for the actuator, and \$33 for the sensor. Compared with the previous case, each cost is increased.

The case of $C_0 = [\frac{1}{10}\ 0]$ is analyzed as well. Since improvement of the sensor becomes costlier, more funds must be invested in the sensor to achieve the optimal allocation of instrumentation cost (see Table 2.3).

Table 2.2: Case $B_0 = [0\ \frac{1}{10}]^T$

	c_A, c_S 10, 100	c_A^*, c_S^* 77, 33	c_A, c_S 100, 10
J	1.7267	0.1684	0.2683
J_s	0.5176	0.0791	0.1667
J_c	1.2091	0.0893	0.1006

Table 2.3: Case $C_0 = [\frac{1}{10}\ 0]$

	c_A, c_S 10, 100	c_A^*, c_S^* 33, 77	c_A, c_S 100, 10
J	0.2683	0.1684	1.7267
J_s	0.1667	0.1314	1.6408
J_c	0.1006	0.0370	0.0859

Next, we analyze optimal allocations when D_2 (sensor noise) or D_1 (disturbance) change. First, as D_1 is increased ten times to $[0\ 10]^T$, more money must go to the actuator (see Table 2.4). On the other hand, if D_2 is increased to 10, the optimal allocation of \$33 for actuator and \$77 for sensor is obtained, implying more money must go to the sensor (Table 2.5).

Table 2.4: Case $D_1 = [0\ 10]^T$

	c_A, c_S 10, 100	c_A^*, c_S^* 77, 33	c_A, c_S 100, 10
J	5.4604	0.5327	0.8485
J_s	1.6368	0.2502	0.5303
J_c	3.8236	0.2825	0.3181

Table 2.5: Case $D_2 = 10$

	c_A, c_S 10, 100	c_A^*, c_S^* 33, 77	c_A, c_S 100, 10
J	0.2683	0.1684	1.7267
J_s	0.1677	0.1314	1.6408
J_c	0.1006	0.0370	0.0859

Example 2.3 Next, the same plant as in Example 2.2 is investigated for different values of E_1. Previously, only the first state was penalized in J. Now, both states are penalized. Therefore, the system under consideration becomes

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ B_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ C_0 = [1\ 0],$$

$$D_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},\ D_2 = 1,\ E_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad (2.26)$$

For this system, the optimal distribution is changed to \$31 for the actuator (see Table 2.6).

When $B_0 = [0\ \frac{1}{10}]^T$, as expected, the optimal allocation decision favors the actuator as shown in Table 2.7. When $C_0 = [\frac{1}{10}\ 0]$, the optimal allocation favors the sensor (see Table 2.8).

Table 2.6: Case $E_2 = I_2$

	c_A, c_S 10, 100	c_A^*, c_S^* 31, 79	c_A, c_S 100, 10
J	0.3025	0.2307	0.7129
J_s	0.2281	0.2081	0.7024
J_c	0.0743	0.0226	0.0105

Table 2.7: Case $B_0 = [0\ \frac{1}{10}]^T$

	c_A, c_S 10, 100	c_A^*, c_S^* 64, 46	c_A, c_S 100, 10
J	2.1923	0.5070	0.9100
J_s	0.8904	0.3713	0.7968
J_c	1.3018	0.1357	0.1131

Changes in D_1 and D_2 bring about the same results as in Example 2. When $D_1 = [0\ 10]^T$, the optimal distribution in Table 2.9 is shifted toward the actuator.

Table 2.8: Case $C_0 = [\frac{1}{10}\ 0]$

	c_A, c_S 10, 100	c_A^*, c_S^* 19, 91	c_A, c_S 100, 10
J	0.9100	0.8560	4.8867
J_s	0.7968	0.7969	4.8575
J_c	0.1131	0.0591	0.0291

Table 2.9: Case $D_1 = [0\ 10]^T$

	c_A, c_S 10, 100	c_A^*, c_S^* 47, 63	c_A, c_S 100, 10
J	16.5445	8.3796	17.5962
J_s	10.1545	7.1480	16.9350
J_c	6.3900	1.2315	0.6612

When D_2 becomes 10, this yields the same results as in the case of $C_0 = [0.1\ 0]$.

3. UNCONSTRAINED IMPROVABILITY

3.1 Bottleneck indicator

The following theorem provides a foundation for identifying the instrumentation bottleneck:

Theorem 3.1 *Given system (1.2)-(1.5), the sensitivities of J with respect to c_A and c_S are given by*

$$\frac{\partial J}{\partial c_A} = \mathrm{Tr}\{\hat{P}(c_A)Q(c_S)Y(c_S)Q(c_S)\}, \qquad (3.27)$$

$$\frac{\partial J}{\partial c_S} = \mathrm{Tr}\{\hat{Q}(c_S)P(c_A)X(c_A)P(c_A)\}, \qquad (3.28)$$

where $\hat{P}(c_A)$ *and* $\hat{Q}(c_S)$ *are negative definite solutions of the following Lyapunov equations*

$$\begin{aligned} 0 &= \hat{P}(c_A)(A - X(c_A)P(c_A)) \\ &\quad +(A^T - P(c_A)X(c_A))\hat{P}(c_A) \\ &\quad -P(c_A)\frac{dX}{dc_A}P(c_A), \qquad (3.29) \\ 0 &= (A - Q(c_S)Y(c_S))\hat{Q}(c_S) \\ &\quad +\hat{Q}(c_S)(A^T - Y(c_S)Q(c_S)) \\ &\quad -Q(c_S)\frac{dY}{dc_S}Q(c_S). \qquad (3.30) \end{aligned}$$

Proof: *See the Appendix.*

Theorem 3.1 provides a method for finding the BN of a given feedback system. Of course, improving the BN component reduces the performance index more effectively than improving the other component.

In section 2, an Improvability Indicator is derived. We now clarify the relation between Improvability and BN Indicators.

Theorem 3.2 *For the feedback system (1.2)-(1.4) and (1.6), the followings hold.*

$$\frac{\partial J}{\partial c_A} = -2\frac{||T_{z_2\ w}||_2^2}{c_A}, \qquad (3.31)$$

$$\frac{\partial J}{\partial c_S} = -2\frac{||T_{z\ w_2}||_2^2}{c_S}. \qquad (3.32)$$

Proof: *See the Appendix.*

Theorem 3.2 has three practical consequences for the feedback system (1.2), (1.4) and (1.6). First, the BN can be found using on-line measurements. Second, BN and Improvability Indicators will point to the same component. Therefore, once the BN is found, improvement is achieved either by re-allocating instrumentation cost from one component to the BN component, or by investing additional instrumentation funds into the BN component. Third, once the system becomes unimprovable, then both sensor and actuator are BNs. Therefore additional instrumentation cost can be invested in any of them.

Based on the above, we have the following Bottleneck Indicator:

Bottleneck Indicator *The actuator is the BN of the feedback system (1.2)-(1.4), (1.6) if*

$$\frac{||T_{z_2\ w}||_2^2}{c_A} > \frac{||T_{z\ w_2}||_2^2}{c_S}. \qquad (3.33)$$

Otherwise, the sensor is the BN.

3.2 Example

Example 3.1 The same plant as in Example 2.1, when re-allocation of instrumentation cost is not possible, is investigated. However, a reduction of the performance index is required, hence additional instrumentation funds of \$20 is arranged to upgrade either sensor or actuator. For this system, the BN is the sensor as calculated in (2.25). When \$20 is invested in upgrading the sensor, J is decreased from 31992 to 3715. If the same funds were invested in the actuator, J would be decreased only to 19257.

4. CONCLUSIONS

We have analyzed improvability of LQG feedback systems in the context of instrumentation cost. We have derived an improvability indicator in the case where the instrumentation budget is fixed, but we are allowed to re-allocate costs between sensors and actuators. We have derived a BN indicator, to identify which component should be upgraded when the instrumentation budget is increased. Most importantly, we have shown that both improvability and BN indicators can be calculated using on-line measurements in the feedback loop, without requiring precise knowledge of the plant data. In the future, we plan to treat performance indices other than LQG, and actuator and sensor models more general than the simple models (1.6).

APPENDIX

Proof of Theorem 2.1
Knowing that $J = \mathrm{Tr}\{QE_1^TE_1 + PAQ + PQA^T + PD_1D_1^T\}$, define the Lagrangian with constraints

$$\begin{aligned} L &= J + \mathrm{Tr}[\Lambda_1(PA + A^TP + E_1^TE_1 - PXP)] \\ &\quad +\mathrm{Tr}[\Lambda_2(AQ + QA^T + D_1D_1^T - QYQ)] \\ &\quad +\lambda_3(c_A + c_S - c_I^*). \qquad (A.34) \end{aligned}$$

Differentiating (A.34) with respect to c_A, c_S, P, Q and simplifying them appropriately result in the following equations

$$-\mathrm{Tr}\{\Lambda_1P\frac{dX(c_A)}{dc_A}P\} + \lambda_3 = 0, \quad (A.35)$$

$$-\mathrm{Tr}\{\Lambda_2Q\frac{dY(c_S)}{dc_S}Q\} + \lambda_3 = 0, \quad (A.36)$$

$$(A - XP)\Lambda_1 + \Lambda_1(A^T - PX) + QYQ = 0, \quad (A.37)$$

$$(A^T - YQ)\Lambda_2 + \Lambda_2(A - QY) + PXP = 0. \quad (A.38)$$

Note that PXP and QYQ are positive semi-definite matrices, and, $A - XP$ and $A - QY$ are asymptotically stable matrices. Thus there exist unique positive semi-definite Λ_1 and Λ_2 satisfying (A.37) and (A.38). From (A.35) and (A.36), eliminating λ_3 yields (2.15). ■

Proof of Corollary 2.1

When (1.6) is assumed, it can be easily shown that (2.15) is simplified to (2.16). ■

Proof of Theorem 2.2

Equations (1.2) and (1.4) can be written in the following form

$$\begin{array}{rcl} \dot{\tilde{x}} & = & \tilde{A}\tilde{x} + \tilde{B}w, \\ z & = & \tilde{C}\tilde{x}, \end{array} \tag{A.39}$$

using $\tilde{x} = [x^T \ x_c^T]^T$, $w = [w_1 \ w_2]^T$, and $z = [z_1 \ z_2]^T$ and $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ as approproate.

Let

$$\tilde{P} = \left[\begin{array}{cc} P_1 & P_{12} \\ P_{12}^T & P_2 \end{array} \right] \tag{A.40}$$

be the positive definite solution of

$$\tilde{P}\tilde{A} + \tilde{A}^T\tilde{P} + \tilde{C}^T\tilde{C} = 0. \tag{A.41}$$

Then,

$$\begin{array}{rl} ||T_{z \ w}||_2^2 & = \mathrm{Tr}\{\tilde{P}\tilde{B}\tilde{B}^T\} \\ & = \mathrm{Tr}\{P_1 D_1 D_1^T\} + \mathrm{Tr}\{P_2 B_c D_2 D_2^T B_c^T\}, \end{array} \tag{A.42}$$

and it is easy to see that

$$||T_{z \ w_2}||_2^2 = \mathrm{Tr}\{P_2 B_c D_2 D_2^T B_c^T\}. \tag{A.43}$$

Using LQG controller $B_c = QC^T(D_2 D_2^T)^{-1}$, $C_c = -(E_2^T E_2)^{-1} B^T P$, and $A_c = A - B_c C + BC_c$, and $P_{12} = -P_2$ [4], the right bottom block of equation (A.41) becomes

$$0 = P_2(A - QY(c_S)) + (A^T - Y(c_S)Q)P_2 + PX(c_A)P. \tag{A.44}$$

Note that (A.44) is identical to (2.14), and, since Λ_2 is unique, $P_2{=}\Lambda_2$ is obtained. Therefore, substituting these in (A.43) yields,

$$||T_{z \ w_2}||_2^2 = \mathrm{Tr}\{\Lambda_2 QYQ\}. \tag{A.45}$$

Equation (2.19) can be similarly derived. ■

Proof of Theorem 2.3

Addition of w_3 modifies the output of the feedback system to

$$y = Cx + D_2 w_2 + D_2 w_3. \tag{A.46}$$

Let

$$\tilde{B}_1 = \left[\begin{array}{ccc} D_1 & 0 & 0 \\ 0 & B_c D_2 & B_c D_2 \end{array} \right], \ \tilde{w} = [w_1 \ w_2 \ w_3]^T. \tag{A.47}$$

Note, the solution of (A.41), $\tilde{P}$, remains the same. Using,

$$\tilde{B}_1\tilde{B}_1^T = \tilde{B}\tilde{B}^T + \left[\begin{array}{cc} 0 & 0 \\ 0 & B_c D_2 D_2^T B_c^T \end{array} \right], \tag{A.48}$$

the following is obtained.

$$\begin{array}{rcl} J & = & ||T_{z \ \tilde{w}}||_2^2 = \mathrm{Tr}\{\tilde{P}\tilde{B}_1\tilde{B}_1^T\} \\ & = & \mathrm{Tr}\{\tilde{P}\tilde{B}\tilde{B}^T\} + \mathrm{Tr}\{P_2 B_c D_2 D_2^T B_c\} \\ & = & J_0 + ||T_{z \ w_2}||_2^2. \end{array} \tag{A.49}$$

■

Proof of Theorem 3.1

Since Q is a function of c_S only and P is a function of c_A only, the following are easily obtained by differentiating the cost J with respect to c_A, and c_S respectively.

$$\frac{\partial J}{\partial c_A} = \mathrm{Tr}\{\frac{dP}{dc_A} QY(c_S)Q\}, \tag{A.50}$$

$$\frac{\partial J}{\partial c_S} = \mathrm{Tr}\{\frac{dQ}{dc_S} PX(c_A)P\}. \tag{A.51}$$

By differentiating (2.11) with respect to c_A, and (2.12) with respect to c_S, and equating $\hat{P} = \frac{dP}{dc_A}$, $\hat{Q} = \frac{dQ}{dc_S}$ gives (3.29) and (3.30). ■

Proof of Theorem 3.2

Pre-multiplying (3.29) by Λ_1, the solution of (2.13), yields

$$\Lambda_1 P \frac{dX}{dc_A} P = \Lambda_1 \hat{P}(A - XP) + \Lambda_1(A^T - PX)\hat{P}, \tag{A.52}$$

and pre-multiplying (2.13) by $\hat{P}$, the solution of (3.29), yields

$$-\hat{P}QY(c_S)Q = \hat{P}(A - XP)\Lambda_1 + \hat{P}\Lambda_1(A^T - PX). \tag{A.53}$$

Taking the trace of (A.52) and (A.53) yields

$$\mathrm{Tr}\{\hat{P}Q(c_S)Y(c_S)Q(c_S)\} = -\mathrm{Tr}\{\Lambda_1 P(c_A) \frac{dX(c_A)}{dc_A} P(c_A)\}. \tag{A.54}$$

Combining equations (3.27), (2.19) and (A.54) result in

$$\frac{\partial J}{\partial c_A} = -\frac{||T_{z_2 \ w}||_2^2}{c_A}. \tag{A.55}$$

Equation (3.32) is similarly derived. ■

References

[1] R. Skelton and M. Delorenzo, "Selection of noisy actuators and sensors in linear stochastic systems," *Large Scale Systems*, vol. 4, pp. 109–136, 1983.

[2] R. Skelton and D. Chiu, "Optimal selection of inputs and outputs in linear stochastic systems," *The Journal of the Astronautical Sciences*, vol. 31, no. 3, pp. 399–414, 1983.

[3] D. Jacobs and S. Meerkov, "A system-theoretic property of serial production lines: improvability," *International Journal of Systems Science*, vol. 26, no. 4, pp. 755–785, 1995.

[4] K. Zhou, J. Doyle, and K. Glover, *Robust and Optimal Control*, Prentice Hall, 1996.