Optimal Fractional Controllers for Commensurate Order Systems: A special case of the Wiener-Hopf Method[1]

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Abstract

In this paper, the authors propose a generalization of the well known Wiener-Hopf design method of optimal controllers and filters, applicable to certain class of systems described by fractional order differential equations, the so called *commensurate order systems,*i.e., in the Laplace domain, systems described by transfer functions which are not quotients of polynomials in s, but in $s^{\alpha}, \alpha = \frac{1}{q}$, being q a positive integer. As can be verified in the literature, such transfer functions arise in the characterization of many industrial processes and physical systems which can be adequately modeled using fractional calculus, or when modelling some distributed parameter systems by finite dimensional models. Taking into account that fractional-order systems and controllers have a limited diffusion, after a brief exposition of the principal results of the traditional Wiener-Hopf method, some fundamental considerations about the dynamical properties of such systems are made. After that, the authors propose a procedure that allows the application of the method to the mentioned class of systems. An illustrative example is given.

1 Introduction

The Wiener-Hopf spectrum-factorization technique is well known and widely used for designing optimal filters and controllers in the frequency domain. When the signals and systems involved are characterized by Laplace transforms which are rational functions, i.e., ratios of polynomials in the complex variable s, the design technique and the closed form solution is straightforward (see, e.g., [1], [2]). However, the problem is not solved when dealing with Laplace transforms which are nonrational. As far as the authors of this paper know, only an old paper by D.A. Pierre [3] deals with this subject, but the solutions proposed in it have some disadvantages: some of them are not closed form expressions always, and other ones reduce the problem to the traditional one by using rational approximations of the original nonrational functions. The necessity of dealing with non-rational functions can arise from two sources: the fact of dealing with systems which have nonrational transfer functions, and the convenience of specifying a system response which can not be modeled by exponential functions. The case of systems with transport delay is the most common of such cases, and it has been treated by some authors yet (see [1], [2] for additional references), but there are many other systems which can be better modeled as commensurate order systems, i.e., as systems whose transfer function is a ratio of polynomials in $s^{\frac{1}{q}}$, being q a positive integer. Among them, certain class of distributed parameter systems (at least, those governed by parabolic partial differential equations) and systems in which some memory phenomena, as diffusion or viscoelasticity, are present, have to be considered (see, e.g., [4], [5], [6] and [7], for aditional references).

The aim of this paper is to generalize the Wiener-Hopf method in order to apply it to these commensurate order systems. After a brief exposition of the Wiener-Hopf method, and a section devoted to the fundamentals of the dynamical properties of commensurate order systems, a procedure for obtaining closed form solutions without the necessity of approximations is proposed in this paper. Finally, an illustrative example is given.

2 Problem Statement and Wiener-Hopf General Solution

In this section, a brief exposition of the method is given. This work is limited to the case of integral square error problems. A more detailed exposition can be seen in [2].

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[1] The authors like to express their gratitude to Dr. Podlubny (Technical University of Kosice, Slovak Republic), for his generous disposition for interchanging information. This work has been partially supported by FEDER Research Grant IFD97-0755-C02-01

2.1 Problem Statement

A description of the system is shown in the block-diagram of the figure 1, were the following conditions are assumed:

1. The functions $F(s), H(s), G(s)$ are transfer functions of linear, stable, single-input, single-output, and physically realizable systems.
2. The functions $G(s), H(s)$ are given functions of s and characterize the plant of the overall system, whereas $F(s)$ is to be selected by the designer.
3. The function $G_0(s)$ is a given transfer function of a linear and stable, but not necessarily physically realizable system, and represents the ideal system for the process at hand with an ideal real input signal $x_0(t)$.
4. The actual system to be designed is subject to a possible nonideal real input signal $x(t)$ and must be physically realizable.
5. The actual system may contain a signal $y_h(t)$- perhaps more than one- which is known to be limited in either power or energy.
6. The input signals $x_0(t), x(t)$ are assumed to be deterministic and to be Laplace transformable.
7. The integral-square values of $y_h(t)$ and $e(t)$ exist.
8. The inverse Laplace transform $\pounds^{-1}[F(s)] = f(t)$ is required to equal zero for all t outside the time interval $[0, \infty]$, that is, $F(s)$ is required to be a stable system.

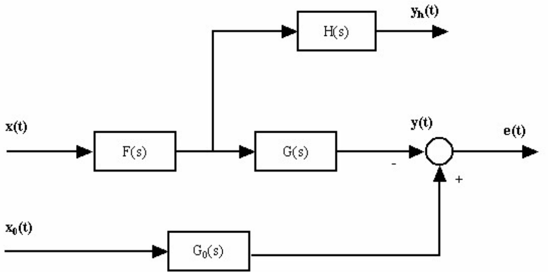

Figure 1: General Problem

The general problem here is to select $F(s)$ so as to minimize the integral-square error J,

$$J = \int_{-\infty}^{\infty} [e(t)]^2 \, dt = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} E(s)E(-s)ds \qquad (1)$$

where $E(s) = \pounds\{e(t)\}$ is the two-sided Laplace transform of $e(t)$, and the right hand term is a statement of Parseval's theorem. A constraint can be given so that integral-square value of $y_h(t)$ must equal a constant K :

$$K = \int_{-\infty}^{\infty} [y_h(t)]^2 \, dt = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} Y_h(s)Y_h(-s)ds \qquad (2)$$

By doing so, an equivalent problem is to find the $F(s)$ which yields an absolute minimum of the augmented functional J_h

$$J_h = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} [E(s)E(-s) + hY_h(s)Y_h(-s)] \, ds \qquad (3)$$

where h is a Lagrange multiplier which is independent of the variable s of integration. This Lagrange multiplier is evaluated ultimately on the basis that the energy constraint (2) must be satisfied. If additional constraints of the form (2) exist for the given problem, they are taken into account in the same way with additional Lagrange multipliers.

2.2 Wiener-Hopf Solution

Applying calculus of variations and the Wiener-Hopf spectral factorization, the general solution is (see [2]):

$$F^*(s) = \frac{\left\{ \frac{G_0(s)G(-s)X_0(s)X(-s)}{[hH(s)H(-s)+G(s)G(-s)]^-[X(s)X(-s)]^-} \right\}_+}{[hH(s)H(-s) + G(s)G(-s)]^+ [X(s)X(-s)]^+} \qquad (4)$$

where any Laplace transform $B(s)$ is factored in the form $B(s) = B(s)^+B(s)^-$, being both $B(s)^+$ and $\frac{1}{B(s)^+}$ analytic in finite regions which lie to the right of the $j\omega$ axis in the s plane, and both $B(s)^-$ and $\frac{1}{B(s)^-}$ analytic in finite regions which lie to the left of the $j\omega$ axis in the s plane, and $B(s)$ is decomposed in the form $B(s) = B(s)_+ + B(s)_-$, being $B(s)_+ = \pounds\{u(t)\pounds^{-1}[B(s)]\}$, $u(t)$ the Heaveside step function, and $B(s)_- = B(s) - B(s)_+$.

3 LTI Systems of Commensurate Order

3.1 System Models

The special class of LTI systems considered in this paper are systems which can be described by fractional order differential equations of the form:

$$\begin{aligned} & b_n(D^\alpha)^n y(t) + b_{n-1}(D^\alpha)^{(n-1)} y(t) + ... + b_0 y(t) \qquad (5) \\ = \quad & a_m(D^\alpha)^m x(t) + a_{m-1}(D^\alpha)^{(m-1)} x(t) + . + a_0 x(t) \end{aligned}$$

where:

$$D^\alpha f(t) \equiv \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(\tau)}{(t-\tau)^\alpha} d\tau \qquad (6)$$

is the fractional derivative of order $\alpha = \frac{1}{q}$, and q, m and n are positive integers.

Knowing that the Laplace transform of $D^{\alpha} f(t)$, with zero initial conditions, is:

$$\pounds[D^{\alpha} f(t)] = s^{\alpha} F(s) \tag{7}$$

the transfer function for the system will be of the form:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{a_m s^{\alpha m} + a_{m-1} s^{\alpha(m-1)} + ... + a_0}{b_n s^{\alpha n} + b_{n-1} s^{\alpha(n-1)} + .. + b_0} \tag{8}$$

3.2 Stable and Minimum-Phase Systems

In order to obtain the factors $[\cdot]^{+}, [\cdot]^{-}, \{\cdot\}_{+}, \{\cdot\}_{-}$ which we need for applying the method, we will study the stability and minimum-phase conditions of these systems.

As can be seen in equation (8), the function $H(s)$ is a nonrational function of s, but a rational function of s^{α}. On the other hand, this function is a multivalued function of s, the domain of which can be viewed as a Riemann surface. It has been proved by several authors and by different ways (see, e.g., [8], [9]) that for a system described by a transfer function of the form of $H(s)$ in equation (8), the stability condition is $|\arg \lambda| > \frac{\alpha\pi}{2}$, being $0 < \alpha < 2$, and λ the roots of the denominator obtained by taking s^{α} as variable. In other words, the principal sheet of the mentioned Riemann surface, defined by $-\pi < \arg(s) < \pi$, is the sheet which contains the strip of convergence for the inversion process.

Taking into account these considerations, the following statements can be made:

1. $h(t) = \pounds^{-1}[H(s)]$ *is a stable real function of time if* $H(s)$ *exist, is real for real* s, *and is analytic in the right half plane of the principal sheet of the Riemann surface and on the imaginary axis of such sheet; i.e., if* $|\arg \lambda| > \frac{\alpha\pi}{2}$, *being* λ *the poles of* $H(s)$.
2. $h(t) = \pounds^{-1}[H(s)]$ *is a stable minimum phase function if both* $H(s), \frac{1}{H(s)}$ *are analytic in the right half plane of the principal sheet of the Riemann surface and on the imaginary axis of such sheet; i.e., if* $|\arg \lambda| > \frac{\alpha\pi}{2}$, *being* λ *the poles of* $H(s)$, *and* $|\arg \gamma| > \frac{\alpha\pi}{2}$, *being* γ *the zeros of* $H(s)$.

4 Method of Solution

Taking into account the results and considerations of sections II and III we will obtain the solution by using the following lemma and theorem.

Lemma 1 *If* $H(s)$ *can be expressed as* $H(s) = P(s)G(s)$, *and* $G(s) = G_{mp}(s)\widehat{G}(s)$, *being* $G_{mp}(s)$ *a stable minimum-phase function, the solution can be expressed as*

$$F^{*}(s) = \frac{1}{G_{mp}(s)} \frac{\left\{ \frac{G_0(s)\widehat{G}(-s)X_0(s)X(-s)}{\left[h\widehat{H}(s)\widehat{H}(-s)+\widehat{G}(s)\widehat{G}(-s)\right]^{-}[X(s)X(-s)]^{-}} \right\}_{+}}{\left[h\widehat{H}(s)\widehat{H}(-s) + \widehat{G}(s)\widehat{G}(-s)\right]^{+}[X(s)X(-s)]^{+}} \tag{9}$$

with $\widehat{H}(s) = P(s)\widehat{G}(s)$.

Proof: *If* $H(s)$ *can be expressed as* $H(s) = P(s)G(s)$, *then*

$$\begin{aligned} & [hH(s)H(-s) + G(s)G(-s)] \\ = & [hP(s)P(-s)G(s)G(-s) + G(s)G(-s)] \\ = & hP(s)P(-s)G(s)G(-s)\left[1 + \frac{1/h}{P(s)P(-s)}\right] \end{aligned} \tag{10}$$

On the other hand, if $G(s)$ *can be factored of the form* $G(s) = G_{mp}(s)\widehat{G}(s)$, *then*

$$\begin{aligned} & [hH(s)H(-s) + G(s)G(-s)]^{+} \\ = & \left[hP(s)P(-s)G(s)G(-s)\left[1 + \frac{1/h}{P(s)P(-s)}\right]\right]^{+} \\ = & [hP(s)P(-s)G_{mp}(s)G_{mp}(-s)\widehat{G}(s)\widehat{G}(-s) \\ & \left[1 + \frac{1/h}{P(s)P(-s)}\right]]^{+} \\ = & G_{mp}(s)[hP(s)P(-s)\widehat{G}(s)\widehat{G}(-s) \\ & \left[1 + \frac{1/h}{P(s)P(-s)}\right]]^{+} \end{aligned} \tag{11}$$

and

$$\begin{aligned} & [hH(s)H(-s) + G(s)G(-s)]^{-} \\ = & \left[hP(s)P(-s)G(s)G(-s)\left[1 + \frac{1/h}{P(s)P(-s)}\right]\right]^{-} \\ = & [hP(s)P(-s)G_{mp}(s)G_{mp}(-s)\widehat{G}(s)\widehat{G}(-s) \\ & \left[1 + \frac{1/h}{P(s)P(-s)}\right]]^{-} \\ = & G_{mp}(-s)[hP(s)P(-s)\widehat{G}(s)\widehat{G}(-s) \\ & \left[1 + \frac{1/h}{P(s)P(-s)}\right]]^{-} \end{aligned} \tag{12}$$

By substitution of (11) and (12) into (4), expression (9) is obtained. ■

Theorem 2 *Let p and q relatively prime positive integers and let $r = \frac{p}{q}, \alpha = \frac{1}{q}$. Let $b = \rho e^{j\psi}, \rho \neq 0, |\psi| < \alpha\frac{\pi}{2}$, and let $b_k, k = 1, 2, ..., p$ be the p, pth roots of b. Then:*

$$\frac{1}{(s^r - b)} = \frac{P_{mp}(s^\alpha)}{Q_{mp}(s^\alpha)Q(s)} \tag{13}$$

with:

$$\begin{aligned} |\arg(roots(P_{mp}(s^\alpha))| &> \alpha\frac{\pi}{2} \\ |\arg(roots(Q_{mp}(s^\alpha))| &> \alpha\frac{\pi}{2} \\ |\arg(roots(Q(s))| &< \frac{\pi}{2} \end{aligned}$$

and $Q(s)$ being a rational function of s.

Proof: By the statements of the theorem, we can express $(s^r - b)$ of the form:

$$\begin{aligned} (s^r - b) &= (s^{\frac{p}{q}} - b) = \prod_{k=1}^{p} (s^{\frac{1}{q}} - b_k) \\ &= (s^{\frac{1}{q}} - b_1)(s^{\frac{p-1}{q}} + b_1 s^{\frac{p-2}{q}} + ... \\ &\quad + b_1^{p-2} s^{\frac{1}{q}} + b_1^{p-1}) \\ &= (s^{\frac{1}{q}} - b_1)Q(s^\alpha) \end{aligned}$$

where b_1 is the principal value of the *pth* root of b, that is,

$$b_1 = \rho^{\frac{1}{p}} e^{j\frac{\psi}{p}}, \qquad \left|\frac{\psi}{p}\right| < \alpha\frac{\pi}{2p} \leq \alpha\frac{\pi}{2}$$

and the roots of $Q(s^\alpha)$ are $b_k, k = 2, 3, ..., p$ with

$$\arg b_k = \frac{\psi + 2l\pi}{p}, \qquad l = 1, 2, ..., p-1$$

The minimum value of $|\arg b_k|$ will be:

$$\min\left(\left|\frac{\psi + 2l\pi}{p}\right|\right) = \left|\frac{-\frac{\pi}{2q} + 2\pi}{p}\right| = \frac{\pi(4q-1)}{2qp}$$

It holds that

$$\frac{\pi(4q-1)}{2qp} < \alpha\frac{\pi}{2} = \frac{\pi}{2q} \Rightarrow p > 4q - 1$$

Assuming that there are n of such roots, it can be stated that:

$$(s^r - b) = \prod_{l=1}^{n+1} (s^{\frac{1}{q}} - b_l) Q_{mp}(s^\alpha)$$

with

$$\begin{aligned} |\arg b_l| &< \alpha\frac{\pi}{2} \\ |\arg roots(Q_{mp}(s^\alpha))| &> \alpha\frac{\pi}{2} \end{aligned}$$

By applying the reduction formulae given and proved in [4]:

$$\frac{1}{s^r - b} = \frac{1}{bp}\sum_{k=1}^{p}\sum_{j=1}^{q} \frac{b_k^j}{s^{j\alpha - 1}(s - b_k^q)} \tag{14}$$

to each factor of the form $(s^{\frac{1}{q}} - b_l)$, with $|\arg b_l| < \alpha\frac{\pi}{2}$, it is obtained:

$$\begin{aligned} \frac{1}{(s^{\frac{1}{q}} - b_l)} &= \frac{1}{b_l}\sum_{j=1}^{q} \frac{b_l^j s^{1-j\alpha}}{(s - b_l^q)} \\ &= \frac{s^{\frac{q-1}{q}} + b_l s^{\frac{q-2}{q}} + ... + b_l^{q-2} s^{\frac{1}{q}} + b_l^{q-1}}{s - b_l^q} \\ &= \frac{P_l(s^\alpha)}{Q_l(s)} \end{aligned}$$

We can see that if $|\arg b_l| < \alpha\frac{\pi}{2}$, then $|\arg b_l^q| < q\alpha\frac{\pi}{2} = \frac{\pi}{2}$, and the numerator is a polynomial of the same form that $Q(s^\alpha)$ but with $p = q$, that is, with $p < 4q - 1$. Then:

$$|\arg roots(P_l(s^\alpha))| > \alpha\frac{\pi}{2}$$

and

$$\frac{1}{\prod_l (s^{\frac{1}{q}} - b_l)} = \frac{\prod_l P_l(s^\alpha)}{\prod_l Q_l(s)} = \frac{P_{mp}(s^\alpha)}{Q(s)}$$

with

$$\begin{aligned} |\arg(roots(P_{mp}(s^\alpha))| &> \alpha\frac{\pi}{2} \\ |\arg(roots(Q(s))| &< \frac{\pi}{2} \end{aligned}$$

Finally, we can state that:

$$\begin{aligned}\frac{1}{(s^r-b)} &= \frac{1}{\prod\limits_{l}(s^{\frac{1}{q}}-b_l)Q_{mp}(s^{\alpha})} \\ &= \frac{P_{mp}(s^{\alpha})}{Q_{mp}(s^{\alpha})Q(s)} \qquad (15)\end{aligned}$$

■

In the light of the former results, we can solve our problem by performing the following procedure:

1. If the function $G(s)$ has factors of the form $(s^r - b)$ with $r = \frac{p}{q}, \alpha = \frac{1}{q}, p$ and q relatively prime positive integers, and $|\arg b| < \alpha\frac{\pi}{2}$, apply *theorem 2* in order to obtain
$$G(s) = G_{mp}(s)\widehat{G}(s)$$
with $\widehat{G}(s)$ a rational function of s.
2. Apply equation (9) to obtain $F^*(s)$.

5 Illustrative Example

The procedure developed in the preceding section is now used to design the optimal filter or feedforward controller $F(s)$ for the system shown in figure 2. The feedback version of this problem is shown in figure 3, where

$$R(s) = \frac{F(s)}{1 - F(s)G(s)} \qquad (16)$$

The transfer function $G(s)$ is of the form $G(s) = e^{-\sqrt{s}}$, and it can be originated by distributed parameter systems described by parabolic partial differential equations, as one-dimensional heat-flow or RC transmission lines (see [10]).

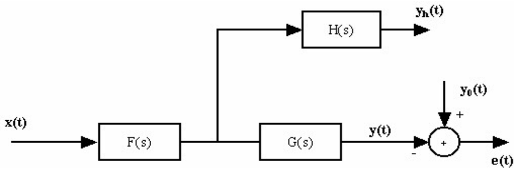

Figure 2: Feedforward control problem

Since the procedure has been developed for commensurate order systems, the first step is the selection of a suitable approximation for the function $e^{-\sqrt{s}}$. Satisfactory results are obtained with the four order Padé approximation

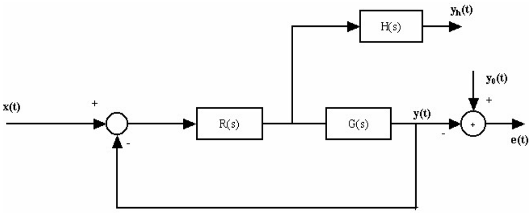

Figure 3: Feedback control problem

$$\widetilde{G}(s) = \frac{s^2 - 20s\sqrt{s} + 180s - 840\sqrt{s} + 1680}{s^2 + 20s\sqrt{s} + 180s + 840\sqrt{s} + 1680} \qquad (17)$$

The poles and zeros of $\widetilde{G}(s)$ are

$$\begin{aligned} z_{1,2} &= 4.2076 \pm j5.3148 = 6.7787e^{\pm j0.9012} \\ z_{3,4} &= 5.7924 \pm j1.7345 = 6.0465e^{\pm j0.2909} \\ p_{1,2} &= -4.2076 \pm j5.3148 = 6.7787e^{\pm j2.2404} \\ p_{3,4} &= -5.7924 \pm j1.7345 = 6.0465e^{\pm j2.8507} \end{aligned}$$

As can be seen, $\widetilde{G}(s)$ is a rational function of the complex variable $s^{\alpha}, \alpha = \frac{1}{2}$. On the other hand, this function corresponds to a non-minimum phase system, since that $|\arg(z_{3,4})| = 0.2909 < \alpha\frac{\pi}{4} = \frac{\pi}{4}$.

5.1 Solution

The solution for the case of $X(s) = X_0(s) = \frac{1}{s}$, and not constraints of the form (2), will be obtained by performing the following steps:

5.1.1 Factorization of $\widetilde{G}(s)$: In view of the zero-pole map of $\widetilde{G}(s)$ the following factorization can be made:

$$\begin{aligned}\widetilde{G}(s) &= \frac{(s - 8.4152\sqrt{s} + 45.9513)}{(s + 8.4152\sqrt{s} + 45.9513)} \\ &\quad \frac{(s - 11.5848\sqrt{s} + 36.5605)}{(s + 11.5848\sqrt{s} + 36.5605)} \\ &= \frac{(\sqrt{s} - z_1)(\sqrt{s} - z_2)}{(\sqrt{s} - p_1)(\sqrt{s} - p_1)} \frac{(\sqrt{s} - z_3)(\sqrt{s} - z_4)}{(\sqrt{s} - p_1)(\sqrt{s} - p_1)}\end{aligned}$$

Applying *theorem 2* to factors $(\sqrt{s} - z_3)(\sqrt{s} - z_4)$, the following result is obtained

$$\begin{aligned} & (\sqrt{s} + z_3)(\sqrt{s} + z_4) \\ = \ & \frac{(\sqrt{s} + z_3)(\sqrt{s} + z_4)(\sqrt{s} - z_3)(\sqrt{s} - z_4)}{(\sqrt{s} - z_3)(\sqrt{s} - z_4)}\end{aligned}$$

$$\begin{aligned} &= \frac{(s-z_3^2)(s-z_4^2)}{(\sqrt{s}-z_3)(\sqrt{s}-z_4)} \\ &= \frac{s^2-61.1087s+1336.6719}{s+11.5848\sqrt{s}+36.5605} = \frac{Q(s)}{P_{mp}(s)} \end{aligned}$$

Now, $\widetilde{G}(s)$ can be expressed as

$$\widetilde{G}(s) = G_{mp}(s)\widehat{G}(s)$$

with

$$\begin{aligned} G_{mp}(s) &= \frac{(\sqrt{s}-z_1)\,(\sqrt{s}-z_2)}{(\sqrt{s}-p_1)\,(\sqrt{s}-p_1)} \\ & \frac{1}{(\sqrt{s}-p_1)\,(\sqrt{s}-p_1)\,P_{mp}(s)} \\ &= \frac{s-8.4152\sqrt{s}+45.9513}{(s+8.4152\sqrt{s}+45.9513)^2} \\ & \frac{1}{(s+11.5848\sqrt{s}+36.5605)} \\ \widehat{G}(s) &= Q(s) = s^2-61.1087s+1336.6719 \end{aligned}$$

5.1.2 Obtention of $F^*(s)$: With

$$X(s) = X_0(s) = \frac{1}{s} \to X(-s) = X_0(-s) = -\frac{1}{s}$$

the factors of expression (9) are:

$$[X(s)X(-s)]^- = X(-s); \quad [X(s)X(-s)]^+ = X(s)$$

$$\left[\widehat{G}(s)\widehat{G}(-s)\right]^- = \widehat{G}(s); \quad \left[\widehat{G}(s)\widehat{G}(-s)\right]^+ = \widehat{G}(-s)$$

$$\left\{\frac{X(-s)\widehat{G}(-s)X_0(s)}{\left[\widehat{G}(s)\widehat{G}(-s)\right]^-\,[X(s)X(-s)]^-}\right\}_+ = \frac{1}{s}$$

and the solution becomes

$$\begin{aligned} F^*(s) &= = \frac{(s+8.4152\sqrt{s}+45.9513)}{(s-8.4152\sqrt{s}+45.9513)} \\ & \frac{(s+11.5848\sqrt{s}+36.5605)^2}{(s^2+61.1087s+1336.6719)} \end{aligned}$$

which is a fractional order filter or controller.

6 Conclusions and further work

In this paper an easy procedure has been developed which allows to use the Wiener-Hopf method for designing optimal filters and controllers in the case of a more general class of LTI systems, the so-called commensurate order systems, described in the Laplace domain by transfer functions which are ratios of polynomials in the complex variable s^α, $\alpha = \frac{1}{q}$, q being a positive integer. It has been shown that the spectral factorization problem can be reduced to the well known case of rational transfer functions without approximations. An example has been given to illustrate the application of the procedure.

A step further can be made at least in two ways: by introducing sensitivity considerations into the problem, and by developing a more general procedure applicable to incommensurate order systems, that is, systems described by generalized fractional differential equations.

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