

Uncertainty, Information and Complexity in Identification and Control

Le Yi Wang, ECE Dept., Wayne State University, Detroit, Michigan 48202

Abstract

The relationship between information acquisition (identification) and information processing (control) in their capability of dealing with uncertainty is studied. It is revealed that such a relationship can be established rigorously from the viewpoint of complexity. A notion of information-based complexity is hence introduced, first in its generality, and then in its special applications to metric spaces in feedback control systems.

1 Introduction

The intriguing relationships among uncertainty, information, complexity, identification, and control have attracted much research effort in the control community during the past decades. Intuitively, it seems quite clear that to achieve a control objective for a plant that belongs to an uncertainty set, one can either utilize the capability of identification in reducing uncertainty, or that of control in providing robustness against uncertainty, or a combination of both. In its generality, this intuition extends beyond feedback control to other applications which involve uncertainty, information acquisition, and information processing. However, it turns out to be a daunting task to establish a rigorous framework in which such understanding can be clearly captured and clarified.

In his persistent pursuit of this task, George Zames developed a general framework and philosophy which characterize capability of modeling, identification, adaptation and feedback in the light of uncertainty, information and complexity, especially in their metric representations. During that period of time, dramatic advancement was witnessed in our understanding of feedback robustness via the development of H^∞ , l^1 , μ theories and of control-oriented identification via information-based methodologies.

On the basis of this understanding, this paper studies the relationship between information acquisition (identification) and information processing (control) in their capability of dealing with uncertainties. It is revealed that such a relationship can be established rigorously from the viewpoint of complexity. A notion of information-based complexity is hence introduced, first

in its generality, and then in its special applications to metric spaces in feedback control systems. Although the complexity notion was primarily motivated by feedback control, it appears that it may have broader applications in other areas as well.

1.1 Related Literature

This paper is a generalization of the basic ideas presented in [10], where a complexity notion is introduced for SISO feedback systems in metric spaces. It appears that the ideas can be extracted to define a complexity notion in a more general information framework. The development of this paper follows the philosophical principles of George Zames on an information-based theory of feedback control, identification, and adaptation [12]. Complexity issues in modeling and identification have been pursued by many researchers, including [11] [7] [5] [4] [8] [6].

The concept of complexity introduced in this paper is an extension of the Kolmogorov ε -entropy [3] [11] which characterizes the complexity of a set of functions by the minimal number of functions which can approximate the set to a precision level ε . While the Kolmogorov entropy has been extensively used in approximation theory, computational complexity theory, information theory, and operator theory, as well as system modeling and identification, it often fails to provide a relevant measure of complexity of uncertainty sets in feedback control problems. The main reason is that the ability of feedback to achieve robust stability and performance depends on the nominal system structure, not merely the size of uncertainty. The Kolmogorov entropy was introduced in the 50's [3]. The book by Carl and Stephani [2] contains many new developments on the theory and applications of the Kolmogorov entropy up to the late 80's.

An extended version of this paper can be found in [9].

2 Information and Complexity

The problem of characterizing complexity of a controlled plant stems from the basic question: What is a "difficult" plant to model, identify, and control? The main idea of this paper is that a plant is considered complex if complex control schemes must be used. Hence,

complexity measures will first be defined on controllers.

Traditionally, system orders or dimensions are perceived as reasonable measures of complexity. This is valid to the extent of numerical computation, computer representation and design of controllers. However, from the view point of uncertainty, information, robustness, as well as control problems which do not admit differential or difference equation models, system orders become inadequate. We are seeking alternative complexity measures which are related to information processing in a more fundamental manner.

2.1 Control Complexity

In this paper, we are considering control schemes S that consist of a finite set of controllers $S = \{F_1, \dots, F_n\}$ selected from a set \mathcal{F} of admissible controllers. Let the set \mathcal{S} be the class of all finite subsets of \mathcal{F} . \mathcal{S} will be the set of *admissible control schemes derived from \mathcal{F}* . For $S \in \mathcal{S}$, we define its *complexity measure* $m(S)$ as the number of controllers it contains, $m(S) = m(\{F_1, \dots, F_n\}) = n$. Accordingly, we will denote the set of admissible schemes by (S, m) . Such control schemes cover more than traditional controllers in differential or difference equations. For instance, the set \mathcal{F} contains all interpreters who can speak one foreign language. Similarly in feedback systems, \mathcal{F} may be as simple as a set of “PI” controllers, or as comprehensive as a set of neural controllers. With such a complexity measure defined on \mathcal{S} , S_1 is considered as more complex than S_2 if $m(S_1) \geq m(S_2)$. If one is to view this from hybrid systems, $m(S)$ will represent the size of discrete-event state of the control scheme.

The *goal of control* is expressed by a statement \mathcal{M} . \mathcal{M} may take the form of a linguistic statement such as “the interpreter can speak the passenger’s language,” or a qualitative requirement such as “the closed-loop system is input/output stable,” or a mathematical expression such as “ $\|W(1 + PF)^{-1}\|_\infty \leq \rho$.”

Let \mathcal{P} be the set of plants under study. Information on a plant $P \in \mathcal{P}$ is given by an uncertainty set $\Omega \subset \mathcal{P}$ that contains P . Consequently, in this paper the concepts of “information” and “uncertainty sets” are essentially equivalent, except that the increase of one is equivalent to the decrease of the other. For a given plant P , a controller $F \in \mathcal{F}$ is said to *achieve nominal performance* if \mathcal{M} is true under the pair (P, F) . Similarly, for a given uncertainty set Ω of plants, a controller $F \in \mathcal{F}$ is said to *achieve robust performance* if \mathcal{M} is true under (P, F) for all $P \in \Omega$. The robustness range R_F of a controller F consists of all plants $P \in \mathcal{P}$ for which (P, F) satisfies \mathcal{M} . Then, Ω is said to be in the robust range of $F \in \mathcal{F}$ if $\Omega \subseteq R_F$. Interchangeably, we say also R_F covers Ω .

For a given uncertainty set $\Omega_a \subset \mathcal{P}$, if there exists a controller F whose range R_F covers Ω_a , then the goal

\mathcal{M} can be achieved without additional information acquisition. Otherwise, further information must be acquired to reduce uncertainty before control can be performed. In this paper, we will use the term “identification” in its broad sense of information acquisition, not necessarily limited to input/output information and difference/differential equation modeling in feedback systems. In an individual identification experiment, the outcome of the identification process is an uncertainty set $\mathbf{U} \subset \mathcal{P}$ that contains the unknown plant. This set depends on Ω_a , the true plant, as well as the process of information acquisition. To understand the essential complexity features of Ω_a under a selected process of information acquisition, we must eliminate the effect of the true plant by considering all possible outputs \mathbf{U} of the identification process. Hence, we denote by Ω_p the set of all possible \mathbf{U} . Ω_p will be called *posterior information*. Accordingly, Ω_a will be called *prior information*. As a result, the *identification process* is a mapping $\mathcal{I} : \Omega_a \rightarrow \Omega_p$.

From the original concept of information (i.e., the uncertainty set), it seems apparent that information acquisition will only increase information (or equivalently reduce uncertainty) since one can always go back to the prior information which is available before acquisition. In this sense, for any $\mathbf{U} \in \Omega_p$, we should have $\mathbf{U} \subseteq \Omega_a$. This leads to the argument of monotonicity of information: Information acquisition results in a sequence of monotone decreasing uncertainty sets. However, if information is represented approximately, such as the metric information discussed in the subsequent sections, monotonicity of information will be lost. This is due to the fact that approximate representation of information inevitably suffers certain loss of information. Consequently, the monotone property derived for total information is no longer applicable. The results of this paper do not depend on the monotone property.

A control scheme S is said to *achieve robust performance on Ω_p* if for every $\mathbf{U} \in \Omega_p$, there exists a controller $F \in S$ such that \mathbf{U} is in the robustness range R_F of F . Intuitively, this will guarantee that the set of controllers in S can collectively provide robustness which is sufficient to cover any possible outcome of the identification process. Mathematically, one may define the class of the robustness ranges as $R(S) = \{R_F : F \in S\}$.¹ The above requirement can be simply stated as “ Ω_p is a refiner of $R(S)$,”² and denoted by

$$\Omega_p \preceq R(S).$$

Furthermore, the notion of refiner leads to a partial ordering of identification processes. Suppose two iden-

¹It should be emphasized that $R(S)$ is not the union of R_F .

²Suppose M_1 and M_2 are two classes of subsets from a universal set \mathbf{M} . M_1 is said to be a refiner of M_2 if for every $U \in M_1$ there exists $V \in M_2$ such that $U \subseteq V$.

tification processes \mathcal{I}_1 and \mathcal{I}_2 are applied to the same Ω_a , producing the posterior uncertainty sets Ω_p^1 and Ω_p^2 , respectively. \mathcal{I}_1 is said to be a better process than \mathcal{I}_2 , denoted by $\mathcal{I}_1 \preceq \mathcal{I}_2$ if $\Omega_p^1 \preceq \Omega_p^2$. Intuitively, this simply implies that \mathcal{I}_1 produces smaller uncertainty sets. Since the refiner is transitional, that is, $\Omega_1 \preceq \Omega_2$ and $\Omega_2 \preceq \Omega_3$ implies $\Omega_1 \preceq \Omega_3$, we conclude that if S can achieve robust performance on Ω_p , then it can achieve robust performance on all identification processes which are better than Ω_p .

Definition 1 Given a goal \mathcal{M} , a prior uncertainty set Ω_a , an identification process $\mathcal{I} : \Omega_a \rightarrow \Omega_p$, and a set (S, m) of control schemes, the control complexity of Ω_a is defined as

$$\mu(\Omega_a; \mathcal{M}, \mathcal{I}, (S, m)) = \inf\{n : \text{there exists } S \in (S, m), \\ m(S) \leq n, \text{ such that } \Omega_p \preceq R(S)\}$$

From the definition, we may also use equivalently the notation

$$\mu(\Omega_p; \mathcal{M}, (S, m)) = \mu(\Omega_a; \mathcal{M}, \mathcal{I}, (S, m)),$$

or simply $\mu(\Omega_p) = \mu(\Omega_a; \mathcal{M}, \mathcal{I}, (S, m))$ if \mathcal{M} and (S, m) are apparent from context.

It is noted that although the complexity measure $m(S)$ is concretely defined as the number of controllers contained in the control scheme S , the definition of $\mu(\Omega_a)$ can be readily extended to other measures with little modification. On the other hand, currently there are only a few complexity measures, such as system orders in LTI systems and the size of an automata in discrete-event or hybrid systems, which are generic and have found concrete applications in engineering and computer science.

2.2 Basic Properties

Some basic properties of the complexity measure can be easily derived from the definition.

A goal \mathcal{M}_1 is said to be more stringent than another goal \mathcal{M}_2 , denoted by $\mathcal{M}_1 \preceq \mathcal{M}_2$, if “ \mathcal{M}_1 is true” implies that “ \mathcal{M}_2 is true.” For example, let \mathcal{M}_i : “ $\|W(1 + PF)^{-1}\|_\infty \leq \rho_i$.” Then, $\mathcal{M}_1 \preceq \mathcal{M}_2$ whenever $\rho_1 \leq \rho_2$. Similarly, \mathcal{M}_1 : “closed-loop system is exponentially stable” is more stringent than \mathcal{M}_2 : “closed-loop system is stable.”

Proposition 1 1. If $\mathcal{M}_1 \preceq \mathcal{M}_2$, then

$$\mu(\Omega_a; \mathcal{M}_1, \mathcal{I}, (S, m)) \geq \mu(\Omega_a; \mathcal{M}_2, \mathcal{I}, (S, m)).$$

Namely, the more stringent the goal, the more complex the uncertainty set Ω_a .

2. If $\mathcal{I}_1 \preceq \mathcal{I}_2$, then

$$\mu(\Omega_a; \mathcal{M}, \mathcal{I}_1, (S, m)) \leq \mu(\Omega_a; \mathcal{M}, \mathcal{I}_2, (S, m)).$$

Namely, the better the identification, the less complex the uncertainty set Ω_a .

3. If $\mathcal{S}_1 \subseteq \mathcal{S}_2$ and m_1 on \mathcal{S}_1 is embedded in m_2 on \mathcal{S}_2 ,³ then

$$\mu(\Omega_a; \mathcal{M}, \mathcal{I}, (\mathcal{S}_1, m_1)) \geq \mu(\Omega_a; \mathcal{M}, \mathcal{I}, (\mathcal{S}_2, m_2)).$$

Namely, the larger the set of admissible control schemes, the less complex the uncertainty set Ω_a .

4. (\mathcal{S}_1, m_1) is said to be more robust than (\mathcal{S}_2, m_2) if for any $S_2 \in \mathcal{S}_2$, there exists $S_1 \in \mathcal{S}_1$ such that

$$m_1(S_1) \leq m_2(S_2) \quad \text{and} \quad R(S_2) \preceq R(S_1).$$

This means that there always exists an equally or less complex control scheme in \mathcal{S}_1 to provide same or larger robustness.

If (\mathcal{S}_1, m_1) is more robust than (\mathcal{S}_2, m_2) , then

$$\mu(\Omega_a; \mathcal{M}, \mathcal{I}, (\mathcal{S}_1, m_1)) \leq \mu(\Omega_a; \mathcal{M}, \mathcal{I}, (\mathcal{S}_2, m_2)).$$

Namely, the better the robustness of control schemes, the less complex the uncertainty set Ω_a .

2.3 Applications

2.3.1 Relations to Identification Complexity: Suppose a priori information of an uncertain discrete-time plant P is given by a general $\Omega_a \subset H^\infty$, and $\delta(n)$ is the identification n -width of Ω_a , i.e., the best achievable worst-case identification error based on n consecutive observations. As a result, the posterior uncertainty $\Omega_p(n)$ contains balls of radius less than or equal to $\delta(n)$. Apparently, since $\delta(n)$ is a monotone decreasing function of n , $\Omega_p(n)$ is a monotone decreasing function of n in the sense of refiner.

The goal \mathcal{M} is stated as a bound on a performance index, which is a weighted H^∞ norm of the closed-loop system sensitivity $f(P, F) = \|W(1 + PF)^{-1}\|_\infty \leq \rho$. It follows that the control complexity $\mu(\Omega_p(n))$ is a monotone decreasing function of n . For a given identification complexity n , $\mu(\Omega_p(n))$ gives the minimum complexity of control required to achieve the specified performance level ρ . Then, the inverse function of $\mu(\Omega_p(n))$

$$n = \phi(N_0) = \inf\{l \in \mathbf{Z}_+ : \mu(\Omega_p(l)) \leq N_0\}$$

gives the minimal time complexity of identification which is needed to achieve the performance specifications when the complexity of control is limited by N_0 .

³This means that if $S \in \mathcal{S}_1$, then $m_1(S) = m_2(S)$. This will obviously be satisfied if both m_1 and m_2 are the cardinality of S .

One possible application of such a tradeoff between complexities of identification and control might be allocation of resources to identification and control. Suppose $c_1(n)$ and $c_2(N)$ are cost functions of implementing identification experiments and control mechanism, respectively. Then, optimization of the combined cost function $c(n) = c_1(n) + c_2(\mu(\Omega_p(n)))$ will provide a candidate for optimal allocation of resources to achieve a given level of performance specifications.

2.3.2 Relations to Model Reduction: Suppose that Ω_a is a set of LTI (either continuous-time or discrete-time) plants which are possibly infinite dimensional. For computer implementation, it is desirable to represent Ω_a by systems of lower orders. Denote by \mathbf{M}_n the set of LTI systems of order n or lower. Define the model reduction error

$$\delta(n) = \sup_{G \in \Omega_a} \inf_{M \in \mathbf{M}_n} d(G, M)$$

where d is a metric on the systems. d can be the H^∞ or l^1 norm for stable systems, or the gap metric for unstable systems. Consequently, the model reduction defines a mapping $\mathcal{I} : \Omega_a \rightarrow \Omega_p(n)$ in which each $U \in \Omega_p(n)$ is represented by $U = G_0 + \Delta$, $d(\Delta, 0) \leq \delta(n)$ for some $G_0 \in \mathbf{M}_n$.

For a given goal \mathcal{M} and (\mathcal{S}, m) , by definition the corresponding control complexity is

$$\mu(\Omega_a; \mathcal{M}, \mathcal{I}, (\mathcal{S}, m)) = \mu(\Omega_p(n)).$$

Clearly, if $n_2 \geq n_1$, then $\Omega_p(n_2) \preceq \Omega_p(n_1)$ which implies that $\mu(\Omega_p(n_2)) \leq \mu(\Omega_p(n_1))$. As a result, the mapping $\mu(\Omega_p(n))$ is a monotone decreasing function of n , indicating that the higher the model complexity, the less complex the control scheme. If one is to view this from hybrid systems' point of view, this conclusion reveals a complexity relationship between its analog state space and its discrete-event automata.

2.3.3 Relations to Communication Capacity: Consider a situation where data about a plant $P \in \Omega_a$, such as its measured parameters, must be communicated to a control site via a communication network \mathcal{I} . Due to limited channel capacity, such as its data length, channel speed or bandwidth, and data/loss ratios, information going through the network will be subject to a level of distortion. As a result, the output of \mathcal{I} will be an uncertainty set $\mathcal{I}(P) = U(P)$ that contains P . It follows that

$$\mathcal{I}(\Omega_a) = \Omega_p(\rho) = \{U(P) : P \in \Omega_a\}$$

where ρ represents communication channel capacity. The larger the capacity, the less the distortion, namely

$$\Omega_p(\rho_1) \preceq \Omega_p(\rho_2) \quad \text{if} \quad \rho_1 \geq \rho_2. \quad (1)$$

Control decisions at the receiving site are made on the basis of $\Omega_p(\rho)$, with complexity $\mu(\Omega_p(\rho))$. The monotone relation (1) implies that $\mu(\Omega_p(\rho))$ is a monotone decreasing function of ρ . This relationship can be used to reach a tradeoff between an appropriate channel capacity (bandwidth, rate distortion, etc.,) and control complexity.

3 Metric Uncertainty and Information in Feedback Control

The notion of control complexity in metric spaces was first introduced in [10] for SISO LTI systems. It appears that the metric notion of complexity is valid for MIMO systems without much alternation, as presented in the this section. When uncertainty sets are subsets of a metric space, it is possible to represent all uncertainty sets by their nominal center and metric sizes. More precisely, for an uncertainty set $\mathbf{U} \subseteq \mathbf{L}$, where \mathbf{L} is a set with norm $\|\cdot\|$, \mathbf{U} can be represented by a ball of some center Δ_0 and minimum norm radius r_0 : $\mathbf{U} \subseteq \text{Ball}(\Delta_0, r_0)$

$$r_0 = \inf \{r : \mathbf{U} \subseteq \text{Ball}(\Delta, r) \text{ for some } \Delta \in \mathbf{L}\}$$

Also, the goal \mathcal{M} will also be expressed as a metric performance index. This leads to the framework of metric information and complexity.

3.1 Notation

$\mathbb{R}, \mathbb{C}, \mathbb{Z}$ denote the real numbers, complex numbers and integers. The absolute value of $x \in \mathbb{C}$ is $|x|$. The spaces of vectors or matrices with elements in \mathbb{R} will be denoted by \mathbb{R}^n or $\mathbb{R}^{n \times m}$. For a matrix $M \in \mathbb{R}^{n \times m}$, M^T is its transpose.

Suppose \mathbf{L} is a normed space with norm $\|\cdot\|$. $\text{Ball}(K, r) \subseteq \mathbf{L}$ denotes the ball of center K and radius r in \mathbf{L} .⁴ For $M_1, \dots, M_n \subseteq \mathbf{L}$, $\cup_{i=1}^n M_i$ is the union of the subsets. For $M \subseteq \mathbf{L}$, $d(M)$ is the diameter of M

$$d(M) = \sup_{G_1, G_2 \in M} \|G_1 - G_2\|.$$

3.2 Systems and Feedback Configurations

\mathbb{B} denotes a normed algebra of MIMO causal stable linear time invariant systems with norm $\|\cdot\|$. We will use K to denote both the system and its transfer function, and k its impulse response. Unstable systems will belong to \mathbb{B}_e , the extended space of \mathbb{B} . The sets of plants and admissible controllers under consideration will be $\mathbb{P} \subseteq \mathbb{B}_e$ and $\mathbb{F} \subseteq \mathbb{B}_e$, respectively.

Consider the feedback system $y = Pe + d$, $e = r - Fy$ where r , d , e , y are reference input, disturbance, actuator signal and plant output, respectively.

⁴It will be specified whether the ball is open or closed, on the basis of specific applications.

The interconnection of a feedback $F \in \mathbb{F}$ and a plant $P \in \mathbb{P}$ in \mathbb{B}_e is *well-posed* if all elements of the closed-loop system $\mathcal{K}(P, F)$ are in \mathbb{B}_e , and *stable* if they are in \mathbb{B} . $F \in \mathbb{F}$ is said to robustly stabilize a subset $\Omega \subseteq \mathbb{P}$ if $\mathcal{K}(F, P)$ is stable for all $P \in \Omega$. We denote by $\Pi(\Omega)$ the set of all feedback controllers $F \in \mathbb{F}$ which can robustly stabilize Ω . $\Pi(\Omega)$ may be empty, implying that Ω is not robustly stabilizable by using controllers in \mathbb{F} . In the special case where Ω contains only one plant P with a coprime factorization representation and $\mathbb{F} = \mathbb{B}_e$, $\Pi(\Omega)$ is given explicitly by the Youla parametrization of all stabilizing controllers for P .

3.3 Metric Information and Robustness

Let $\Omega \subseteq \mathbb{P}$ be an uncertainty set of plants represented by

$$\Omega = \mathcal{G}(\mathbf{U}) = \{\mathcal{G}(\Delta) : \Delta \in \mathbf{U} \subseteq \mathbb{L}\} \quad (2)$$

where $\mathcal{G} : \mathbb{L} \rightarrow \mathbb{P}$ is a structural mapping which relates the plant to its uncertainty variable Δ . For example, systems with gain uncertainty $\Omega = \{kG_0 : k \in [a, b]\}$ will have $\mathbb{L} = \mathbb{R}$, norm $|\cdot|$ (absolute value), $\mathbf{U} = [a, b]$, and $\mathcal{G}(k) = kG_0$. Similarly, uncertainty expressions in unstructured uncertainty $\Omega = \{\mathcal{G}(\Delta) = G_0 + W\Delta : \Delta \in \mathbf{U} \subseteq \mathbb{L} = \mathbb{B}\}$, linear fraction transformation (LFT) type

$$\Omega = \{\mathcal{G}(\Delta) = M_{22} + M_{21}\Delta(1 - M_{11}\Delta)^{-1}M_{12} : \Delta \in \mathbf{U} \subseteq \mathbb{L} = \mathbb{B}\},$$

or structured uncertainties can all be formulated in this expression.

In metric information, uncertainty sets and robustness are expressed by their metric sizes. In other words, they will be expressed as a ball of center Δ_0 and norm radius r in \mathbb{L} , $Ball(\Delta_0, r) \subseteq \mathbb{L}$. We shall start with metric robustness of feedback systems.

Suppose that a plant G has the structure $G = \mathcal{G}(\Delta)$ with $\Delta \in \mathbb{L}$, where \mathbb{L} is equipped with norm $\|\cdot\|$, and controllers are selected from \mathbb{F} . The performance of a feedback system is measured by a metric performance index

$$f : \mathbb{P} \times \mathbb{F} \rightarrow \mathbb{R}_+$$

which assigns a finite non-negative value $f(G, F)$ to a plant-controller pair $(G, F) \in \mathbb{P} \times \mathbb{F}$ if the feedback interconnection $\mathcal{K}(G, F)$ is stable, and ∞ otherwise. For an uncertainty set $\Omega \subseteq \mathbb{P}$, define $f(\Omega, F) = \sup_{G \in \Omega} f(G, F)$.

For a given controller $F \in \mathbb{F}$, a nominal $\Delta_0 \in \mathbf{U}$, and a specified performance level ρ , the metric robustness range of F is defined as

$$r_F(\Delta_0, \rho) = \sup\{r : f(\mathcal{G}(Ball(\Delta_0, r)), F) \leq \rho\}$$

Then,

$$r(\Delta_0, \rho) = \sup_{F \in \mathbb{F}} r_F(\Delta_0, \rho) \quad (3)$$

is the optimal metric robustness for the nominal plant $G_0 = \mathcal{G}(\Delta_0)$ under \mathbb{F} . The notation $Ball(\Delta_0, r; \rho)$ denotes a ball in \mathbb{L} of center Δ_0 , norm radius r and optimal robust performance ρ . Definition (3) implies that

$$\rho = \inf_{F \in \mathbb{F}} \sup_{\Delta \in Ball(\Delta_0, r)} f(\mathcal{G}(\Delta), F)$$

By definition, for each $Ball(\Delta_0, r; \rho)$, there exists a controller $F \in \mathbb{F}$ such that

$$f(\mathcal{G}(Ball(\Delta_0, r)), F) \leq \rho + \varepsilon$$

for any $\varepsilon > 0$. Metric characterization of feedback robustness is exemplified by optimal robustness measures such as H^∞ norm, l^1 norm, μ metric, gap metric, etc.

Next, we define metric expression of posterior uncertainty in an identification process. Suppose prior information about a plant is that it belongs to a given prior uncertainty set $\Omega = \{\mathcal{G}(\Delta) : \Delta \in \mathbf{U} \subseteq \mathbb{L}\}$. Identification experiments provide additional information about the plant and reduce the uncertainty on the plant uncertain variable Δ from \mathbf{U} to a posterior uncertainty set \mathbf{U}_1 . In metric frameworks, \mathbf{U}_1 will be a ball in \mathbb{L} , $Ball(\hat{\Delta}_0, r)$, where $\hat{\Delta}_0$ is the nominal estimate and r the metric estimation error. Denote by \mathbf{U}_p the class of all possible metric posterior uncertainty sets \mathbf{U}_1 ,

$$\mathbf{U}_p = \{Ball(\hat{\Delta}_0, r) : Ball(\hat{\Delta}_0, r) \subseteq \mathbb{L}\}$$

is a possible posterior uncertainty set.

It follows that the set of all possible posterior information on the plant is

$$\Omega_p = \{\mathcal{G}(Ball(\hat{\Delta}_0, r)) : Ball(\hat{\Delta}_0, r) \in \mathbf{U}_p\}. \quad (4)$$

For a required performance level ρ , if for some $Ball(\Delta_0, r; \rho)$ the prior uncertainty set $\mathbf{U} \subseteq Ball(\Delta_0, r; \rho)$, then there exists a single controller \bar{F} which satisfies $f(\mathcal{G}(\mathbf{U}), \bar{F}) \leq \rho$. Hence, robust control is sufficient. Otherwise, identification will be employed and multiple controllers will be used, leading to control schemes $S \in \mathcal{S}$. The main question is: How many controllers are needed? Namely, how complex must be the control schemes?

Definition 2 A collection of n balls of performance level ρ , $\mathbf{V}_n = \{Ball(\Delta_i, r_i; \rho), i = 1, \dots, n\}$ is said to cover \mathbf{U}_p if $\mathbf{U}_p \preceq \mathbf{V}_n$.

Definition 3

(1) The metric control complexity $N_G(\mathbf{U}_p, \rho)$ of \mathbf{U}_p is defined as

$$N_G(\mathbf{U}_p, \rho) = \inf\{n : \mathbf{U}_p \preceq \mathbf{V}_n\}.$$

(2) When the underlying identification is accurate, namely, \mathbf{U}_p is reduced to $\mathbf{U}_p^0 = \{\{K\} : K \in \mathbf{U}\}$, we will use the symbol $n_G(\mathbf{U}, \rho)$ to denote the metric control complexity.

Monotone properties, similar to Proposition 1 can also be derived [10].

The reader is referred to [9] and [10] for many examples, theoretical findings, control design and simulation results on the complexity measures.

4 Concluding Remarks

The starting point of the complexity measure introduced in this paper is a complexity measure defined on control schemes. There are potentially many different approaches of defining complexity measures on control schemes, depending on applications. As a result, the complexity measure of this paper is by no means the ultimate one. The basic approaches and ideas of this paper, namely deriving a complexity measure of plant uncertainty sets from a complexity measure on control schemes, should remain valid when other measures are used. The complexity measure used here, i.e., the number of controllers in control schemes, has some appealing features including its interpretation in terms of sizes of discrete-event state spaces in hybrid systems, potential applications to non-metric, non-analytical control schemes, and its direct link to optimal robustness results obtained in H^∞ , μ , etc.

There are many open issues unresolved. The framework here is developed in deterministic systems. Information theory on stochastic systems has evolved into many fruitful applications in identification, communications, statistics and control. The link between Kolmogorov or Shannon entropy and the complexity issues explored in this paper is fundamentally important and most puzzling at the same time. When communication channels are part of a control system, revealing such a link will provide a vehicle to bring together some understandings on relationships of information acquisition (identification), information networking (communication), and information processing (control). The advancement of information networking such as internet and parallel computers may eventually make it inevitable to study such a relationship.

References

[1] H. Akaike, A new look at the statistical model identification, *IEEE Trans. Automat. Control*, Vol. AC-19, pp. 716-723, 1974.
 [2] B. Carl and I. Stephani, Entropy, Compactness

and the Approximation of Operators, Cambridge University Press, 1990.

[3] A.N. Kolmogorov, On some asymptotic characteristics of completely bounded spaces, *Dokl. Akad. Nauk SSSR*, Vol. 108, No. 3, pp.385-389, 1956.

[4] A. Pinkus, n-widths in Approximation Theory, Springer-Verlag, 1985.

[5] K. Poolla and A. Tikku, On the time complexity of worst-case system identification, *IEEE Trans. Automat. Control*, Vol. AC-39, pp. 944-950, 1994.

[6] J. F. Traub, G. W. Wasilkowski, and H. Wozniakowski, Information-Based Complexity, New York: Academic Press, 1988.

[7] D.C.N. Tse, M.A. Dahleh and J.N. Tsitsiklis, Optimal asymptotic identification under bounded disturbances, *IEEE Trans. Auto. Contr.*, Vol. AC-38, No. 8, pp. 1176-1190, 1993.

[8] A.G. Vitushkin, Theory of the Transmission and Processing of Information, Pergamon Press, 1961.

[9] L.Y. Wang, Uncertainty, information and complexity in identification and control, *International Journal of Robust and Nonlinear Control*, Oct. 2000.

[10] L.Y. Wang and L. Lin, Information-based complexity of uncertainty sets in feedback control, *IEEE Trans. Automat. Control*, to appear.

[11] G.Zames, On the metric complexity of causal linear systems: ε -entropy and ε -dimension for continuous time, *IEEE Trans. Automat. Control*, Vol. AC-24, pp.222-230, 1979.

[12] G.Zames, Information-based theory of identification and adaptation, Plenary Speech, 1996 CDC Conference, Kobe, Japan, December, 1996.