

Scheduling of a limited communication channel for optimal control

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Abstract

In this paper we outline a method for optimal off-line scheduling of a limited bandwidth communication channel that is used for control purposes. We study LQ-control of parallel control processes and describe how the theory for periodic control can be used for defining a cost functional that measures the performances of sampled data systems in relation to desired continuous time performances. This formulation results in a complex combinatorial optimization problem which is solved by exhaustive search and the solution can be viewed as an optimal allocation of the limited communication resources. The resulting optimal sequences are typically such that the sampling is nonuniform but the control law is time varying and computed to take this nonuniform sampling into account. In an example, two inverted pendulums with different closed loop performances that share a common communication resource are studied. The resulting optimal sequences are in agreement with the intuitive idea that the sampling resources should be focused to where they are needed the most, that is, to the controller with the fastest closed loop characteristics.

1 Introduction

In this paper we will present an approach to control systems design where the effects of a communication architecture with limited bandwidth, or the computational constraints of a uniprocessor is modeled and incorporated into the controller design. Traditionally, such aspects are not considered by the control designer and left for a software engineer to take care of during the implementation phase and it is interesting to consider integrated approaches to controller design and implementation. Apart from this motivation, control theory under imperfect communication and computation is a rich and interesting field in its own. The implementation of feedback control in distributed real-time systems is treated in [1] and previous control theoretic treat-

ment of limited communication control has been undertaken by Brockett [2], [3], Wong [4], [5], Tatikonda [6] and Hristu [7], [8]. Related research in the real-time systems community regarding task scheduling has been done by Choi et al [9], by Seto et al. [10] and by Ryu and Hong [11]. This paper is most closely related to [7] and [8] but also to the work by Lincoln and Bernhards-son [12]. We consider N sampled data LQ controllers that share a communication resource. Each controller is designed for one of the N parallel and uncoupled control systems and we show how optimal communication sequences can be computed off-line. What we will do is to impose the following restriction on top of the standard LQ-problems: *Only one group of control signals can be updated at each communication instant.* These instants will be referred to as ticks. The constraint will model the case where all actuators are connected to a common communication network with a limited bandwidth or the case where a single processor is used for all control computations. In order to find sampling frequencies that gives acceptable closed loop performances, rules of thumb are often used (see e.g [13]). We will not take these rules explicitly into account and the effective sampling periods for the different control processes will be given as the result of an optimization procedure. The resulting design procedure is not more complicated for the engineer than a standard LQ-design would be, however computationally more intensive. The sampled data controllers and the optimal sequences that are the result of our algorithm will be time varying and periodic and the sampling non-uniform. This nonuniform sampling is a consequence of the limited communication and the control laws are computed to take this into account. With a numerical experiment we demonstrate how the optimal sequences are in agreement with the intuitive idea that a higher bandwidth controller should be assigned a higher effective sampling frequency. The results have a potential of being useful in practice as well as being theoretically interesting and are new, to the authors best knowledge.

2 Problem formulation and solution

We will consider a number of linear control systems on state space form that are to be stabilized with a desired closed loop performance. As we are interested in finding optimal sequences for the implementation of these controllers, a reasonable and well understood notion of optimality is necessary. We therefore choose to study LQ-control of the processes. This will enable us to calculate costs associated with each sequence in a rigorous way. We will assume full state information and the reason for this is simplicity. The subject of also scheduling sensors is discussed in section 4. The outline of this section is the following: A control designer is assumed to have designed a set of N LQ-controllers for N control systems in such a way that desired closed loop performances has been achieved for the continuous time systems. These controllers are now going to be implemented on the limited resource mentioned above. We will show how the constraints due to the limited resource can be modeled in terms of so called *communication sequences*. Periodic systems theory can be used to calculate a reasonable performance metric associated with these sequences. This performance metric is computed via algebraic Riccati equations. Finding the optimal sequence is formulated as a combinatorial optimization problem which we solve by exhaustive search.

2.1 Periodic sampled-data systems

Consider now the i th of the N underlying continuous time LQ problems

$$\begin{aligned} \min_u \quad & x'(T)Q_0^c x(T) + \int_0^T x'Q_1^c x + u'Q_2^c u dt \\ \text{st} \quad & \dot{x} = A^c x + B^c u \\ & x(0) = x_0 \end{aligned} \quad (1)$$

where $x \in R^n$, $u \in R^m$ and where we consider the weighting matrices Q_1^c , Q_2^c and Q_0^c to be designed for a desired closed loop performance. For notational simplicity we have suppressed the indexing of the system matrices. Associated with the problem is the optimal value of the objective functional

$$V(t, T, x) = x'P^c(t)x \quad (2)$$

where $P^c(t)$ is the positive definite solution to the Riccati equation

$$\begin{aligned} \dot{P} &= -PA^c - A^{c'}P - Q_1^c + PB^{c'}(Q_2^c)^{-1}B^cP \\ P(T) &= Q_0^c \end{aligned} \quad (3)$$

Consider now sampled data control with sampling interval T_s of (1), that is, let $u(t) = u(kT_s)$ for $kT_s \leq t < (k+1)T_s$ and assume for simplicity $T = nT_s$. The sampling interval T_s is given by the time between the previously mentioned ticks of the communication resource. The sampled data representation (see [13]) of

(1) is

$$\begin{aligned} \min_{\bar{u}} \quad & \bar{x}'_n \bar{Q}_0 \bar{x}_n + \sum_{k=0}^{n-1} \begin{bmatrix} \bar{x}_k \\ \bar{u}_k \end{bmatrix}' \begin{bmatrix} \bar{Q}_1 & \bar{Q}_{12} \\ \bar{Q}'_{12} & \bar{Q}_2 \end{bmatrix} \begin{bmatrix} \bar{x}_k \\ \bar{u}_k \end{bmatrix} \\ \text{st} \quad & \bar{x}_{k+1} = \bar{A}\bar{x}_k + \bar{B}\bar{u}_k \end{aligned} \quad (4)$$

where $\bar{x}_k = x(kT_s)$, $\bar{u}_k = u(kT_s)$ and where as a function of the sampling time T_s

$$\begin{aligned} \bar{A} &= e^{A^c T_s} \\ \bar{B} &= \int_0^{T_s} e^{A^c t} B^c dt \\ \bar{Q}_1 &= \int_0^{T_s} e^{A^{c'} t} Q_1^c e^{A^c t} dt \\ \bar{Q}_2 &= \int_0^{T_s} \bar{B}'(t) Q_2^c \bar{B}(t) + Q_2^c dt \\ \bar{Q}_{12} &= \int_0^{T_s} e^{A^{c'} t} Q_1^c \bar{B}(t) dt \\ \bar{Q}_0 &= Q_0^c \end{aligned} \quad (5)$$

where $\bar{B}(s) = \int_0^s e^{A^c t} B^c dt$. Note that this discrete time representation is exact and that there is no approximation involved. Especially, the Q -matrices take into account any inter-sample ripple that might occur. Our model of the limited communication is, as previously mentioned, that only one control signal can be updated at each sampling instance. We can therefore associate numbers to each sample, telling what control signal will be updated at that sampling instance. Note that a ‘‘control signal’’ does not have to be a scalar, it could just as well be a vector. Let the *communication sequence* (previously introduced in [2]) $s = s_1 s_2 s_3 s_4 \dots$ be defined in the natural way

$$s_k = \{i, \text{ if control } i \text{ is updated at } t = kT_s \quad (6)$$

and the *update functions*

$$\sigma_i(j) = \begin{cases} 1, & j = i \\ 0, & \text{otherwise} \end{cases} \quad (7)$$

and

$$\sigma_i^*(j) = 1 - \sigma_i(j). \quad (8)$$

To model that the control signals that are not updated are held constant we introduce the extra state ξ defined by

$$\begin{aligned} \xi_{k+1} &= \sigma(s_k) \bar{u}_k + \sigma^*(s_k) \xi_k \\ \xi_0 &= 0 \end{aligned} \quad (9)$$

Thus ξ is the value of the control command last sent to the actuator. The state equations now assume the form

$$\begin{cases} \bar{x}_{k+1} &= \bar{A}\bar{x}_k + \bar{B}_k [\sigma(s_k) \bar{u}_k + \sigma^*(s_k) \xi_k] \\ \xi_{k+1} &= \sigma(s_k) \bar{u}_k + \sigma^*(s_k) \xi_k \end{cases} \quad (10)$$

Note that for a given communication sequence s , this system is linear and time varying and can be viewed as a function of the communication sequence. In the same way, the LQ-cost can be written in terms of the update functions σ and σ^* and if we augment the state spaces according to

$$x_k = \begin{bmatrix} \bar{x}_k \\ \xi_k \end{bmatrix} \quad \text{and} \quad u_k = \bar{u}_k \quad (11)$$

and let

$$\begin{aligned}
A_k &= \begin{bmatrix} \bar{A}_k & \bar{B}_k \sigma^*(s_k) \\ 0 & I \sigma^*(s_k) \end{bmatrix} \\
B_k &= \begin{bmatrix} \bar{B}_k \sigma(s_k) \\ \sigma(s_k) \end{bmatrix} \\
x_0 &= [\bar{x}'_0 \quad 0']' \\
Q_{1,k} &= \begin{bmatrix} \bar{Q}_{1,k} & \bar{Q}_{12} \sigma^*(s_k) \\ \bar{Q}'_{12} \sigma^*(s_k) & \bar{Q}_2 \sigma^*(s_k) \end{bmatrix} \\
Q_{12,k} &= \begin{bmatrix} \bar{Q}_{12} \sigma(s_k) \\ 0 \end{bmatrix} \\
Q_2 &= Q_2 \\
Q_0 &= \begin{bmatrix} \bar{Q}_0 & 0 \\ 0 & 0 \end{bmatrix}
\end{aligned} \tag{12}$$

then we have the time varying LQ-problem

$$\begin{aligned}
\min_u \quad & x'_n Q_0 x_n + \sum_{k=1}^{n-1} \begin{bmatrix} x \\ u \end{bmatrix}' \begin{bmatrix} Q_1 & Q_{12} \\ Q'_{12} & Q_2 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}_k \\
\text{st} \quad & x_{k+1} = A_k x_k + B_k u_k \\
& x(0) = x_0
\end{aligned} \tag{13}$$

As is well known the cross-term Q_{12} can easily be eliminated and the solution to (13) is the discrete time LQ-feedback

$$u_k = K_k^{LQ} x_k \tag{14}$$

and the associated optimal cost is

$$V_{k,n}(x) = x' P_k x \tag{15}$$

where P_k is the solution to the discrete time Riccati equation associated with (13). From an implementation perspective it is quite natural to consider periodic sequences with bounded periodicity. Let the set of p -periodic sequences be

$$\mathcal{S}_p = \{s = s_1 s_2 \dots | s_i = s_{i+p} \forall i \geq 1\}. \tag{16}$$

For $s \in \mathcal{S}_p$, the N different LQ-problems will be periodic and it can be shown (see [14] for details) that the steady state solution is given by a periodic controller given by the periodic solution to the corresponding periodic Riccati equation provided controllability and observability is fulfilled. It is also shown that the periodic solution can be found via lifting and by solving an algebraic Riccati equation for the lifted system. Let therefore the set $\mathcal{S}_p^{co} \subseteq \mathcal{S}_p$ be the set of such sequences and let from now on $s \in \mathcal{S}_p^{co}$ and consider the case when $T, n \rightarrow \infty$. The LQ costs will thus be $V_k(x, s) = x' P_k(s) x$ where $P_k(s)$ is the periodic solution to the periodic Riccati equation associated with (13). Finding the periodic solution is straightforward but tedious and the expression for the matrices of the lifted system is not included. To summarize, for a given periodic sequence, the solution to the N sampled data LQ-problems basically amounts to solving N algebraic Riccati equations which gives the periodic controllers and the periodic optimal cost. These Riccati equation

solutions are obtained by downsampling and by augmenting the input and output spaces with the signals at the instances between the downsamples. Consequently, the state space dimension is unchanged. Still, there are considerable computations involved in finding the solutions. For the simple pendulums considered in section 3, the computational time needed to compute the cost of one sequence was 0.17 CPU seconds on a Sparc Ultra 5 workstation. The time needed to compute the costs of all 8-periodic sequences was 43 CPU seconds. Also, the computational time grows exponentially with the sequence periodicity.

2.2 Optimality and problem solution

Recall that we are aiming at a problem formulation which will give us the optimal sequence where the ideal situation is that the sampled data closed loop performance is identical to the continuous. We claim that it makes sense to consider the deviations from the optimal continuous time cost as a performance measure. The continuous time optimal cost is given by (2) and is $V^c(x) = x' P^c x$ and the relative deviation from that cost is given by

$$\frac{x' \bar{P}_k(s) x - x' P^c x}{x' P^c x}$$

where \bar{P}_k is used to denote the left upper block of P_k associated with the original states \bar{x}_k (recall that the states were augmented in equation 11). If the deviation is averaged over one period and maximized over all possible non-zero initial conditions, we can write the performance of one process as

$$f_i(s) = \max_{\|x\| \neq 0} \frac{x' \sum_{k=1}^p \frac{1}{p} \bar{P}_k(s) x}{x' P^c x} - 1 \tag{17}$$

which is now only a function of the sequence s . The index i is introduced to denote that everything in the expression is related to the i th system. As performance measure for the entire set of control systems we take the weighted sum

$$f(s) = \sum_{i=1}^N w_i f_i(s). \tag{18}$$

In this way we are able to state the problem of finding the optimal sequence as the combinatorial optimization problem

$$\begin{aligned}
\min \quad & f(s) \\
& s \in \mathcal{S}_p^{co} \\
& p \in \{1 \dots T_p\}
\end{aligned} \tag{19}$$

where $T_p < \infty$ is a predefined maximal periodicity introduced to ensure the existence of a solution to the problem. For example, if the optimal sequence for a problem with two parallel systems is nonperiodic, say

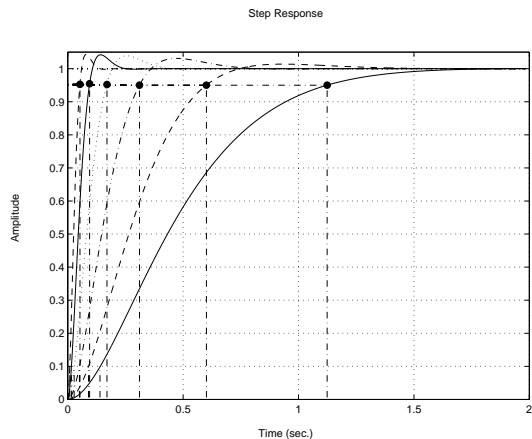


Figure 1: Closed loop specifications for a step change in the reference signal. The rise times are from left to right: 0.05, 0.1, 0.15, 0.25, 0.5 and 1.0 seconds.

$s = 12112111211112 \dots$ there would be no optimal periodic sequence. Introducing T_p eliminates this problem.

It is possible to achieve a solution to (19) by exhaustive search as the set $\{s \in \mathcal{S}_p^c, p \in \{1, \dots, T_p\}\}$ is finite and this is the way the examples in section 3 are obtained. For larger, higher dimensional problems and longer sequence periodicities a more efficient solution method should be developed and we will discuss this matter in section 4.

Remark: $f_i(s)$ can be rewritten in terms of eigenvalues of the involved matrices as

$$f_i(s) = \max_{\text{eig}} \left\{ (P^c)^{-1} \sum_{k=1}^p \frac{1}{P} \bar{P}_k(s) \right\} - 1 \quad (20)$$

To see this, consider the positive definite symmetric matrices U , V and W where $W'W = V$. It holds that

$$\begin{aligned} \max_{\|x\| \neq 0} \frac{x'Ux}{x'Vx} &= \max_{\|z\| \neq 0} \frac{z'W^{-T}UW^{-1}z}{z'Tz} = \\ &= \max_{\text{eig}} \{W^{-T}UW\} = \\ &= \max_{\text{eig}} \{V^{-1}U\} \end{aligned}$$

3 Example

To demonstrate our algorithms we will consider the stabilization of two physically independent inverted pendulums. The controllers for the pendulums share the same communication channel, and the control implementation is in agreement with our problem formulations so only one controller can use the channel at each tick. We will study how different specifications on closed loop performance affect the optimal sequence.

Simulations of the system with two pendulums will also be made.

3.1 Models of the dynamics and computer system

The setup consists of two inverted mathematical pendulums controlled by torques, T_i , at the base joints. The two pendulums have exactly the same dynamics, i.e. the masses and lengths are the same ($l = 1$, $m = 1$). The deviation from the upright position is denoted by the angle ϕ . The dynamics for an inverted pendulum of length l is written in its linearized form as

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ g/l & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ g/l \end{bmatrix} u(t) \quad (21)$$

where $x' = [\phi \ \dot{\phi}]'$ is the state and $u = T/(mgl)$ is the scaled control signal. The model (21) is of the same form as (1) and the methods described in this paper can readily be applied.

3.2 Control design

The control specifications for the two pendulums are given as rise times (5% to 95% of the final value) and overshoots for a set-point change. The overshoots must be lower than 5% and the rise time for the second pendulum 1 s. For the first pendulum we will study the rise times 0.05, 0.1, 0.15, 0.25, 0.5 and 1.0 s and see how the optimal sequence depends on these rise times. To obtain the desired closed loop performances the control weight matrices are taken as $(Q_2^c)_1 = (Q_2^c)_2 = 1$ and the state weight matrices Q_1^c as

$$(Q_1^c)_1 = \begin{bmatrix} \beta^2 & 0 \\ 0 & 0 \end{bmatrix} \quad (Q_1^c)_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad (22)$$

and we can vary β to penalize the angle ϕ . By keeping everything else constant a relation between the specifications and the optimal sequence becomes easier to comprehend. The resulting step responses fulfill the specifications as is shown in figure 1 where the curves have been scaled to yield the same steady state values. The sampling interval is taken to be $T_s = 0.010$ s. This is within the range specified by the rules of thumb in [13] stating that the sampling time should be 4-7

Rise time	β^2	Sequence \hat{s}
1.0	1	12 ...
0.5	10	12 ...
0.25	100	12 ...
0.15	1000	112 ...
0.10	10000	1112 ...
0.05	100000	11112 ...

Table 1: Optimal sequences for two controllers. The rise times indicated in the table is for the first controller. The second has a rise time of 1s.

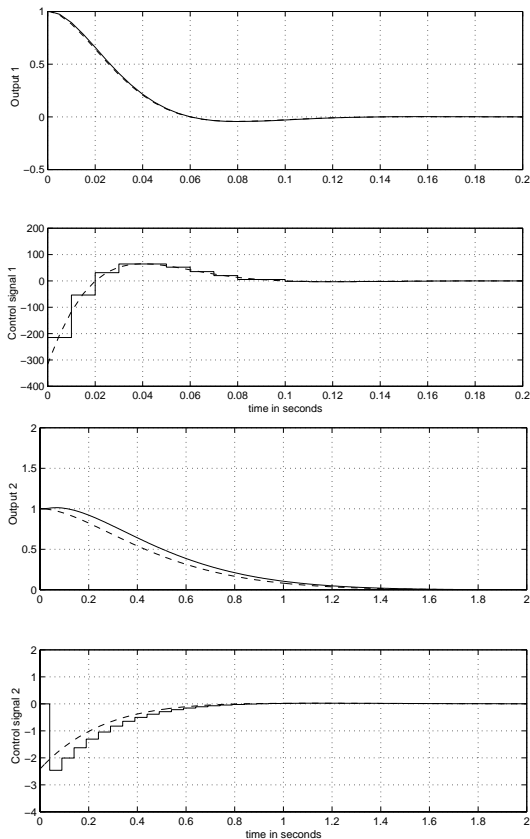


Figure 2: The optimal sequence: Graphs of the output (plot 1 & 3) and the control signal (plot 2 & 4) of two control system, one with a high bandwidth (No 1 - top) and one with a low bandwidth (No 2 - bottom). The sampling time is $T_s = 0.01$ and the optimal sequence is $s = 11112 \dots$

times faster than the rise time of the system. In table 1 the optimal sequence corresponding to the specified rise times for the first pendulum and the corresponding β^2 -values shows a behavior that could be expected from intuition. This behavior is that more communication resources should be allocated to the controller which has the highest bandwidth requirement. Note however that when the fastest system has a rise time one fourth of the slowest system, the sequence still has not changed. In this example we have searched for the optimal sequence of maximal periodicity 8 but the optimal sequence is of length 5.

3.3 Simulation

We have simulated the system (Matlab/Simulink) with the two pendulums to demonstrate the calculated sequences. In figure 2 the initial value response is shown for the case when the rise time for the first pendulum is specified to be 0.05s and 1s for the second pendulum. Note how the effect of the optimal sequence $s = 11112 \dots$ is such that the high performance

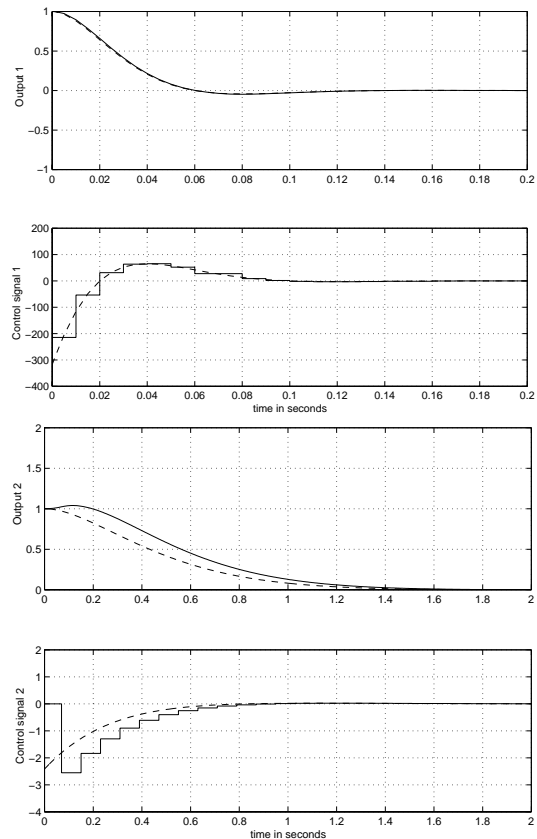


Figure 3: A nonoptimal sequence: $s = 11111112 \dots$. Graphs of the output (plot 1 & 3) and the control signal (plot 2 & 4) of two control system, one with a high bandwidth (No 1 - top) and one with a low bandwidth (No 2 - bottom).

controller (left) follows the closed loop specifications closely due to the intensive sampling and how the low performance controller (right) does not follow the specifications as closely. The time scales in the two graphs differ by an order of magnitude. As a concluding example we will study the sequence $s = 11111112 \dots$. This is an interesting sequence to compare the optimal one with as the ratio of the rise times for the two closed loop systems are $1/0.05 = 20$ and it might be argued that, with such a large difference in rise time, the sequence with the maximal attention to the fast controller should be chosen. In our case of 8-periodic sequences, $s = 11111112 \dots$ is that sequence. In figure 3 it can be seen that the performance of the slow controller is decreased while it for the fast controller hardly is changed at all. It appears that intuition might be misleading in this case.

4 Discussion and summary

In this paper we have outlined a method for optimal off-line scheduling of a limited bandwidth communication

channel that is used for control purposes. The resulting optimal sequences are typically such that the sampling is nonuniform but the control laws are time varying and computed to take this nonuniform sampling into account. Our formulation results in a combinatorial optimization problem of considerable complexity due to the fact that the discrete search space is large and the objective function is expensive to compute. A numerical experiment demonstrates that the optimal sequences are in agreement with the intuitive idea that the most sampling resources should be distributed to the system with the fastest closed loop characteristics. We will continue this work by studying a number of interesting questions and extensions. We have assumed full state information and only studied scheduling of control signals and it would be interesting to incorporate scheduling of measurement signals into the problem. The extension seems straightforward regarding modeling but the solution procedure will be even more computationally expensive. Further, we have considered parallel and independent processes. This is not necessary and a more general formulation would be that of a MIMO-system where sensors and actuators share a common channel. This setting is investigated in [8] but it is not treated in terms of an LQG-problem. In order to be able to solve larger, higher dimensional problem with longer sequence periods an efficient optimization algorithm should be developed. A more efficient way of computing $f(s)$ would be of interest as well as finding out how the problem could be suitably embedded into a continuous and convex problem. This would make a branch and bound solution possible. Regardless of method, $f(s)$ and the structure of S_p^{co} should be investigated, both from a control theoretic and an optimization point of view. To summarize, we have developed a version of sampled data LQ-control where a limited communication resource is taken into account. The resource allocation is automatic and the resulting sequence and controller is integrated in such a way that the overall system performance is maximized and the implementation phase is made easier. The procedure is computationally heavy but not more complicated than a standard LQ-design for the control designer. We are of the opinion that the results and ideas are of practical importance for control systems implementation as well as being theoretically interesting.

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